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FIXED POINT THEOREMS FOR A THIRD POWER TYPE CONTRACTION MAPPINGS IN G-METRIC SPACES

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Abstract

In this paper, we introduce a new third power type contractive condition in the G-metric spaces, and several new fixed point theorems are established in complete G-metric space. The obtained results in this paper extend the recent relative results.

Keywords: G-metric space, third power type contraction mappings, fixed point.

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1. Introduction

Metric fixed point theory is an important mathematical discipline because of its applications in areas as variational and linear inequalities, optimization theory. In 1992, Dhage[2] introduced the concept of D-metric space. Unfortunately, it was shown that certain theorems involving Dhage's D-metric spaces are flawed, and most of the results claimed by Dhage and others are invalid. These errors are pointed out by Mustafa and Sims[7]. In 2006, a new structure of generalized metric spaces was introduced by Mustafa and Sims[8] as appropriate notion of generalized metric space called G-metric spaces. Some other papers dealing with G-metric spaces are those in[1], [3]-[6], [9]-[11]. In this paper, we will prove some general fixed point theorems for third power type contractions mapping in complete G-metric spaces.

Throughout the paper, we mean by $\mathbb{N}$ the set of all natural numbers.

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1.1. Definition (see[8])**.** Let X be a nonempty set, and let $G : X \times X \times X \longrightarrow R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z)$=0 if $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$,for all $x, y, z \in X$ with $z \neq$ y;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)

then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair $(X, G)$is called a G-metric space.

1.2. Definition (see[8])**.** Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

1.3. Proposition (see[8])**.** *Let (X, G) be a G-metric space, then the followings are equivalent:*

(1) x_n is G-convergent to x.
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$.
(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

1.4. Definition (see[8])**.** Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; i.e.if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$

1.5. Definition (see[8])**.** A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

1.6. Proposition (see[8])**.** *Let (X, G) be a G-metric space. Then the following are equivalent.*

(1) The sequence $\{x_n\}$ is G-Cauchy.
(2) For every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.

1.7. Proposition (see[8])**.** *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

1.8. Definition (see[8])**.** Let (X, G) and (X', G') be G-metric space, and $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

1.9. Proposition (see[8])**.** *Let (X, G) and (X', G') be G-metric space. Then $f : X \to X'$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

1.10. Proposition (see[8])**.** *Let (X, G) be a G-metric space. Then , for any x, y, z, a in X it follows that :*

(i) if $G(x, y, z) = 0$, then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(y, x, x)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

2. Main Results

2.1. Theorem. *Let (X,G) be a complete G-metric space. Suppose the map $T : X \to X$ satisfies*

$$G^3(Tx,Ty,Tz) \le qG(x,Tx,Tx)G(y,Ty,Ty)G(z,Tz,Tz) \tag{2.1}$$

for all $x, y, z \in X$, where $0 \le q < 1$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Let $x_0 \in X$ be arbitrary point, and define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$, $n \in \mathbb{N}$. Assume $x_n \ne x_{n+1}$, for each $n \in \mathbb{N}$.

First, we prove the sequence $\{x_n\}$ is a G-Cauchy sequence. In fact, by (2.1), we have

$$G^3(x_n,x_{n+1},x_{n+1}) \le qG(x_{n-1},x_n,x_n)G(x_n,x_{n+1},x_{n+1})G(x_n,x_{n+1},x_{n+1}) = G^3(Tx_{n-1},Tx_n,Tx_n).$$

Thus, we have

$$G(x_n,x_{n+1},x_{n+1}) \le qG(x_{n-1},x_n,x_n) \le \cdots \le q^n G(x_0,x_1,x_1). \tag{2.2}$$

For every $m, n \in \mathbb{N}$, $m > n$, using (G5) and (2.2), we have

$$\begin{aligned} G(x_n,x_m,x_m) &\le G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) + \cdots + G(x_{m-1},x_m,x_m), \\ &\le (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0,x_1,x_1) \\ &< \frac{q^n}{1-q} G(x_0,x_1,x_1). \end{aligned}$$

and so $G(x_n,x_m,x_m) \to 0$, as $n,m \to \infty$. Thus $\{x_n\}$ is G-Cauchy sequence. Due to the completeness of (X,G), there exists $u \in X$ such that $\{x_n\}$ is G -converge to u.

