Hacettepe Journal of Mathematics and Statistics
Volume 42 (5) (2013), 533 – 549

SOME NOTES CONCERNING CHEEGER-GROMOLL METRICS

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Received 04 : 04 : 2012 : Accepted 22 : 08 : 2012

Abstract

The purpose of this paper is to introduce Cheeger-Gromoll type metric on the cotangent bundle of Riemannian manifold and investigate curvature properties and geodesics on the cotangent bundle with respect to the Cheeger-Gromoll metric.

Keywords: Cheeger-Gromoll metric, cotangent bundle, vertical and horizontal lift, curvature tensor, geodesics

2000 AMS Classification: 53C07, 53C22, 53C25

1. Introduction

In [4] Cheeger and Gromoll study complete manifolds of nonnegative curvature and suggest a construction of a new Riemannian metrics ${}^{CG}g$. Musso and Triccerri [9] were the first giving the explicit formula for this metric. In [12] the Levi-Civita connection of ${}^{CG}g$ and its Riemannian curvature tensor are calculated by Sekizawa. In [5] Gudmundsson and Kappos corrected the formulas for curvature of ${}^{CG}g$ in the tangent bundle given by Sekizawa [12]. In [11] Salimov and Kazimova invastigated geodesics of the Cheeger-Gromoll metric on tangent bundle. The geometry of Cheeger-Gromoll metric is well known and intensively studied for the tangent bundle (see for example [1],[2],[7],[8],[10]). The similar metric in theoritical physics has been obtained by Tamm (Nobel Laureate in Physics for the year 1958, see [13]). The main purpose of this paper is to introduce Levi-Civita connection of Cheeger-Gromoll type metric on the cotangent bundle T^*M^n of Riemannian manifold M^n and invastigate curvature properties and geodesics on T^*M^n with respect to the Levi-Civita connection of ${}^{CG}g$. Since the construction of lifts to the cotangent bundle is not similar to the definition of lifts to the tangent bundle, we have some differences for Cheeger-Gromoll metrics on cotangent bundles (see Theorem 3.2 and Theorem 3.5).

Let (M^n, g) be an n-dimensional Riemannian manifold, T^*M^n its cotangent bundle

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and π the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1, ..., n$ on M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i$ $(i = 1, ..., n)$, where $x^{\bar{i}} = p_i$ are the components of covectors p in each cotangent space $T^*_x M^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, $i = 1, ..., n$.

We denote by $\Im^r_s(M^n)(\Im^r_s(T^*M^n))$ the module over $F(M^n)(F(T^*M^n))$ of C^∞ tensor fields of type (r, s), where $F(M^n)(F(T^*M^n))$ is the ring of real-valued C^∞ functions on $M^n(T^*M^n)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M^n$ of a vector and a covector (1-form) field $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively. Then the complete and horizontal lifts ${}^C X, {}^H X \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift ${}^V \omega \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ are given, respectively, by

$$(1.1) \qquad {}^C X = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}},$$

$$(1.2) \qquad {}^H X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}},$$

$$(1.3) \qquad {}^V \omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}},$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, where Γ^h_{ij} are the components of the Levi-Civita connection ∇_g on M^n (see [14] for more details).

1.1. Theorem. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M^n of M^n satisfies the following*

$$(1.4) \qquad \begin{aligned} &i) [{}^V \omega, {}^V \theta] = 0, \\ &ii) [{}^H X, {}^V \omega] = {}^V(\nabla_X \omega), \\ &iii) [{}^H X, {}^H Y] = {}^H [X, Y] + \gamma R(X, Y) = {}^H [X, Y] + {}^V(pR(X, Y)) \end{aligned}$$

for all $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$. (See [14, p.238, p.277] for more details).

1.2. Definition. Let M^n be a Riemannian manifold with metric g. A Riemannian metric $\bar{g}$ on cotangent bundle T^*M^n is said to be natural with respect to g on M^n if

$$(1.5) \qquad \begin{aligned} &i) \quad \bar{g}({}^H X, {}^H Y) = g(X, Y), \\ &ii) \quad \bar{g}({}^H X, {}^V \omega) = 0 \end{aligned}$$

for all $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$.

1.3. Theorem. *Let M^n be a Riemannian manifold with metric g and T^*M^n be the cotangent bundle of M^n. If the Riemannian metric $\bar{g}$ on T^*M^n is natural with respect* *to g on M^n then the corresponding Levi-Civita connection $\bar{\nabla}$ satisfies*

$$
\begin{aligned}
&i)\bar{g}(\bar{\nabla}_{^HX}{}^HY, {}^HZ) = g(\nabla_X Y, Z),\\
&ii)\bar{g}(\bar{\nabla}_{^HX}{}^HY, {}^V\omega) = \frac{1}{2}\bar{g}({}^V\omega, {}^V(pR(X,Y))),\\
&iii)\bar{g}(\bar{\nabla}_{^HX}{}^V\omega, {}^HZ) = \frac{1}{2}\bar{g}({}^V(pR(Z,X)), {}^V\omega,),\\
&iv)\bar{g}(\bar{\nabla}_{^HX}{}^V\omega, {}^V\theta) = \frac{1}{2}({}^HX(\bar{g}({}^V\omega, {}^V\theta)) - \bar{g}({}^V\omega, {}^V(\nabla_X\theta)) + \bar{g}({}^V\theta, {}^V(\nabla_X\omega))),\\
(1.6)\quad &v)\bar{g}(\bar{\nabla}_{^V\omega}{}^HY, {}^HZ) = -\frac{1}{2}\bar{g}({}^V\omega, {}^V(pR(Y,Z))),\\
&vi)\bar{g}(\bar{\nabla}_{^V\omega}{}^HY, {}^V\theta) = \frac{1}{2}({}^HY(\bar{g}({}^V\omega, {}^V\theta)) - \bar{g}({}^V\omega, {}^V(\nabla_Y\theta)) - \bar{g}({}^V\theta, {}^V(\nabla_Y\omega))),\\
&vii)\bar{g}(\bar{\nabla}_{^V\omega}{}^V\theta, {}^HZ) = \frac{1}{2}(-{}^HZ(\bar{g}({}^V\omega, {}^V\theta)) + \bar{g}({}^V\omega, {}^V(\nabla_Z\theta)) + \bar{g}({}^V\theta, {}^V(\nabla_Z\omega))),\\
&viii)\bar{g}(\bar{\nabla}_{^V\omega}{}^V\theta, {}^V\xi) = \frac{1}{2}({}^V\omega(\bar{g}({}^V\theta, {}^V\xi) + {}^V\theta(\bar{g}({}^V\xi, {}^V\omega) - {}^V\xi(\bar{g}({}^V\omega, {}^V\theta))
\end{aligned}
$$

for all $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$.

Proof. We use Koszul Formula for the Levi-Civita connection $\bar{\nabla}$ stating that

$$
\begin{aligned}
2\bar{g}(\bar{\nabla}_{^iX}{}^jY, {}^kZ) = {}& {}^iX(\bar{g}({}^jY, {}^kZ)) + {}^jY(\bar{g}({}^kZ, {}^iX)) - {}^kZ(\bar{g}({}^iX, {}^jY))\\
&- \bar{g}({}^iX, [{}^jY, {}^kZ]) + \bar{g}({}^jY, [{}^kZ, {}^iX]) + \bar{g}({}^kZ, [{}^iX, {}^jY])
\end{aligned}
$$

for all $X, Y, Z \in \Im^1_0(M^n)$ and $i, j, k \in \{H, V\}$. If $i, j, k \in \{V\}$, we write $\omega, \theta, \xi \in \Im^0_1(M^n)$ instead of X, Y, Z in T^*M^n.

i) Using Koszul Formula, Theorem 1.1 and Definition 1.2, we have

$$
\begin{aligned}
2\bar{g}(\bar{\nabla}_{^HX}{}^HY, {}^HZ) &= {}^HX(\bar{g}({}^HY, {}^HZ)) + {}^HY(\bar{g}({}^HZ, {}^HX)) - {}^HZ(\bar{g}({}^HX, {}^HY)),\\
&- \bar{g}({}^HX, [{}^HY, {}^HZ]) + \bar{g}({}^HY, [{}^HZ, {}^HX]) + \bar{g}({}^HZ, [{}^HX, {}^HY]),\\
&= 2g(\nabla_X Y, Z)
\end{aligned}
$$

ii) The statement is obtained as follows:

$$
\begin{aligned}
2\bar{g}(\bar{\nabla}_{^HX}{}^HY, {}^V\omega) &= {}^HX(\bar{g}({}^HY, {}^V\omega)) + {}^HY(\bar{g}({}^V\omega, {}^HX)) - {}^V\omega(\bar{g}({}^HX, {}^HY)),\\
&- \bar{g}({}^HX, [{}^HY, {}^V\omega]) + \bar{g}({}^HY, [{}^V\omega, {}^HX]) + \bar{g}({}^V\omega, [{}^HX, {}^HY]),\\
&= -{}^V\omega^V(g(X,Y)) - \bar{g}({}^HX, {}^V(\nabla_X\omega)),\\
&+ \bar{g}({}^HY, {}^V(-\nabla_X\omega)) + \bar{g}({}^V\omega, {}^H[X,Y] + {}^V(pR(X,Y))),\\
&= \bar{g}({}^V\omega, {}^V(pR(X,Y))).
\end{aligned}
$$

iii) and *v)* are analogues to *ii)*.

