Hacettepe Journal of Mathematics and Statistics
Volume 42 (5) (2013), 501 – 516

APPROXIMATE QUADRATIC FUNCTIONAL EQUATION IN FELBIN'S TYPE NORMED LINEAR SPACES

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Received 11 : 12 : 2011 : Accepted 11 : 06 : 2012

Abstract

In this paper, we prove the generalized Hyers–Ulam–Rassias stability of quadratic functional equation in Felbin's type normed linear spaces by using the direct and fixed point methods. The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

Keywords: Analysis, fixed point, fuzzy numbers, fuzzy normed space, Hyers–Ulam–Rassias stability

2000 AMS Classification: 26E50, 03E72, 39B72, 47H10, 47Jxx

1. Introduction and preliminaries

In [15] Grantner et al., takes the fuzzy real number as a decreasing mapping from the real line to the unit interval or lattice in general. Lowen [28] applies the fuzzy real numbers as non-decreasing, left continuous mapping from the real line to the unit interval so that its supremum over $\mathbb{R}$ is 1. Also fuzzy arithmetic operations on L-fuzzy real line were studied by Rodabaugh [54], where he showed that the binary addition is the only extension of addition to $\mathbb{R}(L)$. Hoehle [17] especially emphasized the role of fuzzy real numbers as modeling a fuzzy threshold softening the notion of Dedekind cut. In this paper a fuzzy real number is taken as a fuzzy normal and convex mapping from the real line to the unit interval.
The concept of the fuzzy metric space has been studied by Kaleva [24, 25] by using fuzzy number as a fuzzy set on the real axis. Kaleva also has recently showed that a fuzzy metric space can be embedded in a complete fuzzy metric space [26]. In [14], Felbin introduced the concept of fuzzy normed linear space (FNLS); Xiao and Zhu [59] studied its linear topological structures and some basic properties of a fuzzy normed linear space. It is

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known that theories of classical normed space and Menger probabilistic normed spaces are special cases of fuzzy normed linear spaces.

In 1940, Ulam [58] brought up a question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If the above problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. In 1941, Ulam's problem was solved by Hyers [18] in Banach spaces. This result was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [49] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [49] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. P. Găvruta [16] generalized the Th.M. Rassias' result in the spirit of Th.M. Rassias's stability approach. Following the techniques of the proof of the corollary of Hyers [18], we observed that Hyers introduced (in 1941) the following Hyers continuity condition: about the continuity of the mapping for each fixed, and then he proved homogenouity of degree one and therefore the linearity of the mapping. This condition has been considered further till now, through the complete Hyers direct method, in order to prove linearity for the generalized Hyers-Ulam stability problem for approximate homomorphisms (see [19]). Beginning around the year 1980, the stability problems of a wide class of functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7], [10], [12], [13] ,[19], [22]–[29], [30]–[45], [51]–[52]).

In 1991 J. A. Baker [5] used the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In 2003, V.Radu [46] applied the fixed point alternative theorem to investigate the Hyers-Ulam-Rassias stability for Caychy functional equation. In 2003 Cădariu and Radu applied the fixed point method to the investigation of the quadratic functional equation [8].

The following functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y), \tag{1.1}$$

is called a *quadratic functional equation,* and every solution of equation (1.1) is said to be a *quadratic mapping.* F. Skof [57] proved the Hyers–Ulam stability of the quadratic functional equation (1.1) for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [9], S. Czerwik proved the Hyers–Ulam stability of the quadratic functional equation (1.1). C. Borelli and G.L. Forti [6] generalized the stability result of the quadratic functional equation (1.1). Jun and Lee [20] proved the Hyers–Ulam stability of the Pexiderized quadratic equation

$$f(x+y)+g(x-y)=2h(x)+2k(y)$$

for mappings f, g, h and k. The stability problem of the quadratic equation has been extensively investigated by some mathematicians.

This paper is organized as follows: In Section 2, we retell some preliminaries in fuzzy real numbers. Also we state some properties of felbin's type normed linear spaces where they will be applied in other sections.

In Section 3, we investigate generalized Hyers-Ulam-Rassias stability of quadratic functional equation in felbin's type normed linear spaces by direct method.

In Section 4, We will adopt the idea of Cădariu and Radu [46], to prove the generalized Hyers–Ulam stability results of quadratic functional equation in Felbin's type normed linear spaces.

2. Fuzzy real number

In this section, we give some preliminaries in the theory of fuzzy real numbers. Furthermore, we give some definitions and prove some theorems, which help to investigate Hyers–Ulam–Rassias stability in Felbin's type normed linear spaces.

Let η be a fuzzy subset on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \to [0,1]$ associating with each real number t its grade of membership $\eta(t)$.

2.1. Definition. [14] A fuzzy subset η on $\mathbb{R}$ is called a fuzzy real number, whose $\alpha-$level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:
(N1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.
(N2) For each $\alpha \in (0,1]$, $[\eta]_\alpha = [\eta_\alpha^-,\ \eta_\alpha^+]$ where $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.
The set of all fuzzy real numbers denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers.

