Hacettepe Journal of Mathematics and Statistics
Volume 42 (5) (2013), 551 – 557

ON BP-ALGEBRAS

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Received 06 : 05 : 2011 : Accepted 25 : 11 : 2012

Abstract

In this paper, we introduce the notion of a BP-algebra, and discuss some relations with several algebras. Moreover, we discuss a quadratic BP-algebra and show that the quadratic BP-algebra is equivalent to several quadratic algebras.

Keywords: B-algebra, 0-commutative, BF-algebra, BP-algebra, BH-algebra, (normal) subalgebra.

2000 AMS Classification: 06F35, 03G25.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([11]) introduced the notion of d-algebras which is another generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Also they introduced the notion of B-*algebras* ([9, 12, 13]), i.e., (I) $x*x = e$; (II) $x*e = x$; (III) $(x * y) * z = x * (z * (e * y))$, for any $x, y, z \in X$. A. Walendziak ([14]) obtained another axiomatization of B-algebras. Y. B. Jun, E. H. Roh and H. S. Kim ([5]) introduced a new notion, called a BH-*algebras* which is a generalization of $BCH/BCI/BCK$-algebras. A. Walendziak ([15]) introduced a new notion, called an BF-*algebra*, i.e., (I); (II) and (IV) $e * (x * y) = y * x$ for any $x, y \in X$. In ([15]) it was shown that a BF-algebra is a generalizations of a B-algebra. H. S. Kim and N. R. Kye ([7]) introduced the notion of a quadratic BF-algebra, and obtained that quadratic BF-algebras, quadratic Q-algebras, BG -algebras and B-algebras are equivalent nations on a field X with $|X| \geq 3$, and hence every quadratic BF-algebra is a BCI-algebra. In this paper, we introduce the notion of

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a BP-algebra, and discuss some relations with several algebras. Moreover, we discuss a quadratic BP-algebra and show that the quadratic BP-algebra is equivalent to several quadratic algebras and hence becomes a BCI-algebra.

2. Preliminaries

2.1. Theorem. *([12]) By a B-algebra we mean a non-empty set X with a constant 0 and a binary operation "$*$" satisfying axioms: for all $x, y, z \in X$,*

(B_1) $x * x = 0$,
(B_2) $x * 0 = x$,
(B_3) $(x * y) * z = x * [z * (0 * y)]$.

2.2. Definition. ([9]) A B-algebra $(X; *, 0)$ is said to be 0-*commutative* if for any $x, y \in X$, $x * (0 * y) = y * (0 * x)$.

2.3. Proposition. *([9]) If $(X; *, 0)$ is a 0-commutative B-algebra, then we have the following properties: for any $x, y, z, w \in X$,*

(i) $(x * z) * (y * w) = (w * z) * (y * x)$,
(ii) $(x * z) * (y * z) = x * y$,
(iii) $(z * y) * (z * x) = x * y$,
(iv) $(x * z) * y = (0 * z) * (y * x)$,
(v) $x * (y * z) = z * (y * x)$,
(vi) $(x * y) * z = (x * z) * y$,
(vii) $[(x * y) * (x * z)] * (z * y) = 0$,
(viii) $(x * (x * y)) * y = 0$,
(ix) $x * (x * y) = y$,
(x) *The left cancellation law holds, i.e., $x * y = x * z$ implies $y = z$.*

3. A BP-algebra

In this section, we define BP-algebra and investigate its properties.

3.1. Definition. An algebra $(X; *, 0)$ of type (2,0) is called a BP-*algebra* if it satisfies (B_1) and

(BP_1) $x * (x * y) = y$,
(BP_2) $(x * z) * (y * z) = x * y$, for any $x, y, z \in X$.

3.2. Example. (1). Let $X := \{0, a, b, c\}$ be a set with the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X; *, 0)$ is a BP-algebra.
(2). Let $X := \{0, a, b, c\}$ be a set with the following table:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X;*,0)$ is a BP-algebra.