iv) Again using Koszul formula, Theorem 1.1 and Definition 1.2, we have

$$
\begin{aligned}
2\bar{g}(\bar{\nabla}_{^HX}{}^V\omega, {}^V\theta) &= {}^HX(\bar{g}({}^V\omega, {}^V\theta)) + {}^V\omega(\bar{g}({}^V\theta, {}^HX)) - {}^V\theta(\bar{g}({}^HX, {}^V\omega)),\\
&- \bar{g}({}^HX, [{}^V\omega, {}^V\theta]) + \bar{g}({}^V\omega, [{}^V\theta, {}^HX]) + \bar{g}({}^V\theta, [{}^HX, {}^V\omega]),\\
&= {}^HX(\bar{g}({}^V\omega, {}^V\theta)) + \bar{g}({}^V\omega, {}^V(-\nabla_X\theta)) + \bar{g}({}^V\theta, {}^V(\nabla_X\omega)),\\
&= {}^HX(\bar{g}({}^V\omega, {}^V\theta)) - \bar{g}({}^V\omega, {}^V(\nabla_X\theta)) + \bar{g}({}^V\theta, {}^V(\nabla_X\omega)).
\end{aligned}
$$

vi) and *vii)* are analogues to *iv)*.

viii)

$$2\bar{g}(\bar{\nabla}_{{}^V\omega}{}^V\theta, {}^V\xi) = {}^V\omega(\bar{g}({}^V\theta, {}^V\xi)) + {}^V\theta(\bar{g}({}^V\xi, {}^V\omega)) - {}^V\xi(\bar{g}({}^V\omega, {}^V\theta)),$$
$$- \bar{g}({}^V\omega, [{}^V\theta, {}^V\xi]) + \bar{g}({}^V\theta, [{}^V\xi, {}^V\omega]) + \bar{g}({}^V\xi, [{}^V\omega, {}^V\theta]),$$
$$= {}^V\omega(\bar{g}({}^V\theta, {}^V\xi)) + {}^V\theta(\bar{g}({}^V\xi, {}^V\omega)) - {}^V\xi(\bar{g}({}^V\omega, {}^V\theta)).$$

□

1.4. Corollary. *Let M^n be a Riemannian manifold with metric g and $\bar{g}$ be a natural metric on the cotangent bundle T^*M^n of M^n. Then the levi-Civita connection $\bar{\nabla}$ satisfies*

$$\bar{\nabla}_{{}^H X}{}^H Y = {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y)) \tag{1.7}$$

for all $X, Y \in \Im^1_0(M^n)$.

Proof. The statement is obtained by combing *i)* and *ii)* of Theorem 1.3. □

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x(M^n)$ by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$$

for all $\omega, \theta \in \Im^0_1(M^n)$.

1.5. Definition. A Cheeger-Gromoll metric ${}^{CG}g$ is defined on T^*M^n by the following three equations

$$ {}^{CG}g({}^H X, {}^H Y) = {}^V(g(X,Y)) = g(X,Y)\circ\pi, \tag{1.8}$$

$$ {}^{CG}g({}^V\omega, {}^H Y) = 0, \tag{1.9}$$

$$ {}^{CG}g({}^V\omega, {}^V\theta) = \frac{1}{1+r^2}\big(g^{-1}(\omega,\theta) + g^{-1}(\omega,p)g^{-1}(\theta,p)\big) \tag{1.10}$$

for any $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$, where $r^2 = g^{ij}p_ip_j$.

Since any tensor field of type (0,2) on T^*M^n is completely determined by its action on vector fields of type ${}^H X$ and ${}^V\omega$, it follows that ${}^{CG}g$ is completely determined by its equations (1.8), (1.9) and (1.10).

We now see, from (1.1) and (1.2), that the complete lift ${}^C X$ of $X \in \Im^1_0(M^n)$ is expressed by

$$ {}^C X = {}^H X - {}^V\big(p(\nabla X)\big), \tag{1.11}$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$.

Using (1.8), (1.9), (1.10) and (1.11), we have

$$ {}^{CG}g({}^C X, {}^C Y) = {}^V\big(g(X,Y)\big) + \frac{1}{1+r^2}\big(g^{-1}(p(\nabla X), p(\nabla Y)) + g^{-1}(p(\nabla X), p)g^{-1}(p(\nabla Y), p)\big), \tag{1.12}$$

where $g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p_l\nabla_i X^l)(p_k\nabla_j Y^k)$, $g^{-1}(p(\nabla X), p) = g^{ij}p_i(p(\nabla X))_j$.

Since the tensor field ${}^{CG}g \in \Im^0_2(T^*M^n)$ is completely determined also by its action on vector fields type ${}^C X$ and ${}^C Y$ (see[14, p.237]), we have an alternative characterization of ${}^{CG}g$ on T^*M^n: ${}^{CG}g$ is completely determined by the condition (1.12).

2. Levi-Civita connection of ${}^{CG}g$

We put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \theta^{(i)} = dx^i, i = 1, ..., n.$$

Then from (1.2) and (1.3) we see that ${}^{H}X_{(i)}$ and ${}^{V}\theta^{(i)}$ have respectively local expressions of the form

$$\tilde{e}_{(i)} = {}^{H}X_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{2.1}$$

$$\tilde{e}_{(\bar{i})} = {}^{V}\theta^{(i)} = \frac{\partial}{\partial x^{\bar{i}}}. \tag{2.2}$$

We call the set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\} = \{{}^{H}X_{(i)}, {}^{V}\theta^{(i)}\}$ the frame adapted to Levi-Civita connection ∇_g. The indices $\alpha, \beta, ... = 1, ..., 2n$ indicate the indices with respect to the adapted frame.

We now, from the equations (1.2), (1.3), (2.1) and (2.2) see that ${}^{H}X$ and ${}^{V}\omega$ have respectively components

$$ {}^{H}X = X^i \tilde{e}_{(i)}, \quad {}^{H}X = ({}^{H}X^\alpha) = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{2.3}$$

$$ {}^{V}\omega = \sum_i \omega_i \tilde{e}_{(\bar{i})}, \quad {}^{V}\omega = ({}^{V}\omega^\alpha) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{2.4}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where X^i and ω_i being local components of $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively.

From (1.8), (1.9) and (1.10) we see that

$$\begin{aligned}
{}^{CG}g_{ij} &= {}^{CG}g\big(\tilde{e}_{(i)}, \tilde{e}_{(j)}\big) = {}^{V}\big(g(\partial_i, \partial_j)\big) = g_{ij}, \\
{}^{CG}g_{\bar{i}j} &= {}^{CG}g\big(\tilde{e}_{(\bar{i})}, \tilde{e}_{(j)}\big) = 0, \\
{}^{CG}g_{\bar{i}\bar{j}} &= {}^{CG}g\big(\tilde{e}_{(\bar{i})}, \tilde{e}_{(\bar{j})}\big) = \frac{1}{1+r^2}\big(g^{-1}(dx^i, dx^j) + g^{-1}(dx^i, p_k) g^{-1}(dx^j, p_l)\big) \\
&= \frac{1}{1+r^2}\big(g^{ij} + g^{ik} g^{lj} p_k p_l\big),
\end{aligned}$$

i.e. ${}^{CG}g$ has components

$$ {}^{CG}g = \begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{1+r^2}\big(g^{ij} + g^{ik} g^{lj} p_k p_l\big) \end{pmatrix} \tag{2.5}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

Cheeger-Gromoll metric is obviously contained in the class of natural metrics. For the Levi-Civita connection of the Cheeger-Gromoll metric we have the following.

2.1. Theorem. *Let M^n be a Riemannian manifold with metric g and ${}^{CG}\nabla$ be the Levi-Civita connection of the cotangent bundle T^*M^n equipped with the Cheeger-Gromoll metric ${}^{CG}g$. Then ${}^{CG}\nabla$ satisfies*

$$(2.6)\quad \begin{aligned} &\text{i)} \quad {}^{CG}\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y), \\ &\text{ii)} \quad {}^{CG}\nabla_{{}^HX}{}^V\omega = {}^V(\nabla_X\omega) + \frac{1}{2\alpha}{}^H(p(g^{-1}\circ R(\ ,X)\tilde\omega), \\ &\text{iii)} \quad {}^{CG}\nabla_{{}^V\omega}{}^HY = \frac{1}{2\alpha}{}^H(p(g^{-1}\circ R(\ ,Y)\tilde\omega), \\ &\text{iv)} \quad {}^{CG}\nabla_{{}^V\omega}{}^V\theta = -\frac{1}{\alpha}\big({}^{CG}g({}^V\omega,\gamma\delta){}^V\theta + {}^{CG}g({}^V\theta,\gamma\delta){}^V\omega\big) \\ &\qquad + \frac{\alpha+1}{\alpha}{}^{CG}g({}^V\omega,{}^V\theta)\gamma\delta - \frac{1}{\alpha}{}^{CG}g({}^V\omega,\gamma\delta){}^{CG}g({}^V\theta,\gamma\delta)\gamma\delta \end{aligned}$$

for all $X,Y \in \Im^1_0(M^n)$, $\omega,\theta \in \Im^0_1(M^n)$, where $\tilde\omega = g^{-1}\circ\omega \in \Im^1_0(M^n)$, $R(\ ,X)\tilde\omega \in \Im^1_1(M^n)$, $g^{-1}\circ R(\ ,X)\tilde\omega \in \Im^2_0(M^n)$, $\alpha = 1+r^2$, R and $\gamma\delta$ denotes respectively the curvature tensor of ∇ and the canonical vertical vector field on T^*M^n with expression $\gamma\delta = p_i e_{(\bar i)}$.

