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A NEW COMPUTATIONAL APPROACH FOR TESTING EQUALITY OF INVERSE GAUSSIAN MEANS UNDER HETEROGENEITY

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Abstract

In this article, a testing procedure based on computational approach test is proposed for the equality of several inverse Gaussian means under heterogeneity. Not requiring the knowledge of any sampling distribution, depending heavily on numerical computations and Monte Carlo simulation, moreover, figuring out the critical region automatically are the advantages of the computational approach test. We compare it with some of the existing tests; the parametric bootstrap and the generalized test variables in terms of type I risks and powers by using Monte Carlo simulation.

Keywords: Computational approach test, Generalized test variables, Hypothesis testing, Inverse gaussian distribution, Maximum likelihood estimate, Parametric bootstrap.

2000 AMS Classification: 62F03, 65C05, 65C60, 62E17.

1. Introduction

Inverse Gaussian (IG) distribution is given in quite a number of books on stochastic processes and probability. The probability distribution of the first passage time in Brownian motion is given by Schrödinger (1915). Since Tweedie (1945) has shown the inverse relationship between the cumulant generating function of the first passage time distribution and that of the normal distribution, it is called as an IG for the first passage time distribution. Further, as Wald (1947) has derived the limiting form of IG distribution, it is also called as Wald's distribution, especially in the Russian literature [4].

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The density function of the two-parameter IG distribution IG (μ,λ) is defined as in equation (1.1).

$$(1.1) \quad f\left(x;\mu,\lambda\right) = \left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\frac{-\lambda(x-\mu)^2}{2\mu^2 x}, \quad x>0, \quad \mu,\, \lambda>0$$

Here, μ is the mean parameter and λ is the scale parameter. For the last three decades, the IG distribution has gained significant attention in describing and analyzing right-skewed data. For example, Doksum and Hóyland (1992) developed a model for variable-stress-accelerated life testing experiments based on the IG distributions, Durham and Padgett (1997) used the IG models to develop a new general method based on cumulative damage for describing the failure of a system. Seshadri's (1999) study is a good reference for other applications of IG distribution in life tests, remote sensing, etc. [11].

The main appeal of IG models lies in these facts:(1) they could accommodate a variety of shapes, from highly skewed to almost normal; (2) they are unique among the distributions for positively right-skewed data (e.g. Weibull, gamma, log normal) on account of its many elegant and convenient properties sharing with Gaussian models [10]. Hence, it could be mentioned that IG distribution has several properties analogous to normal distribution [5, 9, 10].

A very common problem in applied statistics is that of comparing the means of several populations. It is well known that there exists an analysis of variance (ANOVA) F-test for the problem of testing the equality of means from several independent samples under the assumptions of normality. Similarly, for testing equality of IG means when the scale parameters are the same, an analysis of reciprocals (ANORE) F-test is developed by Chhikara and Folks (1989) [7]. Since the number of populations having IG distributions is k, the unknown mean of ith population is μ_i and the general mean is μ, the null and alternative hypotheses could be written as shown in equation (1.2).

$$(1.2) \qquad H_0 : \mu_1 = \mu_2 = \cdots = \mu_k = \mu \; and \; H_1 : \mu_i \neq \mu_j,\; \exists\, i \neq j \;\; (i,j = 1,\cdots,k)$$