On the other hand, using (2.1), we have

$$\begin{aligned} G^3(x_n,x_n,Tu) &= G^3(Tx_{n-1},Tx_{n-1},Tu) \\ &\le qG(x_{n-1},x_n,x_n)G(x_{n-1},x_n,x_n)G(u,Tu,Tu) \end{aligned}$$

Letting $n \to \infty$, and using the fact that G is continuous on its variable, we get that

$$G^3(u,u,Tu) = 0.$$

Therefore, $Tu = u$, hence u is a fixed point of T. Now, let v be an another fixed point of T, then we have

$$\begin{aligned} G^3(u,u,v) &= G^3(Tu,Tu,Tv) \\ &\le qG(u,Tu,Tu)G(u,Tu,Tu)G(v,Tv,Tv) \\ &= 0. \end{aligned}$$

Thus, $u = v$. Then we know the fixed point of T is unique.

To show that T is G-continuous at u, let $\{y_n\}$ be any sequence in X such that $\{y_n\}$ is G-convergent to u. For $n \in \mathbb{N}$, we have

$$G^3(u,u,Ty_n) = G^3(Tu,Tu,Ty_n) \le qG(u,Tu,Tu)G(u,Tu,Tu)G(y_n,Ty_n,Ty_n)$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} G(u,u,Ty_n) = 0$. Hence $\{Ty_n\}$ is G-convergent to $u = Tu$. So T is G-continuous at u. □

2.2. Corollary. *Let (X,G) be a complete G-metric space. Suppose the map $T : X \to X$ satisfies*

$$G^3(T^p x,T^p y,T^p z) \le qG(x,T^p x,T^p x)G(y,T^p y,T^p y)G(z,T^p z,T^p z)$$

for all $x, y, z \in X$, where $0 \le q < 1$, $p \in \mathbb{N}$. Then T has a unique fixed point (say u) and T^p is G-continuous at u.

Proof. From Theorem 2.1 we know that T^p has a unique fixed point (say u), that is, $T^p u = u$, and T^p is G-continuous at u. Since $Tu = TT^p u = T^{p+1}u = T^p Tu$, so Tu is another fixed point for T^p, and by uniqueness, we have $Tu = u$. □

2.3. Theorem. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a G-continuous mapping, which satisfies the following condition*

$$G^3(Tx, T^2x, T^3x) \leq qG(x, Tx, Tx)G(Tx, T^2x, T^2x)G(T^2x, T^3x, T^3x) \tag{2.3}$$

for all $x \in X$, where $0 \leq q < 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$ be arbitrary point, and define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$, $n \in \mathbb{N}$. Assume $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.

First, we prove the sequence $\{x_n\}$ is a G-Cauchy sequence. In fact, by (2.3), we have

$$\begin{aligned} G^3(x_n, x_{n+1}, x_{n+2}) &= G^3(Tx_{n-1}, T^2x_{n-1}, T^3x_{n-1}) \\ &\leq qG(x_{n-1}, x_n, x_n)G(x_n, x_{n+1}, x_{n+1})G(x_{n+1}, x_{n+2}, x_{n+2}). \end{aligned}$$

On the other hand, using (G3), we have

$$\begin{aligned} G(x_{n-1}, x_n, x_n) &\leq G(x_{n-1}, x_n, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) &\leq G(x_n, x_{n+1}, x_{n+2}), \\ G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq G(x_n, x_{n+1}, x_{n+2}). \end{aligned}$$

Thus, we have

$$G^3(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1})G^2(x_n, x_{n+1}, x_{n+2}).$$