Proof. i) The first statement is just Corollary 1.4.
ii) Following Definition 1.2 and Theorem 1.3 we see that

$$\begin{aligned} 2{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^V\omega, {}^HY) &= {}^{CG}g({}^V\big(pR(Y,X)\big), {}^V\omega\big) \\ &= \frac{1}{\alpha}\big(g^{-1}(pR(Y,X),\omega) + g^{-1}(pR(Y,X),p)g^{-1}(\omega,p)\big) \\ &= \frac{1}{\alpha}{}^{CG}g\big({}^H(p(g^{-1}\circ R(\ ,X)\tilde\omega)), {}^HY\big) \end{aligned}$$

and

$$\begin{aligned} &2{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^V\omega, {}^V\theta) = \big({}^HX({}^{CG}g({}^V\omega,{}^V\theta)) - {}^{CG}g({}^V\omega, {}^V(\nabla_X\theta)) \\ &+ {}^{CG}g({}^V\theta, {}^V(\nabla_X\omega))\big) \\ &= {}^{CG}g({}^V\omega, {}^V(\nabla_X\theta)) + {}^{CG}g({}^V\theta, {}^V(\nabla_X\omega)) - {}^{CG}g({}^V\omega, {}^V(\nabla_X\theta)) \\ &+ {}^{CG}g({}^V\theta, {}^V(\nabla_X\omega)) \\ &= 2{}^{CG}g({}^V\theta, {}^V(\nabla_X\omega)) = 2{}^{CG}g({}^V(\nabla_X\omega), {}^V\theta) \end{aligned}$$

Using

$$\begin{aligned} g^{-1}(pR(Y,X),\omega) &= (g^{kl}(pR(Y,X))_k\omega_l) \\ &= (g^{kl}p_sR_{ijk}{}^sY^iX^j\omega_l) = (p_sR_{ijk}{}^sY^iX^jg^{kl}\omega_l) \\ &= (p_sR_{ijk}{}^sY^iX^j\tilde\omega^k) = (g_{ai}p_sR^a_{.jk}{}^sY^iX^j\tilde\omega^k) \\ &= g\big(p(g^{-1}\circ R(\ ,X)\tilde\omega), Y\big) \\ &= {}^{CG}g\big({}^H(p(g^{-1}\circ R(\ ,X)\tilde\omega)), {}^HY\big), \end{aligned}$$

$$\begin{aligned} g^{-1}(pR(Y,X),p) &= (g^{ij}p_sR_{abi}{}^sY^aX^bp_j) \\ &= (p_sg^{ts}R_{abit}Y^aX^b\tilde p^i) = (R_{abit}Y^aX^b\tilde p^i\tilde p^t) \\ &= (R_{itab}Y^aX^b\tilde p^i\tilde p^t) = (g_{fb}R_{ita.}{}^fY^aX^b\tilde p^i\tilde p^t) \\ &= g\big(R(\tilde p,\tilde p)Y,X\big) = 0, \end{aligned}$$

${}^HX(\frac{1}{\alpha}) = 0$
and

${}^{CG}X({}^{CG}g({}^{V}\omega, {}^{V}\theta) = {}^{CG}g({}^{V}\omega, {}^{V}(\nabla_X\theta)) + {}^{CG}g({}^{V}\theta, {}^{V}(\nabla_X\omega))$
we have
${}^{CG}\nabla_{{}^{H}X}{}^{V}\omega = {}^{V}(\nabla_X\omega) + \frac{1}{2\alpha}{}^{H}(p(g^{-1}\circ R(\ ,X)\tilde{\omega})$
iii) Calculations similar to those in *ii)* give

$$\begin{aligned}
&2{}^{CG}g({}^{CG}\nabla_{{}^{V}\omega}{}^{H}Y, {}^{V}\theta) = \big({}^{H}Y({}^{CG}g({}^{V}\omega, {}^{V}\theta)) - {}^{CG}g({}^{V}\omega, {}^{V}(\nabla_Y\theta))\\
&-{}^{CG}g({}^{V}\theta, {}^{V}(\nabla_Y\omega))\big)\\
&= {}^{CG}g({}^{V}\omega, {}^{V}(\nabla_Y\theta)) + {}^{CG}g({}^{V}\theta, {}^{V}(\nabla_Y\omega)) - {}^{CG}g({}^{V}\omega, {}^{V}(\nabla_Y\theta))\\
&-{}^{CG}g({}^{V}\theta, {}^{V}(\nabla_Y\omega))\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
2{}^{CG}g({}^{CG}\nabla_{{}^{V}\omega}{}^{H}Y, {}^{H}Z) &= -{}^{CG}g\big({}^{V}\omega, {}^{V}(pR(Y,Z))\big)\\
&= -\frac{1}{\alpha}\big(g^{-1}(\omega, pR(Y,Z)) + g^{-1}(pR(Y,Z),p)g^{-1}(\omega,p)\big)\\
&= \frac{1}{\alpha}{}^{CG}g\big({}^{H}(p(g^{-1}\circ R(\ ,Y)\tilde{\omega})), {}^{H}Z\big).
\end{aligned}$$

Thus we have

$$ {}^{CG}\nabla_{{}^{V}\omega}{}^{H}Y = \frac{1}{2\alpha}{}^{H}(p(g^{-1}\circ R(\ ,Y)\tilde{\omega}). $$

iv) Applying Theorem 1.3 we yield

$$\begin{aligned}
&2{}^{CG}g({}^{CG}\nabla_{{}^{V}\omega}{}^{V}\theta, {}^{H}Z) = \frac{1}{2}\big(-{}^{H}Z({}^{CG}g({}^{V}\omega, {}^{V}\theta)) + {}^{CG}g({}^{V}\omega, {}^{V}(\nabla_Z\theta))\\
&+{}^{CG}g({}^{V}\theta, {}^{V}(\nabla_Z\omega))\big)\\
&= -{}^{CG}g({}^{V}\omega, {}^{V}(\nabla_Z\theta)) - {}^{CG}g({}^{V}\theta, {}^{V}(\nabla_Z\omega)) + {}^{CG}g({}^{V}\omega, {}^{V}(\nabla_Z\theta))\\
&+{}^{CG}g({}^{V}\theta, {}^{V}(\nabla_Z\omega))\\
&= 0
\end{aligned}$$

Using ${}^{V}\omega(\frac{1}{\alpha}) = -\frac{2}{\alpha^2}g^{-1}(\omega,p)$,

$$\begin{aligned}
{}^{V}\omega({}^{CG}g({}^{V}\theta, {}^{V}\xi)) = &-\frac{2}{\alpha^2}g^{-1}(\omega,p)[g^{-1}(\theta,\xi) + g^{-1}(\theta,p)g^{-1}(\xi,p)]\\
&+\frac{1}{\alpha}\big(g^{-1}(\omega,\theta)g^{-1}(\xi,p) + g^{-1}(\theta,\omega)g^{-1}(\omega,\xi)\big)
\end{aligned}$$

and

$$\begin{aligned}
{}^{CG}g({}^{V}\omega, \gamma\delta) &= \frac{1}{\alpha}\big(g^{-1}(\omega,p) + g^{-1}(\omega,p)g^{-1}(p,p)\big)\\
&= \frac{1}{\alpha}g^{-1}(\omega,p)(1 + g^{-1}(p,p)) = g^{-1}(\omega,p)
\end{aligned}$$