However, the disadvantage of ANORE is that it fails for testing equality of IG means, when the scale parameters are non-homogeneous. Therefore, to deal with this problem of comparing IG means in the case of unequal scale parameters, Tian (2006) developed an approach using the concepts of generalized test variables (GTV) and generalized p-values. The concept of GTV has been widely applied to a variety of practical settings, where standard inference methods do not exist. The GTV was proposed by Weerahandi (1995) in order to test the equality of normal means under heterogeneity and the generalized p-values approach was introduced by Tsui and Weerahandi (1989) for significance testing of hypotheses in the presence of nuisance parameters. In Tian's (2006) article, the concept of generalized p-value is applied for testing equality of several IG means for the general cases without the assumption of homogeneity. Simulation results of his study indicate that the proposed test has excellent type I risk control under both heterogeneity and homogeneity, whereas the type I risk of the ANORE test could be much larger than the nominal level under heterogeneity [7, 11]. Ma and Tian (2009) proposed a parametric bootstrap (PB) approach for testing equality of IG means under heterogeneity. Bootstrap approach is a computer intensive method used frequently in applied statistics. It is a type of Monte Carlo method applied on observed data. [7]. Ye et al. (2010) proposed a mixture method for the common mean problem based on generalized inference and the large sample theory. However, according to their study, if the sample sizes for each group, n$_i$, are not large and/or the scale parameter λ_i is not large compared to μ_i, the approximate distributions don't fit well [6]. Recently, Lin and Wu (2011) have discussed an interval estimation method for the common mean of several heterogeneous IG populations. The proposed method is based on a higher order likelihood-based procedure.

It is important to develop a test procedure for equality of IG means with satisfactory type I risk regardless of number of groups and the sample sizes. In this paper, a new computational approach test (CAT) is proposed for equality of IG means under heterogeneity. CAT method based on simulation and numerical computations uses the maximum likelihood estimates (MLEs), but does not require any asymptotic distribution. This approach provides an algorithmic framework based on the Monte-Carlo simulation and numerical computations, which can be implemented mechanically by applied researchers to draw statistical inferences, when a suitable parametric model is assumed for a given data set [8]. In this article, extensive simulation results are presented to evaluate the type I risk and power of the proposed CAT approach in comparison to that of the, PB and GTV approaches.

This article is organized as follows. In Section 2, the tests based on the PB and the GTV, used for testing the equality of IG means under heterogeneity, are presented. In Section 3, a new CAT based on simulation and numerical computations, used for the equality of IG means, is presented. In Section 4, simulation results on type I risk control and power are presented. Concluding remarks are summarized in Section 5.

2. Tests for Testing Equality of Inverse Gaussian Means under Heterogeneity

In this section, PB approach and GTV developed for testing equality of several IG means, without the assumption of equal scale parameters, are presented.

2.1. The Parametric Bootstrap Approach. In this section, a PB approach is presented that is used for testing equality of several IG means [7].

Let $X_{i1}, X_{i2}, \cdots, X_{in_i}$ be a random sample from an $IG(\mu_i, \lambda_i)$, $i = 1, \cdots, k$ and the terms in equation (2.1) are defined. It is also well known that equation (2.2) could be written for $i = 1, \cdots, k$.

(2.1) $\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$ and $V_i = \sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)$

(2.2) $\bar{X}_i \sim IG(\mu_i, n_i\lambda_i)$ and $\lambda_i V_i \sim \chi^2_{n_i-1}$

The hypotheses of interest are given before as shown in equation (1.2). Then, equation (2.3) is defined with $\hat{\mu} = \frac{\sum_{i=1}^{k} n_i\lambda_i\bar{X}_i}{\sum_{i=1}^{k} n_i\lambda_i}$. Under null hypothesis H_0, $Q(\lambda_1, \lambda_2, \cdots, \lambda_k) \sim \chi^2_{k-1}$ [7, 10].

(2.3) $Q(\lambda_1, \lambda_2, \cdots, \lambda_k) = \sum_{i=1}^{k} n_i\lambda_i\left(\frac{1}{\bar{X}_i} - \frac{1}{\hat{\mu}}\right)$

The PB approach involves sampling from the estimated models. Based on the test statistic $\boldsymbol{Q}$ in equation (2.3), the PB pivot variable could be developed as follows. Firstly, equation (2.4) is written with $\hat{\lambda}_i = n_i/V_i$ and $\lambda_{Bi} \sim \frac{\chi^2_{n_i-1}}{V_i}$, $i = 1, \cdots, k$. Then the PB pivot variable, based on the statistic in equation (2.3), is given by equation (2.5). In equation (2.5), $\hat{\mu}_B = \frac{\sum_{i=1}^{k} n_i\lambda_{Bi}\bar{X}_{Bi}}{\sum_{i=1}^{k} n_i\lambda_{Bi}}$ [7].

