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DYNAMICS AND GLOBAL BEHAVIOR FOR A FOURTH-ORDER RATIONAL DIFFERENCE EQUATION

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Abstract

In this paper we study the behavior of the rational difference equation of the fourth order

$$x_{n+1} = ax_n + \frac{bx_n x_{n-2}}{cx_{n-2} + dx_{n-3}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-3},\ x_{-2}, x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we give the solution of some special cases of this equation.

Keywords: difference equations, stability, boundedness, periodicity, solution of difference equations.

2000 AMS Classification: 39A10

1. Introduction

In this paper we deal with the behavior of the solutions of the following nonlinear difference equation

$$(1) \qquad x_{n+1} = ax_n + \frac{bx_n x_{n-2}}{cx_{n-2} + dx_{n-3}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we give the solution of some special cases of this equation.

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Many researchers have investigated the behavior of the solution of difference equations for example: Camouzis [6] has investigated the global attractivity and the local stability of the difference equation

$$x_{n+1} = \frac{bx_n^2}{1+x_{n-1}^2}$$

Cinar [7] has got the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}.$$

In [9] Elabbasy et al. investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Elabbasy et al. [10] studied the global stability, boundedness, periodicity character and obtained the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [11] investigated the global stability, periodicity character and got the form of the solution of some special cases of the difference equation

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$

Touafek [35] dealt with the behavior of the second order rational difference equation

$$x_{n+1} = \frac{ax_n^4 + bx_nx_{n-1}^3 + cx_n^2x_{n-1}^2 + dx_n^3x_{n-1} + ex_{n-1}^4}{Ax_n^4 + Bx_nx_{n-1}^3 + Cx_n^2x_{n-1}^2 + Dx_n^3x_{n-1} + Ex_{n-1}^4}.$$

In [42] Yalçınkaya dealt with the behavior of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed [44] studied the dynamics of the nonlinear rational difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

See also [1]–[5], [8]. Other related results on rational difference equations can be found in refs. [12]–[20].

Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. Some nonlinear difference equations, especially the boundedness, global attractivity, oscillatory and some other properties of second order (and other order) nonlinear difference equations and systems of difference equations have been investigated by many authors, see [21]–[44]. Let us introduce some basic definitions and some theorems that we need in the sequel. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

1.1. Definition (Equilibrium Point)**.** A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

1.2. Definition (Periodicity)**.** A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

1.3. Definition (Fibonacci Sequence)**.** The sequence $\{F_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, ...\}$ i.e. $F_m = F_{m-1} + F_{m-2}$, $m \geq 0$, $F_{-2} = 0$, $F_{-1} = 1$ is called Fibonacci Sequence.

1.4. Definition (Stability)**.** (i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [27]: Assume that $p_i \in R$, $i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \tag{4}$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-2}, x_{n-3}). \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [28]**:** Let $[a, b]$ be an interval of real numbers and assume that

$$g : [a, b]^3 \to [a, b],$$

is a continuous function satisfying the following properties :

(a) $g(x,y,z)$ is non-decreasing in x and y in $[a,b]$ for each $z\in[a,b]$, and is non-increasing in $z\in[a,b]$ for each x and y in $[a,b]$;

(b) If $(m,M)\in[a,b]\times[a,b]$ is a solution of the system

$$M=g(M,M,m) \quad \textit{and} \quad m=g(m,m,M),$$

then

$$m=M.$$

Then Eq.(5) has a unique equilibrium $\overline{x}\in[a,b]$ and every solution of Eq.(5) converges to $\overline{x}$.

2. Local Stability of Eq.(1)

In this section we investigate the local stability character of the solutions of Eq.(1). Eq.(1) has a unique equilibrium point and is given by

$$\overline{x}=a\overline{x}+\frac{b\overline{x}^2}{c\overline{x}+d\overline{x}},$$

or,

$$\overline{x}^2(1-a)(c+d)=b\overline{x}^2,$$

if $(c+d)(1-a)\neq b$, then the unique equilibrium point is $\overline{x}=0$.