The number $\overline{0}$ stands for the fuzzy real number as:

$$\overline{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\overline{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$ because if $r \in (-\infty, \infty)$, then $\overline{r} \in F(\mathbb{R})$ satisfies $\overline{r}(t) = \overline{0}(t - r)$.

2.2. Theorem. [3] *Let $\{[a_\alpha, b_\alpha];\ \alpha \in (0,1]\}$ be a family of nested bounded closed intervals. Let $\eta : \mathbb{R} \to [0,1]$ be a function defined by*

$$\eta(t) = \sup\{\alpha \in (0,1] :\ t \in [a_\alpha, b_\alpha]\}.$$

Then η is a fuzzy real number.

2.3. Theorem. [4] *Let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, be a given family of non-empty intervals. If*
(i) $[a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$ for all $0 < \alpha_1 \leq \alpha_2$.
(ii) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ whenever $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to α.
Then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number η such that $\eta(t) = \sup\{\alpha \in (0,1] :\ t \in [a_\alpha, b_\alpha]\}$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+] = [a_\alpha, b_\alpha]$.

2.4. Definition. [14] Fuzzy arithmetic operations $\oplus, \ominus, \otimes, \oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

$$(1) \quad (\eta \oplus \delta)(t) = \sup_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(t - s)\}\ ,\ t \in \mathbb{R},$$

$$(2) \quad (\eta \ominus \delta)(t) = \sup_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(s - t)\}\ ,\ t \in \mathbb{R},$$

$$(3) \quad (\eta \otimes \delta)(t) = \sup_{s\in\mathbb{R}, s\neq 0}\{\eta(s) \wedge \delta(t/s)\}\ ,\ t \in \mathbb{R},$$

$$(4) \quad (\eta \oslash \delta)(t) = \sup_{s\in\mathbb{R}}\{\eta(st) \wedge \delta(s)\}\ ,\ t \in \mathbb{R}.$$

The additive and multiplicative identities in $F(\mathbb{R})$ are $\overline{0}$ and $\overline{1}$, respectively. Let $\ominus\eta$ be defined as $\overline{0} \ominus \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus\delta)$.

2.5. Definition. [14] For $k \in \mathbb{R} \setminus \{0\}$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(t/k)$ and $0 \odot \eta$ is defined to be $\overline{0}$.

2.6. Lemma. [32, 56] *Let $\eta,\ \delta$ be fuzzy real numbers. Then*

$$\forall t \in \mathbb{R}\ ,\quad \eta(t) = \delta(t) \Leftrightarrow \forall\alpha \in (0,1]\ ,\quad [\eta]_\alpha = [\delta]_\alpha.$$

2.7. Lemma. [14] *Let* $\eta,\ \delta \in F(\mathbb{R})$ *and* $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. *Then*

(1) $[\eta \oplus \delta]_\alpha = [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+]$,

(2) $[\eta \ominus \delta]_\alpha = [\eta_\alpha^- - \delta_\alpha^+, \eta_\alpha^+ - \delta_\alpha^-]$,

(3) $[\eta \otimes \delta]_\alpha = [\eta_\alpha^- \delta_\alpha^-, \eta_\alpha^+ \delta_\alpha^+]$, $\eta,\ \delta \in F^*(\mathbb{R})$,

(4) $[\bar{1} \oslash \delta]_\alpha = [1/\delta_\alpha^+, 1/\delta_\alpha^-]$, $\delta_\alpha^- > 0$.

2.8. Definition. Let η be a non-negative fuzzy real number and $p \neq 0$ be a real number. Define η^p as:

$$\eta^p(t) = \begin{cases} \eta(t^{\frac{1}{p}}), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Set $\eta^p = \bar{1}$, in case $p = 0$.

We show that η^p is a non-negative fuzzy real number, i.e., $\eta^p \in F^*(\mathbb{R})$, $\forall p \in \mathbb{R}$. We need to investigate Conditions (N1) and (N2) in the definition of fuzzy real numbers.
For Condition (N1), since η is a fuzzy real number, there exists $t_0 \in [0, +\infty)$ such that $\eta(t_0) = 1$. Set $t' = t_0^p$. Then $\eta^p(t') = \eta((t')^{\frac{1}{p}}) = \eta((t_0^p)^{\frac{1}{p}}) = \eta(t_0) = 1$.
For Condition (N2), since for all $\alpha \in (0, 1]$, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, we have

$$\begin{aligned} [\eta^p]_\alpha &= \{t \ ; \ \eta^p(t) \geq \alpha\} \\ &= \{t \ ; \ \eta(t^{\frac{1}{p}}) \geq \alpha\} = \{s^p \ ; \ \eta(s) \geq \alpha\} \\ &= (\{s \ ; \ \eta(s) \geq \alpha\})^p \\ &= ([\eta_\alpha^-, \eta_\alpha^+])^p. \end{aligned}$$

Therefore, η^p is a fuzzy real number. Also it is clear that if $p > 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^-)^p, (\eta_\alpha^+)^p]$ and if $p < 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^+)^p, (\eta_\alpha^-)^p]$.
Note : Here we mean by B^p the set $\{x^p : \ x \in B\}$, where $B \subset \mathbb{R}$ and $p \in \mathbb{R}$.