(2.4) $\bar{X}_{Bi} \sim IG\left(\frac{\sum_{i=1}^{k} n_i\hat{\lambda}_i\bar{X}_i}{\sum_{i=1}^{k} n_i\hat{\lambda}_i}, n_i\hat{\lambda}_i\right)$

(2.5) $Q_B\left(\bar{X}_{B1}, \cdots, \bar{X}_{Bk}, \lambda_{B1}, \cdots, \lambda_{Bk} \,\middle|\, \bar{X}_1, \cdots, \bar{X}_k, V_1, \cdots, V_k\right)$

$= \sum_{i=1}^{k} n_i \lambda_{Bi} \left(\frac{1}{\overline{X}_{Bi}} - \frac{1}{\widehat{\mu}_B} \right)$

For a given data set with $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$ and $(v_1, v_2, \cdots, v_k)$, m bootstrap samples, $Q_B^{(i)}, i = 1, \cdots, m$, are drawn. The PB approach rejects H_0 at significance level α, as shown in equation (2.6). Here, Q_{B0} is an observed value of Q_B in equation (2.5), as given in equation (2.7). Here, $\tilde{\mu}$ and $\tilde{\lambda}_i, i = 1, \cdots, k$ are the MLEs under H_0 from the observed samples [7].

$$(2.6) \qquad p = \frac{\#\left(Q_B^{(i)} \geq Q_{B0}\right)}{m} \leq \alpha$$

$$(2.7) \quad Q_{B0} = \sum_{i=1}^{k} n_i \tilde{\lambda}_i \left(\frac{1}{\bar{x}_i} - \frac{1}{\tilde{\mu}} \right)$$

The EM-algorithm for calculating $\widetilde{\mu}$ and $\widetilde{\lambda}_i$ is described as below. The log-likelihood function under null-hypotheses H_0 is as shown in equation (2.8).

$$(2.8) \log\left(L\left(\mu, \lambda_1, \cdots, \lambda_k\right)\right) = \sum_{i=1}^{k} \frac{n_i}{2} \log\left(\lambda_i\right) - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_i}{2\mu^2 x_{ij}} \left(x_{ij} - \mu\right)^2 + const$$

Here, $x_{ij}, i = 1, \cdots, k, j = 1, \cdots, n_i$ are observed values of X_{ij}. The MLEs of $\mu, \lambda_1, \cdots, \lambda_k$ under H_0 have no closed forms. The estimates can be obtained through the iterations as below: updating the estimates from l-step estimates $\left(\lambda_1^{(l)}, \cdots, \lambda_k^{(l)}, \mu^{(l)}\right)$ by equation (2.9). Here, initial value $\mu^{(0)}$ could set as grand mean $\frac{1}{k}\sum_{i=1}^{k} \bar{x}_i$. $\left(\lambda_1^{(l)}\right.$ $\left., \cdots, \lambda_k^{(l)}, \mu^{(l)}\right)$ converge to the MLEs under H_0 denoted as $\left(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_k, \tilde{\mu},\right)$ [7].

$$(2.9) \qquad \lambda_i^{(l+1)} = n_i / \sum_{i=1}^{n_i} \left(\frac{1}{x_{ij}} - \frac{1}{\mu^{(l)}} \right)^2, i = 1, \cdots, k, \; \mu^{(l+1)} = \frac{\sum_{i=1}^{k} n_i \lambda_i^{(l+1)} \bar{x}_i}{\sum_{i=1}^{k} n_i \lambda_i^{(l+1)}}$$

Moreover, the algorithm for calculating p-value using the PB approach could be given as shown below [7]. Algorithm:

1. For a given data $\left(x_{i1}, x_{i2}, \cdots, x_{in_i}\right),\ i = 1, \cdots, k$, calculate observed statistics $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$ and $(v_1, v_2, \cdots, v_k)$. Then calculate the MLEs $\tilde{\mu}, \tilde{\lambda}_1, \cdots, \tilde{\lambda}_k$ under null hypotheses. Compute Q_{B0} as defined in equation (2.7).