Let $f:(0,\infty)^3\longrightarrow(0,\infty)$ be a function defined by

$$f(u,v,w)=au+\frac{buv}{cv+dw}. \tag{6}$$

Therefore it follows that

$$\begin{aligned} f_u(u,v,w) &= a+\frac{bv}{cv+dw}, \\ f_v(u,v,w) &= \frac{bduw}{(cv+dw)^2}, \\ f_w(u,v,w) &= \frac{-bduv}{(cv+dw)^2}, \end{aligned}$$

we see that

$$\begin{aligned} f_u(\overline{x},\overline{x},\overline{x}) &= a+\frac{b}{c+d}, \\ f_v(\overline{x},\overline{x},\overline{x}) &= \frac{bd}{(c+d)^2}, \\ f_w(\overline{x},\overline{x},\overline{x}) &= \frac{-bd}{(c+d)^2}. \end{aligned}$$

The linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1}-\left(a+\frac{b}{c+d}\right)y_n-\frac{bd}{(c+d)^2}y_{n-2}+\frac{bd}{(c+d)^2}y_{n-3}=0. \tag{7}$$

2.1. Theorem. *Assume that*

$$b(c+3d)<(1-a)(c+d)^2.$$

Then the equilibrium point of Eq.(1) is locally asymptotically stable.

Proof. It is follows by Theorem A that, Eq.(7) is asymptotically stable if

$$\left|a+\frac{b}{c+d}\right|+\left|\frac{bd}{(c+d)^2}\right|+\left|\frac{bd}{(c+d)^2}\right|<1,$$

or,

$$a+\frac{b}{c+d}+\frac{2bd}{(c+d)^2}<1,$$

and so,

$$\frac{bc+3bd}{(c+d)^2}<(1-a).$$

The proof is complete. □

3. Global Attractor of the Equilibrium Point of Eq.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).

3.1. Theorem. *The equilibrium point $\overline{x}$ of Eq.(1) is global attractor if $c(1-a)\neq b$.*

Proof. Let p,q are a real numbers and assume that $g:[p,q]^3\longrightarrow[p,q]$ be a function defined by $g(u,v,w)=au+\dfrac{buw}{cv+dw}$, then we can easily see that the function $g(u,v,w)$ increasing in u,v and decreasing in w.

Suppose that (m,M) is a solution of the system

$$M=g(M,M,m)\quad and\quad m=g(m,m,M).$$

Then from Eq.(1), we see that

$$M=aM+\frac{bM^2}{cM+dm},\quad m=am+\frac{bm^2}{cm+dM},$$

or,

$$M(1-a)=\frac{bM^2}{cM+dm},\quad m(1-a)=\frac{bm^2}{cm+dM},$$

then

$$c(1-a)M^2+d(1-a)Mm=bM^2,\quad c(1-a)m^2+d(1-a)Mm=bm^2,$$

Subtracting we obtain

$$c(1-a)(M^2-m^2)=b(M^2-m^2),\quad c(1-a)\neq b.$$

Thus

$$M=m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete. □

4. Boundedness of solutions of Eq.(1)

In this section we study the boundedness of solutions of Eq.(1).

4.1. Theorem. *Every solution of Eq.(1) is bounded if* $\left(a+\dfrac{b}{c}\right)<1$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1}=ax_n+\frac{bx_nx_{n-2}}{cx_{n-2}+dx_{n-3}}\leq ax_n+\frac{bx_nx_{n-2}}{cx_{n-2}}=(a+\frac{b}{c})x_n.$$

Then

$$x_{n+1}\leq x_n \quad \text{for all} \quad n\geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so are bounded from above by $M=\max\{x_{-3},x_{-2},x_{-1},x_0\}$. □

For confirming the results of this section, we consider numerical example for $x_{-3}=7,\ x_{-2}=2,\ x_{-1}=9,\ x_0=6,\ a=0.2,\ b=0.4,\ c=5,\ d=3$. [See Fig. 1] and for $x_{-3}=7,\ x_{-2}=2,\ x_{-1}=9,\ x_0=6,\ a=0.2,\ b=9,\ c=5,\ d=3.$. [See Fig. 2]