2.9. Theorem. [32, 56] *Let* η *be a non-negative fuzzy real number and* $p,\ q$ *be non-zero integers. Then*
(1) $p > 0 \ \Rightarrow \ \eta^p = \otimes_{i=1}^p \eta$,
(2) $p < 0 \ \Rightarrow \ \eta^p = \bar{1} \oslash (\otimes_{i=1}^{-p} \eta)$,
(3) $\eta^p \otimes \eta^q = \eta^{p+q}$; $pq > 0$,
(4) $(\eta^p)^q = \eta^{pq}$.

2.10. Definition. [14] Define a partial ordering $\preceq$ in $F(\mathbb{R})$ by $\eta \preceq \delta$ if and only if $\eta_\alpha^- \leq \delta_\alpha^-$ and $\eta_\alpha^+ \leq \delta_\alpha^+$ for all $\alpha \in (0, 1]$. The strict inequality in $F(\mathbb{R})$ is defined by $\eta \prec \delta$ if and only if $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^+ < \delta_\alpha^+$ for all $\alpha \in (0, 1]$.

2.11. Definition. [59] Let X be a real linear space; L and R (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $[0, 1] \times [0, 1]$ into $[0, 1]$ satisfying $L(0, 0) = 0$, $R(1, 1) = 1$. Then $\|.\|$ is called a fuzzy norm and $(X, \|.\|, L, R)$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\|.\|$ from X into $F^*(\mathbb{R})$ satisfies the following axioms, where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in X$ and $\alpha \in (0, 1]$:
(A1) $\|x\| = \bar{0}$ if and only if $x = 0$,
(A2) $\|rx\| = |r| \odot \|x\|$ for all $x \in X$ and $r \in (-\infty, \infty)$,
(A3) For all $x,\ y \in X$:
(A3L) if $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$, then
$\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$,
(A3R) if $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$, then
$\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$.

2.12. Lemma. [60] *Let $(X, \|.\|, L, R)$ be an FNLS, and suppose that*
$(R-1)$ $R(a,b) \leq \max(a,b)$,
$(R-2)$ $\forall \alpha \in (0,1]$, $\exists \beta \in (0,\alpha]$ *such that* $R(\beta, y) \leq \alpha$ *for all* $y \in (0,\alpha)$,
$(R-3)$ $\lim_{a\to 0^+} R(a,a) = 0$.
Then $(R-1) \Rightarrow (R-2) \Rightarrow (R-3)$ *but not conversely.*

2.13. Lemma. [60] *Let $(X, \|.\|, L, R)$ be an FNLS. Then we have the following:*
(1) If $R(a,b) \leq \max(a,b)$, then $\forall \alpha \in (0,1]$, $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $x, y \in X$.
(2) If $(R-2)$, then for each $\alpha \in (0,1]$ there is $\beta \in (0,\alpha]$ such that $\|x+y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\alpha^+$ for all $x, y \in X$.
(3) If $\lim_{a\to 0^+} R(a,a) = 0$, then for each $\alpha \in (0,1]$ there is $\beta \in (0,\alpha]$ such that $\|x+y\|_\alpha^+ \leq \|x\|_\beta^+ + \|y\|_\beta^+$ for all $x, y \in X$.

2.14. Lemma. [32, 56] *Let $(X, \|.\|, L, R)$ be an FNLS, and suppose that*
$(L-1)$ $L(a,b) \geq \min(a,b)$,
$(L-2)$ $\forall \alpha \in (0,1]$, $\exists \beta \in [\alpha,1]$ *such that* $L(\beta,\gamma) \geq \alpha$ *for all* $\gamma \in [\alpha,1]$,
$(L-3)$ $\lim_{a\to 1^-} L(a,a) = 1$.
Then $(L-1) \Rightarrow (L-2) \Rightarrow (L-3)$.

2.15. Lemma. [32, 56] *Let $(X, \|.\|, L, R)$ be an FNLS. Then we have the following:*
(1) If $L(a,b) \geq \min(a,b)$, then $\forall \alpha \in (0,1]$, $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $x, y \in X$.
(2) If $(L-2)$, then for each $\alpha \in (0,1]$ there is $\beta \in [\alpha,1]$ such that $\|x+y\|_\alpha^- \leq \|x\|_\beta^- + \|y\|_\alpha^-$ for all $x, y \in X$.
(3) If $\lim_{a\to 1^-} L(a,a) = 1$, then for each $\alpha \in (0,1]$ there is $\beta \in [\alpha,1]$ such that $\|x+y\|_\alpha^- \leq \|x\|_\beta^- + \|y\|_\beta^-$ for all $x, y \in X$.