3.3. Theorem. *If $(X;*,0)$ is a BP-algebra, then the following hold: for any $x,y \in X$,*

(i). $0*(0*x)=x$,
(ii). $0*(y*x)=x*y$,
(iii). $x*0=x$,
(iv). $x*y=0$ *implies* $y*x=0$,
(v). $0*x=0*y$ *implies* $x=y$,
(vi). $0*x=y$ *implies* $0*y=x$,
(vii). $0*x=x$ *implies* $x*y=y*x$.

Proof. (i). Put $x:=0$ and $y:=x$ in (BP_1). Then $0*(0*x)=x$.
(ii). Using (BP_2) and (B_1), we have $x*y=(x*x)*(y*x)=0*(y*x)$. Hence $0*(y*x)=x*y$.
(iii). Put $y:=x$ in (BP_1). Then $x*(x*x)=x$. It follows from (B_1) that $x*0=x$.
(iv). By (ii), we have $0=0*0=0*(x*y)=y*x$. Thus $y*x=0$.
(v). If $0*x=0*y$, we have $0*(0*x)=0*(0*y)$. It follows from (i) that $x=y$.
(vi). Using (i), we have $0*y=0*(0*x)=x$. Thus $0*y=x$.
(vii). By (ii), we have $x*y=0*(x*y)=y*x$. Thus $x*y=y*x$.

□

3.4. Theorem. *If $(X;*,0)$ is a BP-algebra, then $(X;*,0)$ is a BF-algebra.*

Proof. By Theorem 3.3-(iii), (B_2) holds. It follows from Theorem 3.3-(ii) that (BF) holds.

□

The converse of Theorem 3.4 does not hold in general.

3.5. Example. Let $X:=\{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	1	2
2	2	2	0	2
3	3	1	1	0

Then $(X;*,0)$ is a BF-algebra, but not a BP-algebra, because $(1*3)*(2*3)=2*2=0\neq 1=1*2$.

3.6. Definition. A BP-algebra $(X;*,0)$ is said to be 0-*commutative* if $x*(0*y)=y*(0*x)$ for any $x,y\in X$.

3.7. Proposition. *If $(X;*,0)$ is a 0-commutative BP-algebra, then the following hold: for any $x,y,z\in X$,*

(i). $(x*z)*(y*z)=(z*y)*(z*x)$,
(ii). $x*y=(0*y)*(0*x)$.

Proof. (i). By Proposition 3.3-(ii), we have

$$\begin{aligned}(x*z)*(y*z)=&(x*z)*(0*(z*y))\\=&(z*y)*(0*(x*z))\\=&(z*y)*(z*x).\end{aligned}$$

(ii). Put $z:=0$ in Proposition 3.7-(i). Then $(x*0)*(y*0)=(0*y)*(0*x)$. It follows from Proposition 3.3-(iii) that $x*y=(0*y)*(0*x)$.

□

Every abelian group can determine a BP-algebra.

3.8. Theorem. *Let $(X;\circ,0)$ be an abelian group. If we define $x*y:=x\circ y^{-1}$, then $(X;*,\circ)$ is a BP-algebra.*

Proof. For any $x\in X$, we have $x*x=x\circ x^{-1}=0$. Since X is abelian, we have $x*(x*y)=x*(x\circ y^{-1})=x\circ(x\circ y^{-1})^{-1}=x\circ y\circ x^{-1}=x\circ x^{-1}\circ y=0\circ y=y$. Hence, for any $x,y,z\in X$, we have

$$\begin{aligned}(x*y)*(z*y)=&(x\circ y^{-1})*(z\circ y^{-1})\\ =&(x\circ y^{-1})\circ(z\circ y^{-1})^{-1}\\ =&(x\circ y^{-1})\circ(y\circ z^{-1})\\ =&x\circ(y^{-1}\circ y)\circ z^{-1}\\ =&x\circ z^{-1}\\ =&x*z,\end{aligned}$$

proving the theorem.