we have

$$\begin{aligned}\alpha^{2\,CG}g({}^{CG}\nabla_{{}^V\omega}{}^V\theta,{}^V\xi) &= \frac{\alpha^2}{2}\big({}^V\omega({}^{CG}g({}^V\theta,{}^V\xi)) + {}^V\theta({}^{CG}g({}^V\xi,{}^V\omega))\\
&- {}^V\xi({}^{CG}g({}^V\omega,{}^V\theta))\big)\\
&= -g^{-1}(\omega,p)g^{-1}(\theta,\xi) - g^{-1}(\omega,p)g^{-1}(\theta,p)g^{-1}(\xi,p)\\
&+ \frac{\alpha}{2}g^{-1}(\omega,\theta)g^{-1}(\xi,p) + \frac{\alpha}{2}g^{-1}(\theta,p)g^{-1}(\omega,\xi)\\
&- g^{-1}(\theta,p)g^{-1}(\xi,\omega) - g^{-1}(\theta,p)g^{-1}(\xi,p)g^{-1}(\omega,p)\\
&+ \frac{\alpha}{2}g^{-1}(\theta,\xi)g^{-1}(\omega,p) + \frac{\alpha}{2}g^{-1}(\xi,p)g^{-1}(\theta,\omega)\\
&+ g^{-1}(\xi,p)g^{-1}(\omega,\theta) + g^{-1}(\xi,p)g^{-1}(\omega,p)g^{-1}(\theta,p)\\
&- \frac{\alpha}{2}g^{-1}(\xi,\omega)g^{-1}(\theta,p) - \frac{\alpha}{2}g^{-1}(\omega,p)g^{-1}(\xi,\theta)\\
&= -g^{-1}(\omega,p)g^{-1}(\xi,\theta) + \alpha g^{-1}(\omega,\theta)g^{-1}(\xi,p)\\
&- g^{-1}(\theta,p)g^{-1}(\xi,\omega) - g^{-1}(\theta,p)g^{-1}(\xi,p)g^{-1}(\omega,p)\\
&+ g^{-1}(\xi,p)g^{-1}(\omega,\theta)\\
&= -\alpha g^{-1}(\omega,p)^{CG}g({}^V\theta,{}^V\xi) + \alpha g^{-1}(\xi,p)^{CG}g({}^V\omega,{}^V\theta)\\
&- \alpha g^{-1}(\theta,p)^{CG}g({}^V\xi,{}^V\omega) - \alpha g^{-1}(\theta,p)g^{-1}(\xi,p)g^{-1}(\omega,p)\\
&+ \alpha^2 g^{-1}(\xi,p)^{CG}g({}^V\omega,{}^V\theta)\\
&= {}^{CG}g\big(-{}^{CG}g({}^V\omega,\gamma\delta){}^V\theta - \alpha\,{}^{CG}g({}^V\theta,\gamma\delta){}^V\omega\\
&+ \alpha\,{}^{CG}g({}^V\omega,{}^V\theta)\gamma\delta\\
&+ \alpha^{2\,CG}g({}^V\omega,{}^V\theta)\gamma\delta - {}^{CG}g({}^V\theta,\gamma\delta)^{CG}g({}^V\omega,\gamma\delta)\gamma\delta,{}^V\xi\big).\end{aligned}$$

Thus

$$\begin{aligned}{}^{CG}\nabla_{{}^V\omega}{}^V\theta &= -\frac{1}{\alpha}\big({}^{CG}g({}^V\omega,\gamma\delta){}^V\theta + {}^{CG}g({}^V\theta,\gamma\delta){}^V\omega\big) + \frac{\alpha+1}{\alpha}{}^{CG}g({}^V\omega,{}^V\theta)\gamma\delta\\
&- \frac{1}{\alpha}{}^{CG}g({}^V\theta,\gamma\delta)^{CG}g({}^V\omega,\gamma\delta)\gamma\delta.\end{aligned}$$

□

We write ${}^{CG}\nabla_{e_\alpha}e_\beta = {}^{CG}\Gamma^\delta_{\alpha\beta}e_\delta$ with respect to the adapted frame $\{e_\alpha\}$ of T^*M^n, where ${}^{CG}\Gamma^\delta_{\alpha\beta}$ denote the Christoffel symbols constructed by ${}^{CG}g$. From Theorem 2.1, we immediately have the following.

2.2. Corollary. *Let M^n be a Riemannian manifold with metric g and ${}^{CG}\nabla$ be the Levi-Civita connection of the cotangent bundle T^*M^n equipped with the Cheeger-Gromoll metric ${}^{CG}g$. The particular values of ${}^{CG}\Gamma^\delta_{\alpha\beta}$ for different indices, on taking account of (2.6) are then found to be*

$$\begin{aligned}&{}^{CG}\Gamma^k_{ij} = \Gamma^k_{ij}, \qquad {}^{CG}\Gamma^k_{\bar i\bar j} = {}^{CG}\Gamma^{\bar k}_{\bar i j} = 0,\\
&{}^{CG}\Gamma^{\bar k}_{i\bar j} = -\Gamma^j_{ik}, \qquad {}^{CG}\Gamma^{\bar k}_{ij} = \frac{1}{2}p_a R_{ijk}{}^a,\\
&{}^{CG}\Gamma^k_{\bar i j} = \frac{1}{2\alpha}p_a R^{k\ ia}_{.j.}, \qquad {}^{CG}\Gamma^k_{i\bar j} = \frac{1}{2\alpha}p_a R^{k\ ja}_{.i.}, \qquad (2.7)\\
&{}^{CG}\Gamma^{\bar k}_{\bar i\bar j} = -\frac{1}{\alpha}(p^i\delta^j_k + p^j\delta^i_k) + \frac{\alpha+1}{\alpha^2}g^{ij}p_k + \frac{1}{\alpha^2}p^ip^jp_k.\end{aligned}$$

with respect to the adapted frame, where $p^i = g^{it}p_t$, $\quad R^{k\ ia}_{.j.} = g^{kt}g^{is}R_{tjs}{}^a$.

3. Curvature properties of ${}^{CG}g$

We now consider local 1-forms $\tilde{\omega}^{\alpha}$ in $\pi^{-1}(U)$ defined by

$$\tilde{\omega}^{\alpha} = \bar{A}^{\alpha}{}_{B} dx^{B},$$

where

$$(3.1) \qquad A^{-1} = (\bar{A}^{\alpha}{}_{B}) = \begin{pmatrix} \bar{A}^{i}{}_{j} & \bar{A}^{i}{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_{j} & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^{i}_{j} & 0 \\ -p_a \Gamma^{a}_{ij} & \delta^{j}_{i} \end{pmatrix}$$

The matrix (3.1) is the inverse of the matrix

$$(3.2) \qquad A = (A_{\beta}{}^{A}) = \begin{pmatrix} A_{j}{}^{i} & A_{\bar{j}}{}^{i} \\ A_{j}{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta^{i}_{j} & 0 \\ p_a \Gamma^{a}_{ij} & \delta^{j}_{i} \end{pmatrix}$$

of the transformation $\tilde{e}_{\beta} = A_{\beta}{}^{A} \partial_{A}$ (see (2.1) and (2.2)). We easily see that the set $\{\tilde{\omega}^{\alpha}\}$ is the coframe dual to the adapted frame $\{\tilde{e}_{(\beta)}\}$, i.e. $\tilde{\omega}^{\alpha}(\tilde{e}_{(\beta)}) = \bar{A}^{\alpha}{}_{B} A_{\beta}{}^{B} = \delta^{\alpha}_{\beta}$.

Since the adapted frame $\{\tilde{e}_{(\beta)}\}$ is non-holonomic, we put

$$[\tilde{e}_{\gamma}, \tilde{e}_{\beta}] = \Omega_{\gamma\beta}{}^{\alpha} \tilde{e}_{\alpha}$$

from which we have

$$\Omega_{\gamma\beta}{}^{\alpha} = (\tilde{e}_{\gamma} A_{\beta}{}^{A} - \tilde{e}_{\beta} A_{\gamma}{}^{A}) \bar{A}^{\alpha}{}_{A}.$$

According to (2.1), (2.2), (3.1) and (3.2), the components of non-holonomic object $\Omega_{\gamma\beta}{}^{\alpha}$ are given by

$$(3.3) \qquad \begin{cases} \Omega_{l\bar{j}}{}^{\bar{i}} = -\Omega_{\bar{j}l}{}^{\bar{i}} = \Gamma^{j}_{li}, \\ \Omega_{lj}{}^{\bar{i}} = p_a R_{lji}{}^{a}, \end{cases}$$

all the others being zero, where $R_{lji}{}^{a}$ being local components of the curvature tensor R of ∇_{g}.

Let ${}^{CG}R$ be a curvature tensor of ${}^{CG}\nabla$. Then we obtain

$${}^{CG}R(\tilde{e}_{(\alpha)}, \tilde{e}_{(\beta)})\tilde{e}_{(\gamma)} = {}^{CG}\nabla_{\alpha} {}^{CG}\nabla_{\beta} \tilde{e}_{(\gamma)} - {}^{CG}\nabla_{\beta} {}^{CG}\nabla_{\alpha} \tilde{e}_{(\gamma)} - \Omega_{\alpha\beta}{}^{\varepsilon} {}^{CG}\nabla_{\varepsilon} \tilde{e}_{(\gamma)},$$

where ${}^{CG}\nabla_{\alpha} = {}^{CG}\nabla_{\tilde{e}_{(\alpha)}}$. The curvature tensor ${}^{CG}R$ has components

$${}^{CG}R_{\alpha\beta\gamma}{}^{\sigma} = \tilde{e}_{\alpha} {}^{CG}\Gamma^{\sigma}_{\beta\gamma} - \tilde{e}_{\beta} {}^{CG}\Gamma^{\sigma}_{\alpha\gamma} + {}^{CG}\Gamma^{\sigma}_{\alpha\varepsilon} {}^{CG}\Gamma^{\varepsilon}_{\beta\gamma} - {}^{CG}\Gamma^{\sigma}_{\beta\varepsilon} {}^{CG}\Gamma^{\varepsilon}_{\alpha\gamma} - \Omega_{\alpha\beta}{}^{\varepsilon} {}^{CG}\Gamma^{\sigma}_{\varepsilon\gamma}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