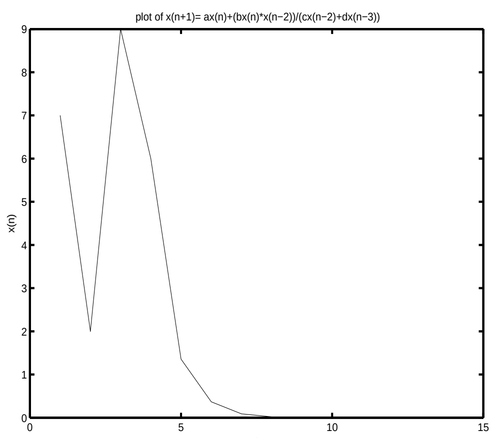

FIGURE 1

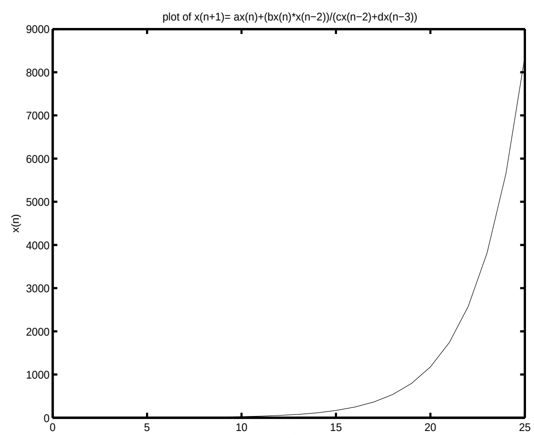

FIGURE 2

5. Special Cases of Eq.(1)

5.1. First Case. In this section we study the following special case of Eq.(1)

$$x_{n+1}=x_n+\frac{x_nx_{n-2}}{x_{n-2}+x_{n-3}}, \tag{8}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers.

5.1. Theorem. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of Eq.(8). Then for* $n=0,1,2,...$

$$\begin{aligned}
x_{3n} &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right),\\
x_{3n+1} &= h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right),\\
x_{3n+2} &= h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right),
\end{aligned}$$

where $x_{-3}=p,\ x_{-2}=r,\ x_{-1}=k,\ x_0=h,\ \{f_m\}_{m=1}^{\infty}=\{1,1,2,3,5,8,13,...\},\ f_{-1}=f_0=1.$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1,\ n-2$. That is;

$$\begin{aligned}
x_{3n-4} &= h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right),\\
x_{3n-3} &= h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right),\\
x_{3n-2} &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right),\\
x_{3n-1} &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right).
\end{aligned}$$

Now, it follows from Eq.(8) that

$$x_{3n}=x_{3n-1}+\frac{x_{3n-1}x_{3n-3}}{x_{3n-3}+x_{3n-4}}=h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)$$

$$+\frac{\left(h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\right)\left(h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\right)}{\left(h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\right)+\left(h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\right)}$$

$$\begin{aligned}
=\ & h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\\
&+\frac{h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2n-1}h+f_{2n-2}k}{f_{2n-2}h+f_{2n-3}k}\right)}{\left(\frac{f_{2n-1}h+f_{2n-2}k}{f_{2n-2}h+f_{2n-3}k}\right)+1}
\end{aligned}$$

$$\begin{aligned}
=\ & h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\\
&\left(1+\frac{f_{2n-1}h+f_{2n-2}k}{f_{2n-1}h+f_{2n-2}k+f_{2n-2}h+f_{2n-3}k}\right)
\end{aligned}$$

$$=h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2n+1}h+f_{2n}k}{f_{2n}h+f_{2n-1}k}\right).$$