□

3.9. Theorem. *Let $(X;*,0)$ be a BP-algebra. Then X is 0-commutative if and only if $(0*x)*(0*y)=y*x$ for any $x,y\in X$.*

Proof. Assume that $(0*x)*(0*y)=y*x$ for any $x,y\in X$. By Theorem 3.3-(i), we have $x*(0*y)=(0*(0*x))*(0*y)=y*(0*x)$.

The converse follows immediately from Proposition 3.7.

□

3.10. Proposition. *If $(X;*,0)$ is a BP-algebra with $(x*y)*z=x*(z*y)$ for any $x,y,z\in X$, then $0*x=x$ for any $x\in X$.*

Proof. Let $x=z=0$ in $(x*y)*z=x*(z*y)$. Then $(0*y)*0=0*(0*y)$. By Theorem 3.3-(i) and (iii), we have $0*y=y$.

□

3.11. Theorem. *If $(X;*,0)$ is a 0-commutative B-algebra, then $(X;*,0)$ is a BP-algebra.*

Proof. Clearly, (B_1) holds. It follows from Proposition 2.3-(ix) and (ii) that (BP_1) and (BP_2) hold. Thus $(X;*,0)$ is a BP-algebra.

□

3.12. Theorem. *If $(X;*,0)$ is a BP-algebra with $(x*y)*z=x*(z*y)$ for any $x,y,z\in X$, then $(X;*,0)$ is a B-algebra.*

Proof. Using Theorem 3.3-(iii), we have $x*(z*(0*y))=x*((z*y)*0)=x*(z*y)=(x*y)*z$ for any $x,y,z\in X$. Thus $(X;*,0)$ is a B-algebra.

□

In general, B-algebras need not be a BP-algebra. See the following example.

3.13. Example. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X; *, 0)$ is a B-algebra, but not a BP-algebra with $(x * y) * z = x * (z * y)$, since $0 * (1 * 0) = 0 * 1 = 2 \neq 1 = 0 * (0 * 1)$ and $(3 * 1) * 2 = 1 * 2 = 3 \neq 0 = 3 * 3 = 3 * (2 * 1)$.

3.14. Theorem. *If $(X; *, 0)$ is a BP-algebra, then it is a BH-algebra.*

Proof. Let $x * y = 0$ and $y * x = 0$ for any $x, y \in X$. Using Theorem 3.3-(iii) and (BP_1), we have $x = x * 0 = x * (x * y) = y$. Thus X is a BH-algebra.

□

The converse of Theorem 3.14 need not be true in general.

3.15. Example. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X; *, 0)$ is a BH-algebra, but not a BP-algebra, since $1 * (1 * 2) = 1 * 0 = 1 \neq 2$.

4. A quadratic BP-algebra

Let X be a field with $|X| \geq 3$. An algebra $(X; *)$ is said to be *quadratic* if $x * y$ is defined by $x * y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$, where $a_1, a_2, a_3, a_4, a_5, a_6 \in X$, for any $x, y \in X$. A quadratic algebra $(X; *)$ is said to be a *quadratic BP-algebra* if it satisfies the conditions (B_1), (BP_1) and (BP_2).

4.1. Theorem. *Let X be a field with $|X| \geq 3$. Then every quadratic BP-algebra $(X; *, e)$ has of the form $x * y = x - y + e$, where $x, y, z \in X$.*

Proof. Define $x * y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, where $A, B, C, D, F \in X$ and $x, y \in X$. Consider (B_1).

$$\begin{aligned} e &= x * x \\ &= (A + B + C)x^2 + (D + E)x + F \end{aligned}$$

It follows that $F = e, A + B + C = 0 = D + E$, i.e, $D = -E$.
Consider (B_2).

$$x = x * e = Ax^2 + Bxe + Ce^2 + Dx + Ee + e.$$

It follows that $A = 0, Be + D = 1$ and $Ce^2 + Ee + e = 0$. Thus $B + C = 0, D = 1 - Be$. Since $D = -E$, we have $E = -1 + Be$. From this information, we have the following more simpler form:

$$\begin{aligned} x * y &= Bxy + Cy^2 + Dx + Ey + e \\ &= Bxy + (-B)y^2 + (1 - Be)x + (Be - 1)y + e \\ &= B(xy - y^2 - ex + ey) + (x - y + e) \\ &= B(x - y)(y - e) + (x - y + e) \end{aligned}$$