Taking account of (2.7) and (3.3), we find

$$
\begin{aligned}
{}^{CG}R_{kij}{}^{l} &= R_{kij}{}^{l} - \frac{1}{2\alpha}p_m p_a R_{kit}{}^{a} R^{l\ tm}_{.j.} + \frac{1}{4\alpha}p_m p_a (R^{l\ tm}_{.k.} R_{ijt}{}^{a} - R^{l\ tm}_{.i.} R_{kjt}{}^{a}),\\
{}^{CG}R_{k\bar{i}j}{}^{l} &= \frac{1}{2\alpha}p_m \nabla_k R^{l\ im}_{.j.},\\
{}^{CG}R_{ki\bar{j}}{}^{l} &= \frac{1}{2\alpha}p_m (\nabla_k R^{l\ jm}_{.i.} - \nabla_i R^{l\ jm}_{.k.}),\\
{}^{CG}R_{kij}{}^{\bar{l}} &= \frac{1}{2}p_m (\nabla_k R_{ijl}{}^{m} - \nabla_i R_{kjl}{}^{m}),\\
{}^{CG}R_{ki\bar{j}}{}^{\bar{l}} &= R_{ikl}{}^{j} + \frac{1}{4\alpha}p_m p_a (R_{ktl}{}^{a} R^{t\ ja}_{.i.} - R_{itl}{}^{m} R^{t\ ja}_{.k.})\\
&\quad + \frac{1}{\alpha}p_a p^j R_{kil}{}^{a} - \frac{\alpha+1}{\alpha^2}p_l p_a R_{kt.}{}^{ja},\\
{}^{CG}R_{\bar{k}ij}{}^{\bar{l}} &= \frac{1}{2}R_{ijl}{}^{k} - \frac{1}{4\alpha}p_m p_a R_{itl}{}^{m} R^{t\ ka}_{.j.}\\
&\quad - \frac{1}{2\alpha}p_a p^k R_{ijl}{}^{a} + \frac{\alpha+1}{2\alpha^2}p_l p_a R_{ij.}{}^{ka},\\
{}^{CG}R_{\bar{k}\bar{i}j}{}^{l} &= \frac{1}{\alpha^2}p_a (p^i R^{l\ ka}_{.j.} - p^k R^{l\ ia}_{.j.}) + \frac{1}{2\alpha}(R^{l\ ik}_{.j.} - R^{l\ ki}_{.j.})\\
&\quad + \frac{1}{4\alpha^2}p_m p_a (R^{l\ km}_{.t.} R^{t\ ia}_{.j.} - R^{l\ im}_{.t.} R^{t\ ka}_{.j.}),\\
{}^{CG}R_{\bar{k}i\bar{j}}{}^{l} &= \frac{1}{2\alpha}R^{l\ jk}_{.i.} + \frac{1}{2\alpha^2}p_a (p^j R^{l\ ka}_{.i.} - p^k R^{l\ ja}_{.i.}) + \frac{1}{4\alpha^2}p_m p_a R^{l\ km}_{.t.} R^{t\ ja}_{.i.},\\
{}^{CG}R_{\bar{k}\bar{i}\bar{j}}{}^{\bar{l}} &= \frac{\alpha^2+\alpha+1}{\alpha^3}(g^{ij}\delta^k_l - g^{jk}\delta^i_l) + \frac{\alpha+2}{\alpha^3}(g^{kj}p^i p_l - g^{ij}p^k p_l)\\
&\quad + \frac{\alpha-1}{\alpha^3}(\delta^i_l p^k p^j - \delta^k_l p^i p^j),
\end{aligned}
$$

$$
{}^{CG}R_{k\bar{i}\bar{j}}{}^{\bar{l}} = {}^{CG}R_{\bar{k}\bar{i}j}{}^{\bar{l}} = {}^{CG}R_{\bar{k}\bar{i}\bar{j}}{}^{l} = {}^{CG}R_{\bar{k}i\bar{j}}{}^{\bar{l}} = 0. \tag{3.4}
$$

It is known (see [6, p.200]) that the sectional curvature on $(T^*M^n, {}^{CG}g)$ for $P(U,V)$ is given by

$$
{}^{CG}K(P) = -\frac{{}^{CG}R_{kmji}U^k V^m U^i V^j}{({}^{CG}g_{ki}{}^{CG}g_{mj} - {}^{CG}g_{kj}{}^{CG}g_{mi})U^k V^m U^i V^j}, \tag{3.5}
$$

where $P(U,V)$ denotes the plane spanned by (U,V). Let $\{X_i\}$ and $\{\omega^i\}$, $i = 1,...,n$ be a local ortonormal frame and coframe on M^n, respectively. Then from (1.8)-(1.10) we see that $\{{}^HX_1,...,{}^HX_n,{}^V\omega^1,...,{}^V\omega^n\}$ is a local ortonormal frame on T^*M^n. Let ${}^{CG}K({}^HX,{}^HY)$, ${}^{CG}K({}^HX,{}^V\theta)$ and ${}^{CG}K({}^V\omega,{}^V\theta)$ denote the sectional curvature of the plane spanned by $({}^HX,{}^HY)$, $({}^HX,{}^V\theta)$ and $({}^V\omega,{}^V\theta)$ on $(T^*M^n, {}^{CG}g)$, respectively. Then, using (2.3), (2.4), (2.5) and (3.4), we have from (3.5)

$$
\begin{aligned}
i)\quad {}^{CG}K({}^HX,{}^HY) &= -\frac{{}^{CG}R_{kijs}{}^H\tilde{X}^k{}^H\tilde{Y}^i{}^H\tilde{X}^j{}^H\tilde{Y}^s}{({}^{CG}g_{kj}{}^{CG}g_{is} - {}^{CG}g_{ks}{}^{CG}g_{ij}){}^H\tilde{X}^k{}^H\tilde{Y}^i{}^H\tilde{X}^j{}^H\tilde{Y}^s}\\
&= -\frac{{}^{CG}R_{kij}{}^{l}\,{}^{CG}g_{sl}{}^H\tilde{X}^k{}^H\tilde{Y}^i{}^H\tilde{X}^j{}^H\tilde{Y}^s + {}^{CG}R_{kij}{}^{\bar{l}}\,{}^{CG}g_{s\bar{l}}{}^H\tilde{X}^k{}^H\tilde{Y}^i{}^H\tilde{X}^j{}^H\tilde{Y}^s}{({}^{CG}g_{kj}{}^{CG}g_{is} - {}^{CG}g_{ks}{}^{CG}g_{ij}){}^H\tilde{X}^k{}^H\tilde{Y}^i{}^H\tilde{X}^j{}^H\tilde{Y}^s}\\
&= \frac{\big(-R_{kij}{}^{l} + \frac{1}{2\alpha}p_m p_a R_{kit}{}^{a} R^{l\ tm}_{.j.} - \frac{1}{4\alpha}p_m p_a (R^{l\ tm}_{.k.} R_{ijt}{}^{a} - R^{l\ tm}_{.i.} R_{kjt}{}^{a})\big)g_{sl}X^k Y^i X^j Y^s}{({}^{CG}g_{kj}{}^{CG}g_{is} - {}^{CG}g_{ks}{}^{CG}g_{ij})X^k Y^i X^j Y^s}\\
&= K(X,Y) - \frac{\frac{1}{2\alpha}g^{tf}\big(pR(X,Y)\big)_t\big(pR(X,Y)\big)_f}{g(X,X)g(Y,Y) - g(X,Y)g(X,Y)}\\
&\quad -\frac{\frac{1}{4\alpha}g^{tf}\big(pR(X,Y)\big)_t\big(pR(X,Y)\big)_f}{g(X,X)g(Y,Y) - g(X,Y)g(X,Y)} + \frac{\frac{1}{4\alpha}g^{tf}\big(pR(Y,Y)\big)_t\big(pR(X,X)\big)_f}{g(X,X)g(Y,Y) - g(X,Y)g(X,Y)}
\end{aligned}
$$

$$= K(X,Y) - \frac{3}{4\alpha}\big|(pR(X,Y)\big|^2.$$

$$ii)\quad {}^{CG}K({}^HX,{}^V\theta) = -\frac{{}^{CG}R_{k\bar{i}j\bar{s}}\,{}^H\tilde{X}^{k\,V}\tilde{\omega}^{\bar{i}\,H}\tilde{X}^{j\,V}\tilde{\omega}^{\bar{s}}}{({}^{CG}g_{kj}{}^{CG}g_{\bar{i}\bar{s}} - {}^{CG}g_{\bar{k}\bar{s}}{}^{CG}g_{\bar{i}j})^H\tilde{X}^{k\,H}\tilde{\omega}^{\bar{i}\,H}\tilde{X}^{j\,V}\tilde{\omega}^{\bar{s}}}$$

$$= -\frac{{}^{CG}R_{k\bar{i}j}{}^{l\,CG}g_{\bar{s}l}X^k\omega_iX^j\omega_s + {}^{CG}R_{k\bar{i}j}{}^{\bar{l}\,CG}g_{\bar{s}\bar{l}}X^k\omega_iX^j\omega_s}{\big(g_{kj}\big(\frac{1}{\alpha}(g^{is} + g^{ia}g^{sb}p_ap_b)\big)\big)X^k\omega_iX^j\omega_s}$$