Therefore

$$x_{3n}=h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right).$$

Also, from Eq.(8), we see that

$$\begin{aligned} x_{3n+1} &= x_{3n} + \frac{x_{3n}x_{3n-2}}{x_{3n-2}+x_{3n-3}} \\ &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right) \\ &\quad + \frac{\begin{array}{c}\left(h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\right) \\ \left(h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\right)\end{array}}{\begin{array}{c}\left(h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\right) \\ +\left(h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\right)\end{array}} \end{aligned}$$

$$\begin{aligned} x_{3n+1} &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right) \\ &\quad + \frac{h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(\frac{f_{2n+1}r+f_{2n}p}{f_{2n}r+f_{2n-1}p}\right)}{\left(\frac{f_{2n+1}r+f_{2n}p}{f_{2n}r+f_{2n-1}p}\right)+1} \end{aligned}$$

$$\begin{aligned} &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right) \\ &\quad + \frac{h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)(f_{2n}r+f_{2n-1}p)}{f_{2n+1}r+f_{2n}p+f_{2n}r+f_{2n-1}p} \\ &= h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(1+\frac{f_{2n+1}r+f_{2n}p}{f_{2n+2}r+f_{2n+1}p}\right). \end{aligned}$$

Thus

$$x_{3n+1} = h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right).$$

Also,

$$\begin{aligned} x_{3n+2} &= x_{3n+1} + \frac{x_{3n+1}x_{3n-1}}{x_{3n-1}+x_{3n-2}} \\ &= h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right) \\ &\quad + \frac{h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2n+1}k+f_{2n}r}{f_{2n}k+f_{2n-1}r}\right)}{\left(\frac{f_{2n+1}k+f_{2n}r}{f_{2n}k+f_{2n-1}r}\right)+1} \end{aligned}$$

$$= h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r+f_{2i+2}p}{f_{2i+2}r+f_{2i+1}p}\right)\left(\frac{f_{2i+1}k+f_{2i}r}{f_{2i}k+f_{2i-1}r}\right)\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(1+\frac{f_{2n+1}k+f_{2n}r}{f_{2n+1}k+f_{2n}r+f_{2n}k+f_{2n-1}r}\right)$$

$$= h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r + f_{2i+2}p}{f_{2i+2}r + f_{2i+1}p}\right)\left(\frac{f_{2i+1}k + f_{2i}r}{f_{2i}k + f_{2i-1}r}\right)\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(1 + \frac{f_{2n+1}k+f_{2n}r}{f_{2n+2}k+f_{2n+1}r}\right)$$

Therefore

$$x_{3n+2} = h\prod_{i=0}^{n}\left(\frac{f_{2i+3}r + f_{2i+2}p}{f_{2i+2}r + f_{2i+1}p}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right).$$

Hence, the proof is completed. □

For confirming the results of this section, we consider numerical example for $x_{-3} = 3,\ x_{-2} = 7,\ x_{-1} = 2,\ x_0 = 6$. [See Fig. 3].

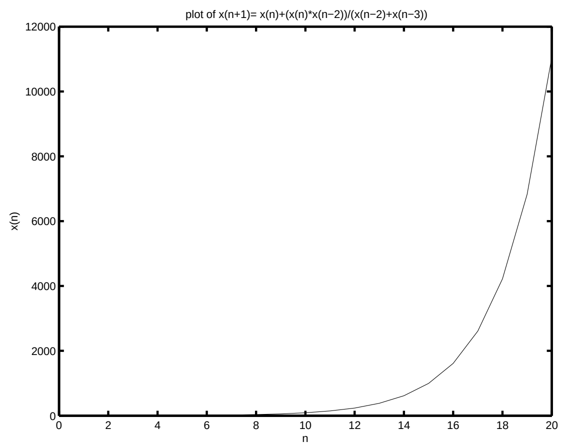

FIGURE 3

5.2. Second Case. In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = x_n + \frac{x_n x_{n-2}}{x_{n-2} - x_{n-3}}, \tag{9}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers with $x_{-3} \neq x_{-2} \neq x_{-1} \neq x_0$.

5.2. Theorem. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(9). Then for $n = 0, 1, 2, ...$*

$$x_{3n} = h\prod_{i=0}^{n-1}\left(\frac{f_{i+3}r - f_{i+1}p}{f_{i+1}r - f_{i-1}p}\right)\left(\frac{f_{i+3}k - f_{i+1}r}{f_{i+1}k - f_{i-1}r}\right)\left(\frac{f_{i+3}h - f_{i+1}k}{f_{i+1}h - f_{i-1}k}\right),$$