2.16. Lemma. [59] *Let $(X, \|.\|, L, R)$ be an FNLS. Then:*
(1) If $R(a,b) \geq \max(a,b)$ and $\forall \alpha \in (0,1]$, $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $x, y \in X$, then (A3R).
(2) If $L(a,b) \leq \min(a,b)$ and $\forall \alpha \in (0,1]$, $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $x, y \in X$, then (A3L).

2.17. Theorem. [55] *Let $(X, \|.\|, L, R)$ be an FNLS and $\lim_{a\to 0^+} R(a,a) = 0$. Then (X,‖.‖,L,R) is a Hausdorff topological vector space, whose neighborhood base of origin θ is $\{N(\varepsilon,\alpha) : \varepsilon > 0,\ \alpha \in (0,1]\}$, where $N(\varepsilon,\alpha) = \{x;\ \|x\|_\alpha^+ \leq \varepsilon\}$.*

2.18. Definition. Let $(X, \|.\|, L, R)$ be an FNLS and $\lim_{a\to 0^+} R(a,a) = 0$. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$, and is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\|_\alpha^+ = 0$ for every $\alpha \in (0,1]$. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A, converges in A. The fuzzy normed space $(X, \|.\|, L, R)$ is said to be a fuzzy Banach space if it is complete.

2.19. Theorem. [60] *Let $(X, \|.\|, L, R)$ be an FNLS satisfying $(R-2)$. Then:*
(1) For each $\alpha \in (0,1]$, $\|.\|_\alpha^+$ is a continuous mapping from X into $\mathbb{R}$ at $x \in X$.
(2) For any $n \in \mathbb{Z}^+$ and $\{x_i\}_{i=1}^n$ we have

$$\forall \alpha \in (0,1], \quad \exists \beta \in (0,\alpha]; \quad \|\sum_{i=1}^n x_i\|_\alpha^+ \leq \sum_{i=1}^n \|x_i\|_\beta^+.$$

2.20. Theorem. *Let $(X, \|.\|, L, R)$ be an FNLS satisfying $(L-2)$. Then:*
(1) For each $\alpha \in (0,1]$, $\|.\|_\alpha^-$ is a continuous mapping from X into $\mathbb{R}$ at $x \in X$.
(2) For any $n \in \mathbb{Z}^+$ and $\{x_i\}_{i=1}^n$ we have

$$\forall \alpha \in (0,1], \quad \exists \beta \in [\alpha,1]; \quad \|\sum_{i=1}^n x_i\|_\alpha^- \leq \sum_{i=1}^n \|x_i\|_\beta^-.$$

Proof. The proof is completely the same as Theorem 2.19. □

3. Stability of the quadratic equation, direct method

In this section, by using the direct method, we investigate the fuzzy stability of quadratic functional equation in the spirit of Hyers, Ulam and Rassias.

3.1. Theorem. *Let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Let $f : X \longrightarrow Y$ be a mapping for which there is a function $\varphi : X \times X \to F^*(\mathbb{R})$ such that*

$$\sum_{i=0}^{\infty} \frac{1}{4^i} (\varphi(2^i x, 2^i y))_\alpha^+ < \infty, \tag{3.1}$$

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \preceq \varphi(x, y), \tag{3.2}$$

for all $x, y \in X$ and all $\alpha \in (0,1]$. Then there exists a unique quadratic mapping $T : X \longrightarrow Y$ such that

$$\forall \alpha \in (0,1], \exists \beta \in (0,\alpha], \ s.t. \ \|f(x) - T(x)\|_\alpha^+ \leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{4^i} (\varphi(2^i x, 2^i x))_\beta^+, \tag{3.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (3.2), we get

$$\|f(2x) - 4f(x)\| \preceq \varphi(x, x), \tag{3.4}$$

for all $x \in X$. If we replace x in (3.4) by $2^n x$ and multiply the both sides of (3.4) to $\frac{1}{4^{n+1}}$ in the fuzzy scalar multiplication sense, then we have

$$\left\| \frac{1}{4^{n+1}} f(2^{n+1} x) - \frac{1}{4^n} f(2^n x) \right\| \preceq \frac{1}{4^{n+1}} \odot \varphi(2^n x, 2^n x), \tag{3.5}$$

for all $x \in X$ and all non-negative integers $n \in \mathbb{N}$. By Theorem 2.19 and inequality (3.5), we conclude that for all $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that

$$\begin{aligned} \left\| \frac{1}{4^{n+1}} f(2^{n+1} x) - \frac{1}{4^m} f(2^m x) \right\|_\alpha^+ &\leq \sum_{i=m}^{n} \left\| \frac{1}{4^{i+1}} f(2^{i+1} x) - \frac{1}{4^i} f(2^i x) \right\|_\beta^+ \\ &\leq \sum_{i=m}^{n} \frac{1}{4^{i+1}} (\varphi(2^i x, 2^i x))_\beta^+, \end{aligned} \tag{3.6}$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. Now (3.1) and (3.6) imply that $\{\frac{1}{4^n} f(2^n x)\}$ is a fuzzy Cauchy sequence in Y for all $x \in X$. Since Y is a fuzzy Banach space, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges for all $x \in X$. So we can define the mapping $T : X \to Y$ by:

$$T(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x),$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.6), by continuity of $\|.\|_\alpha^+$ we get

$$\|f(x) - T(x)\|_\alpha^+ \leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{4^i} (\varphi(2^i x, 2^i x))_\beta^+, \tag{3.7}$$

for all $x \in X$. Therefore, we obtain (3.3). Now we must show that T is quadratic and unique. Applying (3.1) and continuity of $\|.\|_\alpha^+$, we have

$$\begin{aligned}
&\|T(x+y)+T(x-y)-2T(x)-2T(y)\|_\alpha^+\\
&= \lim_{n\to\infty}\frac{1}{4^n}\Big\|f(2^nx+2^ny)+f(2^nx-2^ny)-2f(2^nx)-2f(2^ny)\Big\|_\alpha^+\\
&\le \lim_{n\to\infty}\frac{1}{4^n}(\varphi(2^nx,2^ny))_\alpha^+ = 0,
\end{aligned}$$

for all $x, y \in X$. Therefore, the mapping $T : X \to Y$ is quadratic.

To prove the uniqueness of T, let $T' : X \to Y$ be another quadratic mapping satisfying (3.3). By Theorem 2.19, since

$$\|T(x)-T'(x)\|_\alpha^+ = \lim_{n\to\infty}\frac{1}{4^n}\|f(2^nx)-f(2^nx)\|_\alpha^+ = 0,$$

for all $x \in X$, $T = T'$. □

3.2. Remark. The above Theorem remains true if $\|.\|_\alpha^+$ is replaced by $\|.\|_\alpha^-$ in (3.3) and the fuzzy Banach space Y satisfies in $(L-2)$ and $(R-2)$.

The following Theorem is an alternative result of Theorem 3.1.

3.3. Theorem. *Let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space such that $R(a,b) \le \max(a,b)$ and $L(a,b) \ge \min(a,b)$. Let $f : X \longrightarrow Y$ be a mapping for which there is a function $\varphi : X \times X \to F^*(\mathbb{R})$ satisfying (3.1) and (3.2) for all $x, y \in X$ and all $\alpha \in (0,1]$. Then there exists a unique quadratic mapping $T : X \longrightarrow Y$ such that*

$$\text{(3.8)} \qquad \|f(x)-T(x)\| \preceq \overline{\varphi}(x,x),$$

for all $x \in X$, where $\overline{\varphi}(x,x)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that

$$\begin{aligned}
a_\alpha &= \frac{1}{4}\sum_{i=0}^{\infty}\frac{1}{4^i}(\varphi(2^ix,2^ix))_\alpha^-,\\
b_\alpha &= \frac{1}{4}\sum_{i=0}^{\infty}\frac{1}{4^i}(\varphi(2^ix,2^ix))_\alpha^+,
\end{aligned}$$

for all $x \in X$.

3.4. Theorem. *Let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Let $f : X \longrightarrow Y$ be a mapping for which there is a function $\varphi : X \times X \to F^*(\mathbb{R})$ such that*

$$\text{(3.9)} \qquad \sum_{i=1}^{\infty}4^i\Big(\varphi(\frac{x}{2^i},\frac{y}{2^i})\Big)_\alpha^+ < \infty,$$

$$\text{(3.10)} \qquad \|f(x+y)+f(x-y)-2f(x)-2f(y)\| \preceq \varphi(x,y),$$

for all $x, y \in X$ and all $\alpha \in (0,1]$. Then there exists a unique quadratic mapping $T : X \longrightarrow Y$ such that

$$\text{(3.11)} \qquad \forall\alpha\in(0,1],\ \exists\beta\in(0,\alpha],\ s.t.\ \|f(x)-T(x)\|_\alpha^+ \le \frac{1}{4}\sum_{i=1}^{\infty}4^i(\varphi(\frac{x}{2^i},\frac{x}{2^i}))_\beta^+,$$

for all $x \in X$.

Proof. Letting $y = x$ in (3.10), we get

$$\|f(2x) - 4f(x)\| \preceq \varphi(x, x), \tag{3.12}$$

for all $x \in X$. If we replace x in (3.12) by $\frac{x}{2^{n+1}}$ and multiply the both sides of (3.12) by 4^n in the fuzzy scalar multiplication sense, then we have

$$\left\|4^n f(\frac{x}{2^n}) - 4^{n+1} f(\frac{x}{2^{n+1}})\right\| \preceq 4^n \odot \varphi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}), \tag{3.13}$$

for all $x \in X$ and all non-negative integers $n \in \mathbb{N}$. By Theorem 2.19 and inequality (3.13), we conclude that for all $\alpha \in (0, 1]$ there exists $\beta \in (0, \alpha]$ such that