$$= \Bigg(\frac{\frac{1}{2}R_{kjl}{}^i - \frac{1}{4\alpha}p_mp_aR_{ktl}{}^mR^{t\ ia}_{.j.} - \frac{1}{2\alpha}p_ap^iR_{kjl}{}^a}{\big(\frac{1}{\alpha}(g^{is}g_{kj} + g^{ia}g^{sb}g_{kj}p_ap_b)\big)X^k\omega_iX^j\omega_s}$$

$$+\frac{\frac{\alpha+1}{2\alpha^2}p_lp_aR_{kj.}{}^{ia}}{\big(\frac{1}{\alpha}(g^{is}g_{kj} + g^{ia}g^{sb}g_{kj}p_ap_b)\big)X^k\omega_iX^j\omega_s}\Bigg)\Big(\frac{1}{\alpha}(g^{sl} + g^{su}g^{lv}p_up_v)\Big)X^k\omega_iX^j\omega_s$$

$$= \frac{\frac{\alpha+1}{2\alpha^3}p_lp_aR_{kj.}{}^{ia}g^{sl}X^k\omega_iX^j\omega_s + \frac{\alpha+1}{2\alpha^3}p_lp_aR_{kj.}{}^{ia}g^{su}g^{lv}p_up_vX^k\omega_iX^j\omega_s}{\big(\frac{1}{\alpha}(g^{is}g_{kj} + g^{ia}g^{sb}g_{kj}p_ap_b)\big)X^k\omega_iX^j\omega_s}$$

$$= \frac{\frac{1}{4\alpha^2}g^{tf}\big(pR(\ ,X)\tilde{\omega}\big)_t\big(pR(\ ,X)\tilde{\omega}\big)_f}{\frac{1}{\alpha}\big(g(X,X)g^{-1}(\omega,\omega) + g(X,X)(g^{-1}(\omega,p))^2\big)}$$

$$= \frac{1}{4\alpha}\frac{\big|(pR(\ ,X)\tilde{\omega})\big|^2}{\big(1 + (g^{-1}(\omega,p))^2\big)}$$

$$iii)\,{}^{CG}K({}^V\omega,{}^V\theta) = -\frac{{}^{CG}R_{\varepsilon\gamma\alpha\beta}\,{}^V\tilde{\omega}^{\varepsilon\,V}\tilde{\theta}^{\gamma\,V}\tilde{\omega}^{\alpha\,V}\tilde{\theta}^{\beta}}{({}^{CG}g_{\varepsilon\alpha}{}^{CG}g_{\gamma\beta} - {}^{CG}g_{\varepsilon\beta}{}^{CG}g_{\gamma\alpha})^V\tilde{\omega}^{\varepsilon\,V}\tilde{\theta}^{\gamma\,V}\tilde{\omega}^{\alpha\,V}\tilde{\theta}^{\beta}}$$

$$= -\frac{{}^{CG}R_{\bar{k}\bar{i}\bar{j}\bar{s}}\,{}^V\tilde{\omega}^{\bar{k}\,V}\tilde{\theta}^{\bar{i}\,V}\tilde{\omega}^{\bar{j}\,V}\tilde{\theta}^{\bar{s}}}{({}^{CG}g_{\bar{k}\bar{j}}{}^{CG}g_{\bar{i}\bar{s}} - {}^{CG}g_{\bar{k}\bar{s}}{}^{CG}g_{\bar{i}\bar{j}})^V\tilde{\omega}^{\bar{k}\,V}\tilde{\theta}^{\bar{i}\,V}\tilde{\omega}^{\bar{j}\,V}\tilde{\theta}^{\bar{s}}}$$

$$= -\frac{{}^{CG}R_{\bar{k}\bar{i}\bar{j}}{}^{l\,CG}g_{\bar{s}l}\omega_k\theta_i\omega_j\theta_s + {}^{CG}R_{\bar{k}\bar{i}\bar{j}}{}^{\bar{l}\,CG}g_{\bar{s}\bar{l}}\omega_k\theta_i\omega_j\theta_s}{({}^{CG}g_{\bar{k}\bar{j}}{}^{CG}g_{\bar{i}\bar{s}} - {}^{CG}g_{\bar{k}\bar{s}}{}^{CG}g_{\bar{i}\bar{j}})\omega_k\theta_i\omega_j\theta_s}$$

$$= \Bigg[\frac{\frac{\alpha^2+\alpha+1}{\alpha^3}(g^{jk}\delta^i_l - g^{ij}\delta^k_l) - \frac{\alpha+2}{\alpha^3}(g^{kj}p^ip_l - g^{ij}p^kp_l)}{P}$$

$$-\frac{\frac{\alpha-1}{\alpha^3}(\delta^i_lp^kp^j - \delta^k_lp^ip^j)}{P}\Bigg]\Big(\frac{1}{\alpha}(g^{sl} + g^{sa}g^{lb}p_ap_b)\Big)\omega_k\theta_i\omega_j\theta_s$$

$$= \frac{\frac{\alpha^2+\alpha+1}{\alpha^4} + \frac{1-\alpha}{\alpha^4}(g^{-1}(\theta,p))^2 + \frac{1-\alpha}{\alpha^4}(g^{-1}(\omega,p))^2}{\frac{1}{\alpha^2}\big(1 + (g^{-1}(\theta,p))^2 + (g^{-1}(\omega,p))^2\big)}$$

$$= \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{\big(1 + (g^{-1}(\theta,p))^2 + (g^{-1}(\omega,p))^2\big)},$$

where

$$
\begin{aligned}
P &= ({}^{CG}g_{\bar{k}\bar{j}}\,{}^{CG}g_{\bar{i}\bar{s}} - {}^{CG}g_{\bar{k}\bar{s}}\,{}^{CG}g_{\bar{i}\bar{j}})\omega_k\theta_i\omega_j\theta_s \\
&= \Big(\frac{1}{\alpha}(g^{kj} + g^{ka}g^{jb}p_ap_b)\frac{1}{\alpha}(g^{is} + g^{it}g^{sf}p_tp_f) \\
&\quad - \frac{1}{\alpha}(g^{ks} + g^{kc}g^{sd}p_cp_d)\frac{1}{\alpha}(g^{ij} + g^{iu}g^{jv}p_up_v)\Big)\omega_k\theta_i\omega_j\theta_s \\
&= \frac{1}{\alpha^2}\Big(g^{-1}(\omega,\omega)g^{-1}(\theta,\theta) + g^{-1}(\omega,\omega)\big(g^{-1}(\theta,p)\big)^2 \\
&\quad + g^{-1}(\theta,\theta)\big(g^{-1}(\omega,p)\big)^2 + \big(g^{-1}(\omega,p)\big)^2\big(g^{-1}(\theta,p)\big)^2 - \big(g^{-1}(\omega,\theta)\big)^2 \\
&\quad - g^{-1}(\omega,\theta)g^{-1}(\omega,p)g^{-1}(\theta,p) - g^{-1}(\omega,\theta)g^{-1}(\omega,p)g^{-1}(\theta,p) \\
&\quad + \big(g^{-1}(\omega,p)\big)^2\big(g^{-1}(\theta,p)\big)^2\Big) \\
&= \frac{1}{\alpha^2}\big(1 + \big(g^{-1}(\omega,p)\big)^2 + \big(g^{-1}(\theta,p)\big)^2\big).
\end{aligned}
$$

Thus we have the following.

3.1. Theorem. *Let (M^n,g) be a Riemannian manifold and T^*M^n be its cotangent bundle equipped with the Cheeger-Gromoll metric ${}^{CG}g$. Then the sectional curvature ${}^{CG}K$ of $(T^*M^n, {}^{CG}g)$ satisfy the following:*

$$
\begin{aligned}
&i)\,{}^{CG}K({}^HX,{}^HY) = K(X,Y) - \frac{3}{4\alpha}\big|(pR(X,Y)\big|^2, \\
&ii)\,{}^{CG}K({}^HX,{}^V\omega) = \frac{1}{4\alpha}\frac{\big|(pR(\ ,X)\tilde{\omega})\big|^2}{\big(1+(g^{-1}(\omega,p))^2\big)}, \\
&iii)\,{}^{CG}K({}^V\omega,{}^V\theta) = \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{\big(1+(g^{-1}(\theta,p))^2+(g^{-1}(\omega,p))^2\big)},
\end{aligned}
$$

where K is a sectional curvature of (M^n,g) and $\tilde{\omega} = g^{-1}\circ\omega = (g^{ij}\omega_j) \in \Im^1_0(M^n)$, $R(\ ,X)\tilde{\omega} \in \Im^1_1(M^n)$.