$$x_{3n+1} = h\left(\frac{2r-p}{r-p}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+4}r - f_{i+2}p}{f_{i+2}r - f_{i}p}\right)\left(\frac{f_{i+3}k - f_{i+1}r}{f_{i+1}k - f_{i-1}r}\right)\left(\frac{f_{i+3}h - f_{i+1}k}{f_{i+1}h - f_{i-1}k}\right),$$

$$x_{3n+2} = h\left(\frac{2r-p}{r-p}\right)\left(\frac{2k-r}{k-r}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+4}r - f_{i+2}p}{f_{i+2}r - f_{i}p}\right)\left(\frac{f_{i+4}k - f_{i+2}r}{f_{i+2}k - f_{i}r}\right)\left(\frac{f_{i+3}h - f_{i+1}k}{f_{i+1}h - f_{i-1}k}\right),$$

where $x_{-3} = p,\ x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, ...\},\ f_{-1} = 1,\ f_0 = 0.$

Proof. As the proof of Theorem 5.1 and will be omitted. □

Assume that $x_{-3} = 9,\ x_{-2} = 7,\ x_{-1} = 11,\ x_0 = 6$. [See Fig. 4], and for $x_{-3} = 7$, $x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 8$. [See Fig. 5].

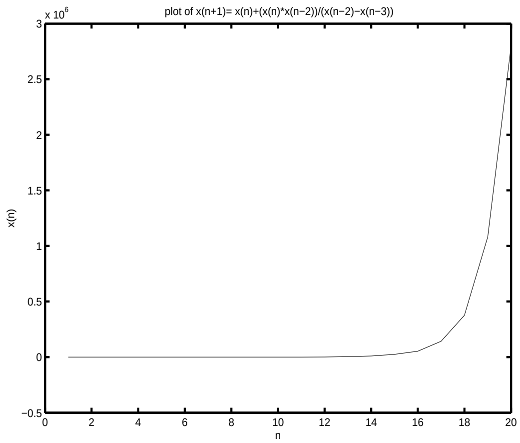

FIGURE 4

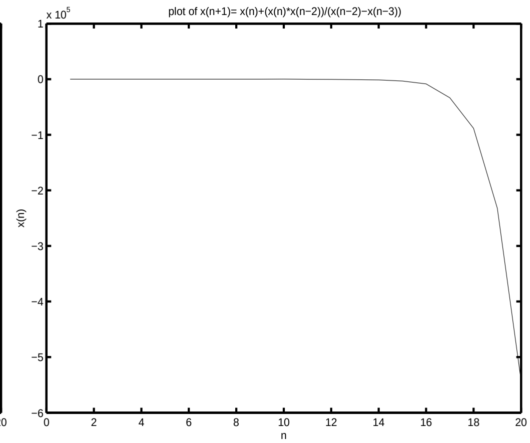

FIGURE 5

5.3. Third Case. In this section we obtain the solution of the following special case of Eq.(1)

$$x_{n+1} = x_n - \frac{x_n x_{n-2}}{x_{n-2} + x_{n-3}}, \tag{10}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers.

5.3. Theorem. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(10). Then for $n = 0, 1, 2, ...$*

$$\begin{aligned}
x_{3n} &= \frac{hkrp}{(f_n r + f_{n+1} p)(f_n k + f_{n+1} r)(f_n h + f_{n+1} k)}, \\
x_{3n+1} &= \frac{hkrp}{(f_{n+1} r + f_{n+2} p)(f_n k + f_{n+1} r)(f_n h + f_{n+1} k)}, \\
x_{3n+2} &= \frac{hkrp}{(f_{n+1} r + f_{n+2} p)(f_{n+1} k + f_{n+2} r)(f_n h + f_{n+1} k)}
\end{aligned}$$

where $x_{-3} = p,\ x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{f_m\}_{m=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, ...\},\ f_0 = 1$.