$$\begin{aligned} \left\|4^{n+1} f(\frac{x}{2^{n+1}}) - 4^m f(\frac{x}{2^m})\right\|_\alpha^+ &\leq \sum_{i=m}^{n} \left\|4^{i+1} f(\frac{x}{2^{i+1}}) - 4^i f(\frac{x}{2^i})\right\|_\beta^+ \\ &\leq \frac{1}{4} \sum_{i=m}^{n} 4^{i+1} \Big(\varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})\Big)_\beta^+, \end{aligned} \tag{3.14}$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. So by (3.9) and (3.14), the sequence $\{4^n f(\frac{x}{2^n})\}$ is a fuzzy Cauchy sequence in Y for all $x \in X$. Since Y is fuzzy Banach space, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges for all $x \in X$. The rest of this proof is similar to the proof of Theorem 3.1. □

3.5. Remark. The above Theorem remains true if $\|.\|_\alpha^+$ is replaced by $\|.\|_\alpha^-$ in (3.11) and the fuzzy Banach space Y satisfies in $(L - 2)$ and $(R - 2)$.

The following Theorem is an alternative result of Theorem 3.4.

3.6. Theorem. *Let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \leq \max(a, b)$ and $L(a, b) \geq \min(a, b)$. Let $f : X \longrightarrow Y$ be a mapping for which there is a function $\varphi : X \times X \to F^*(\mathbb{R})$ satisfying (3.9) and (3.10) for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique quadratic mapping $T : X \longrightarrow Y$ such that*

$$\|f(x) - T(x)\| \preceq \overline{\varphi}(x, x), \tag{3.15}$$

for all $x \in X$, where $\overline{\varphi}(x, x)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that

$$a_\alpha = \frac{1}{4} \sum_{i=1}^{\infty} 4^i (\varphi(\frac{x}{2^i}, \frac{x}{2^i}))_\alpha^-,$$

$$b_\alpha = \frac{1}{4} \sum_{i=1}^{\infty} 4^i (\varphi(\frac{x}{2^i}, \frac{x}{2^i}))_\alpha^+,$$

for all $x \in X$.

3.7. Corollary. *Let μ be a non-negative fuzzy real number and let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \leq \max(a, b)$. Suppose that a mapping $f : X \longrightarrow Y$ satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \preceq \mu, \tag{3.16}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ satisfying:

$$\forall \alpha \in (0, 1],\ \exists \beta \in (0, \alpha],\ s.t.\ \|f(x) - T(x)\|_\alpha^+ \leq \frac{1}{3} \mu_\beta^+,$$

for all $x \in X$.

Proof. Let $\varphi(x, y) := \mu$ for all $x, y \in X$. By Theorem 3.1 we get the desired result. □

3.8. Corollary. *Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $p, q > 2$ or $0 < p, q < 2$. Let X be a fuzzy normed linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Suppose that a mapping $f : X \longrightarrow Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_Y \preceq \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q), \tag{3.17}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\forall \alpha \in (0,1],\ \exists \beta \in (0,\alpha],\ s.t. \|f(x) - T(x)\|_\alpha^+ \leq \mu_\beta^+ \Big(\frac{(\|x\|_\beta^+)^p}{|2^p - 4|} + \frac{(\|x\|_\beta^+)^q}{|2^q - 4|} \Big),$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(x, y) := \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q),$$

for all $x, y \in X$. □

3.9. Corollary. *Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $\lambda = p + q \in (0, 2) \bigcup (2, \infty)$. Let X be a fuzzy normed linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Suppose that a mapping $f : X \longrightarrow Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \preceq \mu \otimes \|x\|_X^p \otimes \|y\|_X^q, \tag{3.18}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\forall \alpha \in (0,1],\ \exists \beta \in (0,\alpha],\ s.t. \|f(x) - T(x)\|_\alpha^+ \leq \frac{(\|x\|_\beta^+)^\lambda \mu_\beta^+}{|2^\lambda - 4|},$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(x, y) := \mu \otimes \|x\|_X^p \otimes \|y\|_X^q,$$

for all $x, y \in X$. □

4. Stability of quadratic equation, fixed point method

In this section we will adopt the idea of Cădariu and Radu [46], to prove the generalized Hyers–Ulam–Rassias stability of quadratic functional equation in Felbin's type fuzzy normed linear spaces.

We recall two fundamental results in fixed point theory.

4.1. Theorem. *(Banach's contraction principle) Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,*

$$d(Jx, Jy) \leq Lf(x, y), \qquad \forall x, y \in X$$

for some Lipschitz constant $L < 1$. Then

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point x^* is globally attractive, i.e.,*

$$\lim_{n \to \infty} J^n x = x^*$$

for any starting point $x \in X$;

(3) *one has the following estimation inequalities:*

$$d(J^n x, x^*) \leq L^n d(x, x^*),$$

$$d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1} x),$$

$$d(x, x^*) \leq \frac{1}{1-L} d(x, Jx)$$

for all nonnegative integers n and all $x \in X$.