By Theorem 3.3-(ii), every BP-algebra satisfies the condition $e*(x*y) = y*x$ for any $x, y \in X$. Consider $e*(x*y)$.

$$\begin{aligned} e*(x*y) &= B(e - x*y)(x*y - e) + (e - x*y + e) \\ &= B[e - B(x-y)(y-e) - (x-y+e)] \\ &\quad [B(x-y)(y-e) + (x-y+e) - e] \\ &\quad + [e - B(x-y)(y-e) - (x-y)] \\ &= B[-B(x-y)(y-e) - (x-y)][B(x-y)(y-e) + (x-y)] \\ &\quad + [e - B(x-y)(y-e) - (x-y)] \\ &= -B[B(x-y)(y-e) + (x-y)]^2 \\ &\quad - [B(x-y)(y-e) + (x-y)] + e \\ &= -B(x-y)^2[B(y-e)+1]^2 - (x-y)[B(y-e)+1] + e. \end{aligned}$$

Since $y*x = B(y-x)(x-e) + (y-x+e) = B(y-x)(x-e) + (y-x) + e$, we obtain

$$-B(x-y)^2[B(y-e)+1]^2 - (x-y)[B(y-e)+1] + e = B(y-x)(x-e) + (y-x) + e.$$

If we let $x := e$ in the above identity, then we have

$$-B(e-y)^2[B(y-e)+1]^2 - (e-y)[B(y-e)+1] + e = (y-e) + e.$$

It follows that $B = 0$. Hence $C = 0, E = -1$ and $D = 1$. Thus $x*y = x - y + e$. It is easy to check that this binary operation satisfies (BP_1) and (BP_2). This completes the proof. □

4.2. Example. (1) Let $\mathbb{R}$ be the set of all real numbers. Define $x*y := x - y + \sqrt{2}$. Then $(\mathbb{R}; *, \sqrt{2})$ is a quadratic BP-algebra.
(2) Let $\kappa := GF(p^n)$ be a Galois field. Define $x*y := (x-y) + e, e \in \kappa$. Then $(\kappa; *, e)$ is a quadratic BP-algebra.

H. K. Park and H. S. Kim ([13]) proved that every quadratic B-algebra $(X; *, e)$, $e \in X$, has the form $x*y = x - y + e$, where X is a field with $|X| \geq 3$. J. Neggers, S. S. Ahn and H. S. Kim ([10]) introduced the notion of a Q-algebra, and obtained that every quadratic Q-algebra $(X; *, e)$, $e \in X$, has the form $x*y = x - y + e$, $x, y \in X$, where X is a field with $|X| \geq 3$. Also, H. S. Kim and H. D. Lee ([8]) showed that every quadratic BG-algebra $(X; *, e)$, $e \in X$, has the form $x*y = x - y + e$, $x, y \in X$, where X is a field with $|X| \geq 3$. H. S. Kim and N. R. Kye ([7]) introduced the notion of a quadratic BF-algebra.

4.3. Theorem. *([7]) Let X be a field with $|X| \geq 3$. Then the following are equivalent :*
(1) $(X; *, e)$ is a quadratic BF-algebra,
(2) $(X; *, e)$ is a quadratic BG-algebra,
(3) $(X; *, e)$ is a quadratic Q-algebra,
(4) $(X; *, e)$ is a quadratic B-algebra.

4.4. Theorem. *([13]) Let X be a field with $|X| \geq 3$. Then every quadratic B-algebra on X is a BCI-algebra.*

4.5. Theorem. *Let X be a field with $|X| \geq 3$. Then every quadratic BP-algebra on X is a BCI-algebra.*

Proof. It is an immediate consequence of Theorem 4.3 and Theorem 4.4. □

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