Proof. For $n = 0, 1$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1,\ n - 2$. That is;

$$\begin{aligned}
x_{3n-4} &= \frac{hkrp}{(f_{n-1} r + f_n p)(f_{n-1} k + f_n r)(f_{n-2} h + f_{n-1} k)}, \\
x_{3n-3} &= \frac{hkrp}{(f_{n-1} r + f_n p)(f_{n-1} k + f_n r)(f_{n-1} h + f_n k)}, \\
x_{3n-2} &= \frac{hkrp}{(f_n r + f_{n+1} p)(f_{n-1} k + f_n r)(f_{n-1} h + f_n k)}, \\
x_{3n-1} &= \frac{hkrp}{(f_n r + f_{n+1} p)(f_n k + f_{n+1} r)(f_{n-1} h + f_n k)}.
\end{aligned}$$

Now, it follows from Eq.(10) that

$$\begin{aligned}
x_{3n} &= x_{3n-1}-\frac{x_{3n-1}x_{3n-3}}{x_{3n-3}+x_{3n-4}}=\frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_{n-1}h+f_nk)} \\
&\quad -\frac{\left(\frac{hkrp}{(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}h+f_nk)}\right)\left(\frac{hkrp}{(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-1}h+f_nk)}\right)}{\frac{hkrp}{(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-1}h+f_nk)}+\frac{hkrp}{(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-2}h+f_{n-1}k)}} \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_{n-1}h+f_nk)} \\
&\quad -\frac{\left(\frac{hkrp}{(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}h+f_nk)}\right)\left(\frac{1}{f_{n-1}h+f_nk}\right)}{\frac{1}{(f_{n-1}h+f_nk)}+\frac{1}{(f_{n-2}h+f_{n-1}k)}} \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_{n-1}h+f_nk)} \\
&\quad -\frac{\left(\frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_{n-1}h+f_nk)}\right)(f_{n-2}h+f_{n-1}k)}{f_{n-2}h+f_{n-1}k+f_{n-1}h+f_nk} \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_{n-1}h+f_nk)}\left(1-\frac{f_{n-2}h+f_{n-1}k}{f_nh+f_{n+1}k}\right) \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_{n-1}h+f_nk)}\left(\frac{f_{n-1}h+f_nk}{f_nh+f_{n+1}k}\right).
\end{aligned}$$

Therefore

$$x_{3n}=\frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}.$$

Also, from Eq.(10), we see that

$$\begin{aligned}
x_{3n+1} &= x_{3n}-\frac{x_{3n}x_{3n-2}}{x_{3n-2}+x_{3n-3}}=\frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)} \\
&\quad -\frac{\frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\frac{1}{(f_nr+f_{n+1}p)}}{\frac{1}{(f_nr+f_{n+1}p)}+\frac{1}{(f_{n-1}r+f_np)}} \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_nh+f_{n+1}k)}\left(1-\frac{(f_{n-1}r+f_np)}{(f_{n-1}r+f_np)+(f_nr+f_{n+1}p)}\right) \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(1-\frac{f_{n-1}r+f_np}{f_{n+1}r+f_{n+2}p}\right) \\
&= \frac{hkrp}{(f_nr+f_{n+1}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(\frac{f_{n+1}r+f_{n+2}p-f_{n-1}r-f_np}{f_{n+1}r+f_{n+2}p}\right).
\end{aligned}$$

Thus

$$x_{3n+1}=\frac{hkrp}{(f_{n+1}r+f_{n+2}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}.$$

Also,

$$\begin{aligned} x_{3n+2} &= x_{3n+1} - \frac{x_{3n+1}x_{3n-1}}{x_{3n-1}+x_{3n-2}} = \frac{hkrp}{(f_{n+1}r+f_{n+2}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)} \\ &\quad - \frac{\frac{hkrp}{(f_{n+1}r+f_{n+2}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\frac{1}{(f_nk+f_{n+1}r)}}{\frac{1}{(f_nk+f_{n+1}r)}+\frac{1}{(f_{n-1}k+f_nr)}} \\ &= \frac{hkrp}{(f_{n+1}r+f_{n+2}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(1-\frac{(f_{n-1}k+f_nr)}{(f_{n-1}k+f_nr)+(f_nk+f_{n+1}r)}\right) \\ &= \frac{hkrp}{(f_{n+1}r+f_{n+2}p)\,(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(1-\frac{f_{n-1}k+f_nr}{f_{n+1}k+f_{n+2}r}\right). \end{aligned}$$

Therefore

$$x_{3n+2} = \frac{hkrp}{(f_{n+1}r+f_{n+2}p)\,(f_{n+1}k+f_{n+2}r)\,(f_nh+f_{n+1}k)}.$$

Hence, the proof is completed. □

Fig. 6 shows the solution of Eq. (10) when $x_{-3}=5,\ x_{-2}=8,\ x_{-1}=4,\ x_0=9$.