4.2. Definition. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

4.3. Theorem. [11] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$*;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\mathcal{Y} = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \mathcal{Y}$.*

4.4. Theorem. *Let X be a liner space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to F^*(\mathbb{R})$ such that*

$$\lim_{i \to \infty} \frac{1}{4^i} \varphi(2^i x . 2^i y)^+_\alpha = 0 \tag{4.1}$$

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \| \preceq \varphi(x, y) \tag{4.2}$$

for all $x, y \in X$ and all $\alpha \in (0, 1]$. If there exists a $L < 1$ such that $\varphi(x, x) \preceq 4L \odot \varphi(\frac{x}{2}, \frac{x}{2})$ for all $x \in X$, then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \|^+_\alpha \leq \frac{1}{4-4L} \varphi(x, x)^+_\alpha$$

for all $x \in X$ and all $\alpha \in (0, 1]$.

Proof. Consider the set

$$K := \{g : X \to Y\}$$

and introduce the *generalized metric* on K:

$$d(g, h) = inf\{C \in \mathbb{R}_+ : \| g(x) - h(x) \|^+_\alpha \leq C\varphi(x, x)^+_\alpha, \ \forall x \in X, \ \forall \alpha \in (0, 1]\}.$$

It is easy to show that (K, d) is complete. Now we consider the linear mapping $J : K \to K$ such that

$$Jg(x) := (\frac{1}{4}) g(2x)$$

for all $x \in X$. For any $g, h \in K$, we have

$$\begin{aligned} d(g,h) < C &\Longrightarrow \|g(x) - h(x)\|_\alpha^+ \leq C\varphi(x,x)_\alpha^+, \qquad \forall x \in X \ \& \ \forall \alpha \in (0,1] \\ &\Longrightarrow \left\|\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\right\|_\alpha^+ \leq \frac{1}{4}C\varphi(2x,2x)_\alpha^+ \leq LC\varphi(x,x)_\alpha^+ \\ &\Longrightarrow d(Jg, Jh) \leq LC. \end{aligned}$$

Therefore, we see that

$$d(Jg, Jh) \leq Ld(g,h), \qquad \forall g, h \in K.$$

This means J is a strictly contractive self-mapping of K, with the Lipschitz constant L.

Letting $y = x$ in (4.2), we get

$$\| f(2x) - 4f(x) \| \preceq \varphi(x,x) \tag{4.3}$$

for all $x \in X$. So

$$\| f(x) - \frac{1}{4}f(2x) \| \preceq \frac{1}{4} \odot \varphi(x,x).$$

Therefore we can conclude that

$$\| f(x) - \frac{1}{4}f(2x) \|_\alpha^+ \leq \frac{1}{4}\varphi(x,x)_\alpha^+ \tag{4.4}$$

for all $x \in X$ and all $\alpha \in (0,1]$. Hence $d(f, Jf) \leq \frac{1}{4}$.

By Theorem 4.3, there exists a mapping $T : X \to Y$ such that

(1) T is a fixed point of J, i.e.,

$$T(2x) = 4T(x) \tag{4.5}$$

for all $x \in X$. The mapping T is a unique fixed point of J in the set

$$\mathcal{Y} = \{g \in X : d(f,g) < \infty\}.$$

This implies that T is a unique mapping such that there exists $C \in (0, \infty)$ satisfying

$$\| T(x) - f(x) \|_\alpha^+ \leq C\varphi(x,x)_\alpha^+$$

for all $x \in X$.

(2) $d(J^n f, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = T(x) \tag{4.6}$$

for all $x \in X$.

(3) $d(f,T) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f,T) \leq \frac{1}{4-4L}.$$

It follows from (4.1) and (4.6) and continuity of $\|.\|_\alpha^+$ that

$$\begin{aligned} &\|T(x+y) + T(x-y) - 2T(x) - 2T(y)\|_\alpha^+ \\ &= \lim_{n\to\infty} \frac{1}{4^n} \| f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y) \|_\alpha^+ \\ &\leq \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y)_\alpha^+ = 0 \end{aligned}$$

for all $x, y \in X$. Therefore, the mapping T is quadratic. $\square$

4.5. Corollary. *Let μ be a non-negative fuzzy real number and let X be a linear space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \| \preceq \mu,$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{3}\mu_\alpha^+$$

for all $x \in X$.

Proof. Let $\varphi(x, y) = \mu$ for all $x, y \in X$ and use the Theorem 4.4. We can choose $L = \dfrac{1}{4}$ to get the desired result. result. □

4.6. Corollary. *Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $0 < p, q < 2$. Let X be a fuzzy normed liner space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \|_Y \preceq \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q) \tag{4.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{2^2 - 2^{\max(p,q)}}\mu_\alpha^+ (\|x\|_X^p \oplus \|x\|_X^q)_\alpha^+$$

for all $x \in X$ and all $\alpha \in (0, 1]$.