3.2. Theorem. *Let (M^n,g) be a Riemannian manifold of constant sectional curvature K. Let T^*M^n be its cotangent bundle equipped with the Cheeger-Gromoll metric ${}^{CG}g$. Then the sectional curvature ${}^{CG}K$ of $(T^*M^n, {}^{CG}g)$ satisfy the following:*

$$
\begin{aligned}
&i)\,{}^{CG}K({}^HX,{}^HY) = K - \frac{3}{4\alpha}K^2\big((g^{-1}(p,\tilde{X}))^2 + (g^{-1}(p,\tilde{Y}))^2\big), \\
&ii)\,{}^{CG}K({}^HX,{}^V\omega) = \begin{cases} \dfrac{K^2\Big(r^2-2g^{-1}(\tilde{X},p)g^{-1}(\omega,p)+\big(g^{-1}(\tilde{X},p)\big)^2\Big)}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)}, & g(X,\tilde{\omega})=1, \\ \dfrac{K^2\Big(\big(g^{-1}(\tilde{X},p)\big)^2\Big)}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)}, & g(X,\tilde{\omega})=0, \end{cases} \\
&iii)\,{}^{CG}K({}^V\omega,{}^V\theta) = \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{\big(1+(g^{-1}(\theta,p))^2+(g^{-1}(\omega,p))^2\big)},
\end{aligned}
$$

where $\tilde{\omega} = g^{-1}\circ\omega = (g^{ij}\omega_j) \in \Im^1_0(M^n)$ and $X^i = g^{ij}X_j = g^{-1}\circ\tilde{X} \in \Im^1_0(M^n)$.

Proof. Let ${R_{kmj}}^{s} = K(\delta^s_k g_{mj} - \delta^s_m g_{kj})$.Using Theorem 3.1, we have

$$
\begin{aligned}
i)^{CG}K(^HX, ^HY) &= K(X,Y) - \frac{3}{4\alpha}\left|(pR(X,Y)\right|^2 \\
&= K - \frac{3}{4\alpha}g^{ij}\big(pR(X,Y)\big)_i\big(pR(X,Y)\big)_j, \\
&= K - \frac{3}{4\alpha}K^2\big((g^{-1}(p,\tilde{X}))^2 + (g^{-1}(p,\tilde{Y}))^2\big)
\end{aligned}
$$

$$
ii)^{CG}K(^HX, ^V\omega) = \frac{1}{4\alpha}\frac{\left|(pR(\ ,X)\tilde{\omega})\right|^2}{\big(1+(g^{-1}(\omega,p))^2\big)} = \frac{g^{tf}\big(pR(\ ,X)\tilde{\omega}\big)_t\big(pR(\ ,X)\tilde{\omega}\big)_f}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)},
$$

$$
= \frac{g^{tf}p_a {R_{tij}}^a X^i\tilde{\omega}^j p_b {R_{fkm}}^b X^k\tilde{\omega}^m}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)}
$$

$$
= \frac{g^{tf}p_a\big(K(\delta^a_t g_{ij} - \delta^a_i g_{tj})\big)X^i\tilde{\omega}^j p_b\big(K(\delta^b_f g_{km} - \delta^b_k g_{fm})\big)X^k\tilde{\omega}^m}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)}
$$

$$
= \frac{K^2\big(g^{tf}p_t g_{ij}p_f g_{km} - g^{tf}p_t g_{ij}p_k g_{fm} - g^{tf}p_i g_{tj}p_f g_{km} + g^{tf}p_i g_{tj}p_k g_{fm}\big)X^i\tilde{\omega}^j X^k\tilde{\omega}^m}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)}
$$

$$
= \frac{K^2\big(r^2(g(X,\tilde{\omega}))^2 - 2g(X,\tilde{\omega})g^{-1}(\tilde{X},p)g^{-1}(\omega,p) + g(\tilde{\omega},\tilde{\omega})\big(g^{-1}(\tilde{X},p)\big)^2\big)}{4\alpha\big(1+(g^{-1}(\omega,p))^2\big)}
$$

$$
= \begin{cases} \frac{K^2\left(r^2 - 2g^{-1}(\tilde{X},p)g^{-1}(\omega,p) + \left(g^{-1}(\tilde{X},p)\right)^2\right)}{4\alpha\left(1+(g^{-1}(\omega,p))^2\right)}, & g(X,\tilde{\omega}) = 1, \\ \frac{K^2\left(\left(g^{-1}(\tilde{X},p)\right)^2\right)}{4\alpha\left(1+(g^{-1}(\omega,p))^2\right)}, & g(X,\tilde{\omega}) = 0, \end{cases}
$$

where

$$
\begin{aligned}
&K^2(r^2 g_{ij}g_{km}X^i\tilde{\omega}^j X^k\tilde{\omega}^m - p_t g_{ij}p_k\delta^t_m X^i\tilde{\omega}^j X^k\tilde{\omega}^m - p_i\delta^f_j p_f g_{km}X^i\tilde{\omega}^j X^k\tilde{\omega}^m \\
&\quad + p_i\delta^f_j p_k g_{fm}X^i\tilde{\omega}^j X^k\tilde{\omega}^m) \\
&= K^2(r^2(g(X,\tilde{\omega}))^2 - p_m g(X,\tilde{\omega})p_k X^k\tilde{\omega}^m - p_i g(X,\tilde{\omega})p_j X^i\tilde{\omega}^j \\
&\quad + p_i g(\tilde{\omega},\tilde{\omega})p_k X^k X^i) \\
&= K^2(r^2(g(X,\tilde{\omega}))^2 - g(X,\tilde{\omega})g^{-1}(\tilde{X},p)g^{-1}(\omega,p) - g(X,\tilde{\omega})g^{-1}(\tilde{X},p)g^{-1}(\omega,p) \\
&\quad + g(\tilde{\omega},\tilde{\omega})g^{-1}(\tilde{X},p)g^{-1}(\tilde{X},p)) \\
&= K^2\big(r^2(g(X,\tilde{\omega}))^2 - 2g(X,\tilde{\omega})g^{-1}(\tilde{X},p)g^{-1}(\omega,p) + g(\tilde{\omega},\tilde{\omega})\big(g^{-1}(\tilde{X},p)\big)^2\big),
\end{aligned}
$$

$$
\begin{aligned}
g(X_a,\tilde{\omega}^b) &= g_{ij}X^i_a(\tilde{\omega}^b)^j = g_{ij}X^i_a g^{jk}\omega^b_k = \delta^k_i X^i_a\omega^b_k \\
&= X^k_a\omega^b_k = \omega^b(X_a) = \delta^b_a = \begin{cases} 1, & a = b, \\ 0, & a \neq b, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
g(\tilde{\omega},\tilde{\omega}) &= g_{ij}\tilde{\omega}^i\tilde{\omega}^j = g_{ij}g^{is}\omega_s g^{jk}\omega_k = \delta^s_j\omega_s g^{jk}\omega_k \\
&= g^{sk}\omega_s\omega_k = g^{-1}(\omega,\omega) = 1.
\end{aligned}
$$

iii) The statement is obtained by iii) of Theorem 3.1.
The theorem is proved. □

Let now the Riemannian manifold (M^n, g) be a flat manifold. Then, using Theorem 3.2, we have the following.

3.3. Theorem. *If the Riemannian manifold (M^n, g) is flat, then the Cheeger-Gromoll metric ${}^{CG}g$ of the cotangent bundle T^*M^n has non-negative sectional curvature, which are nonwhere constant.*

Let (x, p) be a point on T^*M^n with $p \neq 0$ and $\{e_1, , ..., e_n\}$ be an orthonormal basis for the tangent space T_xM^n of M^n at x. Also, let $\{\omega^1, ..., \omega^n\}$ be a dual orthonormal basis for the cotangent spaces $T^*_xM^n$ of M^n at x such that $\omega^1 = \frac{p}{|p|}$, where $|p|$ is the norm of p with respect to the metric g on M^n. Then for $i \in \{1, ..., n\}$ and $k \in \{2, ..., n\}$ define the horizontal and vertical lifts by $f_i = {}^He_i$, $\quad f_{n+1} = {}^V\omega^1$ and $f_{n+k} = \sqrt{\alpha}({}^V\omega^k)$, $\alpha = 1 + r^2$, $r^2 = g^{-1}(p, p)$. Then $\{f_1, ..., f_{2n}\}$ is an orthonormal basis for the cotangent space $T^*_{(x,p)}M^n$ with respect to the Cheeger-Gromoll metric ${}^{CG}g$.

Using Theorem 3.1, we have

$$i)^{CG}K(f_i, f_j) = {}^{CG}K({}^He_i, {}^He_j) = K(e_i, e_j) - \frac{3}{4\alpha}\left|pR(e_i, e_j)\right|^2,$$

$$ii)^{CG}K(f_i, f_{n+1}) = {}^{CG}K({}^He_i, {}^V\omega^1) = \frac{1}{4\alpha}\frac{\left|(pR(\ , e_i)\tilde{\omega}^1)\right|^2}{\left(1 + (g^{-1}(\omega^1, p))^2\right)} = 0$$

by virtue of
$pR(\ , e_i)\tilde{\omega}^1 = \left(p_m R_{\ ,ks}{}^m e_i^k (\frac{p}{|p|})^s\right) = \left(R_{\ ,ksl} e_i^k (\frac{p}{|p|})^s p^l\right) = \frac{1}{|p|}(R_{\ ,ksl} e_i^k p^s p^l) = 0.$

$$iii)^{CG}K(f_i, f_{n+k}) = {}^{CG}K({}^He_i, \sqrt{\alpha}{}^V\omega^k) = \frac{1}{4\alpha}\frac{\left|(pR(\ , e_i)\sqrt{\alpha}\tilde{\omega}^k)\right|^2}{\left(1 + (g^{-1}(\sqrt{\alpha}\omega^k, p))^2\right)}$$
$$= \frac{1}{4}\left|(pR(\ , e_i)\tilde{\omega}^k)\right|^2,$$
$$iv)^{CG}K(f_{n+1}, f_{n+k}) = {}^{CG}K({}^V\omega^1, \sqrt{\alpha}{}^V\omega^k)$$
$$= \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{\left(1 + (g^{-1}(\omega^1, p))^2 + (g^{-1}(\sqrt{\alpha}\omega^k, p))^2\right)},$$
$$= \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{\left(1 + r^2\right)} = \frac{3}{\alpha^2},$$
$$v)^{CG}K(f_{n+k}, f_{n+l}) = {}^{CG}K(\sqrt{\alpha}{}^V\omega^k, \sqrt{\alpha}{}^V\omega^l)$$
$$= \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha}\frac{1}{\left(1 + (g^{-1}(\sqrt{\alpha}\omega^k, p))^2 + (g^{-1}(\sqrt{\alpha}\omega^l, p))^2\right)},$$
$$= \frac{1-\alpha}{\alpha^2} + \frac{\alpha+2}{\alpha} = \frac{\alpha^2+\alpha+1}{\alpha^2}.$$

Thus we have the following.