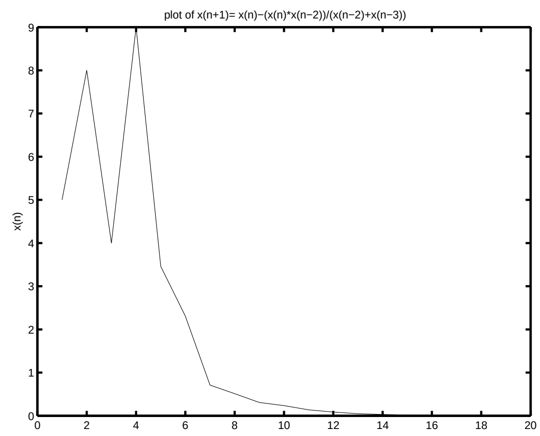

FIGURE 6

5.4. Fourth Case. In this section we study the following special case of Eq.(1)

$$x_{n+1} = x_n - \frac{x_nx_{n-2}}{x_{n-2}-x_{n-3}}, \tag{11}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers.with $x_{-3}\neq x_{-2}\neq x_{-1}\neq x_0$.

5.4. Theorem. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(11). Then every solution of Eq.(11) is periodic with period 18. Moreover $\{x_n\}_{n=-3}^{\infty}$ takes the form*

$$\left\{\begin{array}{c} p,r,k,h,\frac{hp}{(p-r)},\frac{hpr}{(p-r)(r-k)},\frac{hprk}{(p-r)(r-k)(k-h)}, \\ \frac{-hpk}{(r-k)(k-h)},\frac{hp}{(k-h)},-p,-r,-k,-h,\frac{-hp}{(p-r)},\frac{-hpr}{(p-r)(r-k)}, \\ \frac{-hprk}{(p-r)(r-k)(k-h)},\frac{-hpk}{(r-k)(k-h)},\frac{-hp}{(k-h)},p,r,k,h,... \end{array}\right\},$$

or,

$$\begin{aligned}
x_{18n-3} &= p,\ x_{18n-2}=r,\ x_{18n-1}=k,\ x_{18n}=h,\ x_{18n+1}=\frac{hp}{(p-r)},\\
x_{18n+2} &= \frac{hpr}{(p-r)(r-k)},\ x_{18n+3}=\frac{hprk}{(p-r)(r-k)(k-h)},\\
x_{18n+4} &= \frac{-hpk}{(r-k)(k-h)},\ x_{18n+5}=\frac{hp}{(k-h)},\ x_{18n+6}=-p,\\
x_{18n+7} &= -r,\ x_{18n+8}=-k,\ x_{18n+9}=-h,\ x_{18n+10}=\frac{-hp}{(p-r)},\\
x_{18n+11} &= \frac{-hpr}{(p-r)(r-k)},\ x_{18n+12}=\frac{-hprk}{(p-r)(r-k)(k-h)},\\
x_{18n+13} &= \frac{-hpk}{(r-k)(k-h)},\ x_{18n+14}=\frac{-hp}{(k-h)},
\end{aligned}$$

where $x_{-3}=p,\ x_{-2}=r,\ x_{-1}=k,\ x_0=h.$

Proof. The proof is left to the reader. □

Fig. 7 shows the solution of Eq.(11) when $x_{-3}=6,\ x_{-2}=3,\ x_{-1}=7,\ x_0=9$ and Fig. 8 shows the solution when $x_{-3}=-9,\ x_{-2}=6,\ x_{-1}=-5,\ x_0=8.$