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x, y) := \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q)$$

for all $x, y \in X$. We can choose $L = 2^{\max(p,q)-2}$ to get the desired result. □

4.7. Corollary. *Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $\lambda = p + q \in (0, 2)$. Let X be a fuzzy normed liner space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_Y \preceq \mu \otimes \|x\|_X^p \otimes \|y\|_X^q \tag{4.8}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{2^2 - 2^{p+q}}(\mu \otimes \|x\|_X^p \otimes \|x\|_X^q)_\alpha^+$$

for all $x \in X$ and all $\alpha \in (0, 1]$.

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x, y) := \mu \otimes \|x\|_X^p \otimes \|y\|_X^q$$

for all $x, y \in X$. We can choose $L = 2^{p+q-2}$ to get the desired result. □

4.8. Theorem. *Let X be a liner space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to F^*(\mathbb{R})$ such that*

$$\lim_{i\to\infty} 4^i \varphi(\frac{x}{2^i}, \frac{y}{2^i})_\alpha^+ = 0 \tag{4.9}$$

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \| \preceq \varphi(x, y) \tag{4.10}$$

for all $x, y \in X$ *and all* $\alpha \in (0, 1]$. *If there exists a* $L < 1$ *such that* $\varphi(x, x) \preceq \frac{L}{4} \odot \varphi(2x, 2x)$ *for all* $x \in X$, *then there exists a unique quadratic mapping* $T : X \to Y$ *such that*

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{L}{4 - 4L} \varphi(x, x)_\alpha^+$$

for all $x \in X$ *and all* $\alpha \in (0, 1]$.

Proof. Similar to proof of Theorem 4.4, we consider the linear mapping $J : K \to K$ such that

$$Jg(x) := 4g(\frac{x}{2})$$

for all $x \in X$. It is easy to see that, J is a strictly contractive self-mapping of K, with the Lipschitz constant L.

Letting $y = x = \frac{x}{2}$ in (4.10), we get

$$\| f(x) - 4f(\frac{x}{2}) \| \preceq \varphi(\frac{x}{2}, \frac{x}{2}) \preceq \frac{L}{4} \odot \varphi(x, x) \tag{4.11}$$

for all $x \in X$. Then we have

$$\| f(x) - 4f(\frac{x}{2}) \|_\alpha^+ \leq \varphi(\frac{x}{2}, \frac{x}{2})_\alpha^+ \leq \frac{L}{4} \varphi(x, x)_\alpha^+ \tag{4.12}$$

for all $x \in X$ and all $\alpha \in (0, 1]$. Hence $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 4.3, there exists a mapping $T : X \to Y$ such that

(1) T is a fixed point of J, i.e.,

$$4T(\frac{x}{2}) = T(x) \tag{4.13}$$

for all $x \in X$. The mapping T is a unique fixed point of J in the set

$$\mathcal{Y} = \{g \in X : d(f, g) < \infty\}.$$

This implies that T is a unique mapping such that there exists $C \in (0, \infty)$ satisfying

$$\| T(x) - f(x) \|_\alpha^+ \leq C \varphi(x, x)_\alpha^+$$

for all $x \in X$.

(2) $d(J^n f, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 4^n f(\frac{x}{2^n}) = T(x) \tag{4.14}$$

for all $x \in X$.

(3) $d(f, T) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, T) \leq \frac{L}{4 - 4L}.$$

The rest of the proof is similar to proof of Theorem4.4. □

4.9. Corollary. *Let* μ *be a non-negative fuzzy real number and let* p, q *be non-negative real numbers such that* $p, q > 2$. *Let* X *be a fuzzy normed liner space and* $(Y, \|.\|, L, R)$ *be a fuzzy Banach space satisfying* $(R - 1)$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \preceq \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q) \tag{4.15}$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $T : X \to Y$ *such that*

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{2^{\min(p,q)} - 2^2} \mu_\alpha^+ (\|x\|_X^p \oplus \|x\|_X^q)_\alpha^+$$

for all $x \in X$ *and all* $\alpha \in (0, 1]$.

Proof. The proof follows from Theorem 4.8 by taking

$$\varphi(x,y) := \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q)$$

for all $x, y \in X$. We can choose $L = 2^{2-\min(p,q)}$ to get the desired result. □

4.10. Corollary. *Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $\lambda = p + q \in (2, \infty)$. Let X be a fuzzy normed liner space and $(Y, \|.\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_Y \preceq \mu \otimes \|x\|_X^p \otimes \|y\|_X^q \tag{4.16}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \|_\alpha^+ \leq \frac{1}{2^{(p+q)} - 2^2} (\mu \otimes \|x\|_X^p \otimes \|x\|_X^q)_\alpha^+$$

for all $x \in X$ and all $\alpha \in (0, 1]$.

Proof. The proof follows from Theorem 4.8 by taking

$$\varphi(x,y) := \mu \otimes \|x\|_X^p \otimes \|y\|_X^q$$

for all $x, y \in X$. We can choose $L = 2^{2-(p+q)}$ to get the desired result. □

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