3.4. Theorem. *Let (x, p) be a point on T^*M^n and $\{f_1, ..., f_{2n}\}$ be an orthonormal basis for the cotangent spaces $T^*_xM^n$ as above. Then the sectional curvature ${}^{CG}K$ satisfy the following equation*

$$i)^{CG}K(f_i, f_j) = K(e_i, e_j) - \frac{3}{4\alpha}\left|pR(e_i, e_j)\right|^2,$$
$$ii)^{CG}K(f_i, f_{n+1}) = 0,$$
$$iii)^{CG}K(f_i, f_{n+k}) = \frac{1}{4}\left|(pR(\ , e_i)\tilde{\omega}^k)\right|^2,$$
$$iv)^{CG}K(f_{n+1}, f_{n+k}) = \frac{3}{\alpha^2},$$
$$v)^{CG}K(f_{n+k}, f_{n+l}) = \frac{\alpha^2+\alpha+1}{\alpha^2}.$$

where K is a sectional curvature of (M^n, g) and $\tilde{\omega}^k = g^{-1} \circ \omega^k$, for $i \in \{1, ..., n\}$ and $k, l \in \{2, ..., n\}$.

Let now $\{f_1, ..., f_{2n}\}$ be an orthonormal basis for the cotangent space $T^*_x M^n$ as above, then the scalar curvature ${}^{CG}r = \sum_{i \neq j} {}^{CG}K(f_i, f_j)$ is given by

$$\begin{aligned}
{}^{CG}r &= \sum_{i \neq j} {}^{CG}K(f_i, f_j) \\
&= 2\sum_{\substack{i,j=1\\ i<j}}^{n} {}^{CG}K(f_i, f_j) + 2\sum_{i,j=1}^{n} {}^{CG}K(f_i, f_{n+j}) + 2\sum_{\substack{i,j=1\\ i<j}}^{n} {}^{CG}K(f_{n+i}, f_{n+j}) \\
&= \sum_{i \neq j}^{n} K(e_i, e_j) - \frac{3}{4\alpha}\sum_{i,j=1}^{n} \left|pR(e_i, e_j)\right|^2 + \frac{1}{2}\sum_{i,j=1}^{n} \left|(pR(\ , e_i)\tilde{\omega}^j)\right|^2 \\
&\quad + 2\sum_{i=2}^{n} \frac{3}{\alpha^2} + \sum_{\substack{i,j=2\\ i\neq j}}^{n} \frac{\alpha^2 + \alpha + 1}{\alpha^2} \\
&= r - \frac{3}{4\alpha}\sum_{i,j=1}^{n} \left|pR(e_i, e_j)\right|^2 + \frac{1}{2}\sum_{i,j=1}^{n} \left|(pR(\ , e_i)\tilde{\omega}^j)\right|^2 \\
&\quad + 2(n-1)\frac{3}{\alpha^2} + (n-1)(n-2)\frac{\alpha^2 + \alpha + 1}{\alpha^2} \\
&= r - \frac{3}{4\alpha}\sum_{i,j=1}^{n} \left|pR(e_i, e_j)\right|^2 + \frac{1}{2}\sum_{i,j=1}^{n} \left|(pR(\ , e_i)\tilde{\omega}^j)\right|^2 \\
&\quad + \frac{(n-1)}{\alpha^2}\big(6 + (n-2)(\alpha^2 + \alpha + 1)\big)
\end{aligned}$$

from which we have the following.

3.5. Theorem. *If the Riemannian manifold (M^n, g) is flat, then the scalar curvature of $(T^*M^n, {}^{CG}g)$ is given by*

$$ {}^{CG}r = \frac{(n-1)}{\alpha^2}\big(6 + (n-2)(\alpha^2 + \alpha + 1)\big). $$

4. Geodesics of ${}^{CG}g$

Let C be a curve in M^n expressed locally by $x^h = x^h(t)$ and $\omega_h(t)$ be a covector field along C. Then, in the cotangent bundle T^*M^n, we defined a curve $\tilde{C}$ by

$$ x^h = x^h(t), \quad x^{\bar{h}} \overset{def}{=} p_h = \omega_h(t) \tag{4.1} $$

If the curve C satisfies at all the points the relation

$$ \frac{\delta \omega_h}{dt} = \frac{d\omega_h}{dt} - \Gamma^i_{jh}\frac{dx^j}{dt}\omega_i = 0, $$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve C in M^n. Thus, if the initial condition $\omega_h = \omega^0_h$ for $t = t_0$ is given, there exists a unique horizontal lift expressed by (4.1). We now consider differential equations of the geodesic in the cotangent bundle T^*M^n with the metric ${}^{CG}g$. If t is the arc length of a curve $x^A = x^A(t)$, $A = (i, \bar{i})$ in T^*M^n, then equations of geodesic in T^*M^n have the usual form

$$ \frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + {}^{CG}\Gamma^A_{CB}\frac{dx^C}{dt}\frac{dx^B}{dt} = 0 \tag{4.2} $$

with respect to the induced coordinates $(x^i, x^{\bar{i}}) = (x^i, p_i)$ in T^*M^n, where ${}^{CG}\Gamma^A_{CB}$ are components of ${}^{CG}\nabla$ defined by (2.7). We find it more convenient to refer equations (4.2) to the adapted frame $\{e_\alpha\}$. From (2.1) and (2.2) we see that the matrix of change of frames $e_\beta = A_\beta{}^H \partial_H$ has components of the form (3.2). Using (3.1), now we write

$$\theta^\alpha = \bar{A}^\alpha{}_A dx^A,$$

i.e.

$$\theta^h = \bar{A}^h{}_A dx^A = \delta^h_i dx^i = dx^h$$

for $\alpha = h$ and

$$\theta^{\bar{h}} = \bar{A}^{\bar{h}}{}_A dx^A = -p_a \Gamma^a_{hj} dx^j + \delta^h_j dx^j = \delta p_h$$

for $\alpha = \bar{h}$. Also we put

$$\frac{\theta^h}{dt} = \bar{A}^h{}_A \frac{dx^A}{dt} = \frac{dx^h}{dt},$$
$$\frac{\theta^{\bar{h}}}{dt} = \bar{A}^{\bar{h}}{}_A \frac{dx^A}{dt} = \frac{\delta p_h}{dt}$$

along a curve $x^A = x^A(t)$ in T^*M^n. If we therefore write down the form equivalent to (4.2), namely,

$$\frac{d}{dt}\Big(\frac{\theta^\alpha}{dt}\Big) + {}^{CG}\Gamma^\alpha_{\gamma\beta} \frac{\theta^\gamma}{dt} \frac{\theta^\beta}{dt} = 0$$

with respect to adapted frame and taking account of (2.7), then we have

$$(4.3) \quad \begin{cases} (a) \quad \frac{\delta^2 x^h}{dt^2} + \frac{1}{\alpha} p_a R^{k\ ja}_{.i.} \frac{dx^i}{dt} \frac{\delta p_j}{dt} = 0, \\ (b) \quad \frac{\delta^2 p_h}{dt^2} + \big[-\frac{1}{\alpha}(p^i \delta^j_h + p^j \delta^i_h) + \frac{\alpha+1}{\alpha^2} g^{ij} p_h + \frac{1}{\alpha^2} p^i p^j p_h \big] \frac{\delta p_i}{dt} \frac{\delta p_j}{dt} = 0. \end{cases}$$

Thus the equations (4.3) are the equations of the geodesic in T^*M^n with the metric ${}^{CG}g$. Let now $\tilde{C} : x^h = x^h(t), \quad x^{\bar{h}} = p_h(t) = \omega_h(t)$ be a horizontal lift $\big(\frac{\delta p_h}{dt} = \frac{\delta \omega_h}{dt} = 0\big)$ of the geodesic $C : x^h = x^h(t)$ $(\frac{\delta^2 x^h}{dt^2} = 0)$ in M^n of ∇_g. Then by virtue of (4.3), we have the following.

4.1. Theorem. *The horizontal lift of a geodesic in (M^n, g) is always geodesic in T^*M^n with the Cheeger-Gromoll metric ${}^{CG}g$.*

Acknowledgements

This paper is supported by The Scientific and Technological Research Council of Turkey (Project number : 112T111).

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