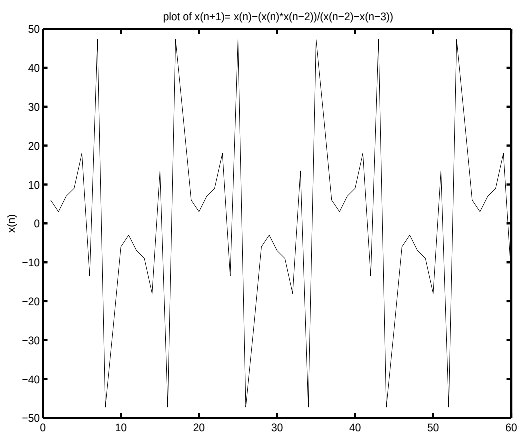

FIGURE 7

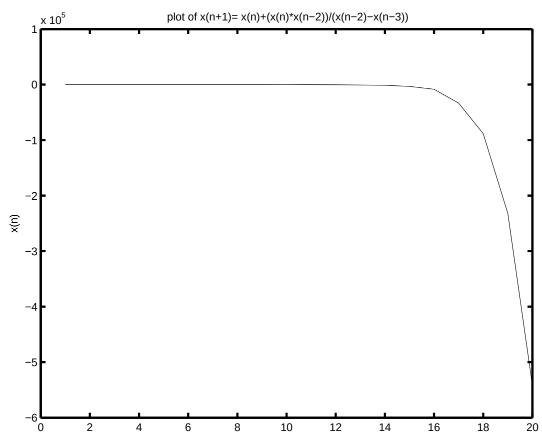

FIGURE 8

5.5. Fifth Case. In this section we obtain the periodic solution of the following special case of Eq.(1)

$$x_{n+1}=x_n+\frac{x_nx_{n-2}}{x_{n-2}-2x_{n-3}}, \tag{12}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are positive.

5.5. Theorem. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(12). If $x_{-3}=7,\ x_{-2}=2,\ x_{-1}=9,\ x_0=6$. Then the solution of Eq.(12) is periodic with period 6 and $\{x_n\}_{n=-3}^{\infty}=\{7,2,9,6,5,14,7,2,9,6,5,...\}$. See Fig. 9.*

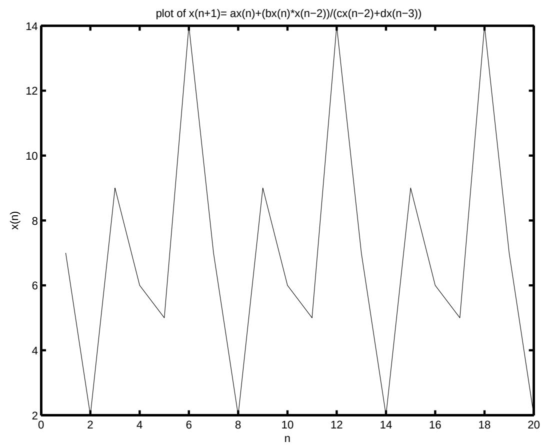

FIGURE 9

5.6. Lemma. *Let* $x_{-3} = 2,\ x_{-2} = 9,\ x_{-1} = 6,\ x_0 = 5$. *Then the solution of Eq.(12) is periodic with period* 6 *and* $\{x_n\}_{n=-3}^{\infty} = \{2, 9, 6, 5, 14, 7, 2, 9, 6, 5, 14, ...\}$.

5.7. Lemma. *If* $x_{-3} = 9,\ x_{-2} = 2,\ x_{-1} = 5,\ x_0 = 3$. *Then the solution of Eq.(12) is periodic with period* 6 *and* $\{x_n\}_{n=-3}^{\infty} = \left\{9, 2, 5, 3, \frac{21}{8}, \frac{63}{4}, 9, 2, 5, 3, ...\right\}$.

Conclusion. This paper discussed about boundedness, global stability and the solutions of some special cases of Eq. (1). In Section 2 we proved when $b(c + 3d) < (1-a)(c+d)^2$, Eq. (1) local stability. In Section 3 we showed that when $c(1-a) \neq b$ the unique equilibrium of Eq. (1) is globally asymptotically stable. In Section 4 we proved that every solution is bounded if $(a + \dfrac{b}{c}) < 1$. In Section 5 we have given the solutions of some special cases of Eq. (1) and given a numerical examples of each case.

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