Therefore, we can get

$$G(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1}) \leq \cdots \leq q^n G(x_0, x_1, x_2). \tag{2.4}$$

Moreover, for all $n, m \in \mathbb{N}$, $n < m$, by $(G3)$, $(G5)$ and (2.4) we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m), \\ &\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_{m+1}), \\ &\leq (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0, x_1, x_2) \\ &< \frac{q^n}{1-q} G(x_0, x_1, x_2). \end{aligned}$$

That means the sequence $\{x_n\}$ is a G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that $\{x_n\}$ is G -converge to u. Furthermore, since T is G-continuous, from $x_{n+1} = Tx_n$, letting $n \to \infty$ at both sides, we have $u = Tu$. Thus, u is a fixed point of T. □

2.4. Theorem. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a G-continuous mapping, which satisfies the following condition:*

$$G^3(Tx, T^2y, T^3z) \leq qG(x, Tx, Tx)G(Tx, T^2y, T^2y)G(Ty, T^3z, T^3z) \tag{2.5}$$

for all $x, y, z \in X$, where $0 \leq q < 1$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Let $x_0 \in X$ be arbitrary point, and define the sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$, $n \in \mathbb{N}$. Assume $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.

First, we prove the sequence $\{x_n\}$ is a G-Cauchy sequence. In fact, by (2.5), we have

$$\begin{aligned} G^3(x_n, x_{n+1}, x_{n+1}) &= G^3(Tx_{n-1}, T^2x_{n-1}, T^3x_{n-2}) \\ &\leq qG(x_{n-1}, x_n, x_n)G(x_n, x_{n+1}, x_{n+1})G(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

Therefore, we can get

(2.6) $$G(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1}) \leq \cdots \leq q^n G(x_0, x_1, x_2).$$

Moreover, for all $n, m \in N, n < m$, by $(G3)$, $(G5)$ and (2.6) we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m), \\ &\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_{m+1}), \\ &\leq (q^n + q^{n+1} + \cdots + q^{m-1}) G(x_0, x_1, x_2) \\ &< \frac{q^n}{1-q} G(x_0, x_1, x_2). \end{aligned}$$

That means the sequence $\{x_n\}$ is a G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that $\{x_n\}$ is G -converge to u.

On the other hand, using (2.5), we have

$$\begin{aligned} G^3(Tu, x_{n+1}, x_{n+1}) &= G^3(Tu, T^2 x_{n-1}, T^3 x_{n-2}) \\ &\leq qG(u, Tu, Tu) G(Tu, x_{n+1}, x_{n+1}) G(x_n, x_{n+1}, x_{n+1}) \end{aligned}$$

Letting $n \to \infty$, and using the fact that G is continuous on its variable, we get that

$$G^3(Tu, u, u) = 0.$$

Therefore, $Tu = u$, hence u is a fixed point of T. Now, let v be an another fixed point of T, then we have

$$\begin{aligned} G^3(u, u, v) &= G^3(Tu, T^2 u, T^3 v) \\ &\leq qG(u, Tu, Tu) G(Tu, T^2 u, T^2 u) G(Tu, T^3 v, T^3 v) \\ &= qG(u, u, u) G(u, u, u) G(u, v, v) = 0. \end{aligned}$$

Thus, $u = v$. Then we know the fixed point of T is unique.

To show that T is G-continuous at u, let $\{y_n\}$ be any sequence in X such that $\{y_n\}$ is G-convergent to u. For $n \in \mathbb{N}$, we have

$$G^3(Ty_n, u, u) = G^3(Ty_n, T^2 u, T^3 u) \leq qG(y_n, Ty_n, Ty_n) G(Ty_n, T^2 u, T^2 u) G(Tu, T^3 u, T^3 u)$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} G(Ty_n, u, u) = 0$. Hence $\{Ty_n\}$ is G-convergent to $u = Tu$. So T is G-continuous at u. □

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