2. Independently generate $\bar{X}_{Bi} \sim IG\left(\frac{\sum_{i=1}^{k} n_i \hat{\lambda}_i \bar{x}_i}{\sum_{i=1}^{k} n_i \hat{\lambda}_i}, n_i \hat{\lambda}_i\right)$ and $\lambda_{Bi} \sim \frac{\chi^2_{(n_i-1)}}{v_i},\ \ i = 1, \cdots, k$. Then compute Q_B in equation (2.5).

3. Repeat step 2 a total m times and obtain m values of Q_B, denoted as $Q_B^{(i)}, i = 1, 2, \cdots, m$.

4. The percentage that $Q_B^{(i)\prime}s\ \ (i = 1, 2, \cdots, m)$ are greater than or equal to Q_{B0} is a Monte Carlo estimate of the p-value for testing H_0 vs. H_1.

2.2. A Generalized Test Variable Approach. Weerahandi (1995) presented a GTV for the equality of several Gaussian means under heterogeneity. A parallel test for IG distribution is developed as follows [11].

Let $X_{i1}, X_{i2}, \cdots, X_{in_i}$ be a random sample from an $IG\left(\mu_i, \lambda_i\right)$ population, with $\bar{X}_i$ as the ith sample mean for $i = 1, \cdots, k$. Let $\bar{x}_i$ denotes the ith observed sample mean for $i = 1, \cdots, k$. It is well known that both of the equation (2.1) and equation (2.2) could be written as it is done before in PB approach section. The generalized pivot for λ_i could be written as shown in equation (2.10). Here, v_i is the observed value of V_i. Obviously, R_{λ_i} coincides with the traditional pivot for λ_i [7, 11].

$$R_{\lambda_i} = \frac{\lambda_i V_i}{v_i} \sim \frac{\chi^2_{n_i-1}}{v_i} \tag{2.10}$$

Under the null hypothesis that is given in equation (1.2), equation (2.1) could be written. Here, $\hat{\mu} = \frac{\sum_{i=1}^{k} n_i \lambda_i \bar{X}_i}{\sum_{i=1}^{k} n_i \lambda_i}$ is as given before in Section 2.1. In the same vein of the GTV for the equality of normal means, a potential GTV is defined for the equality of IG means as in equation (2.12). Here, $q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right) = \sum_{i=1}^{k} n_i R_{\lambda_i}\left(\frac{1}{\bar{x}_i} - \frac{1}{R_{\hat{\mu}}}\right)$ with $R_{\hat{\mu}} = \frac{\sum_{i=1}^{k} n_i R_{\lambda_i} \bar{x}_i}{\sum_{i=1}^{k} n_i R_{\lambda_i}}$. That is, $q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right)$ is $Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right)$ with $\bar{X}_i$ replaced by the observed value $\bar{x}_i$ and λ_i, replaced by the corresponding pivot R_{λ_i} for $i = 1, \cdots, k$ [7, 10, 11].

$$Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right) = \sum_{i=1}^{k} n_i \lambda_i \left(\frac{1}{\bar{X}_i} - \frac{1}{\hat{\mu}}\right) \sim \chi^2_{k-1} \tag{2.11}$$

$$T = \frac{Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right)}{q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right)} \tag{2.12}$$

The proposed test statistic $\boldsymbol{T}$ satisfies the three conditions in order to be an actual GTV:

First of all, as $\bar{X}_i = \bar{x}_i$, $V_i = v_i$, $R_{\lambda_i} = \lambda_i \left(i = 1, \cdots, k\right)$ and $Q\left(\lambda_1, \cdots, \lambda_k\right) = q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right)$. Hence, the observed value of $\boldsymbol{T}$ is $t_{obs} = 1$. Secondly, under null hypothesis, $Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right) \sim \chi^2_{k-1}$ and $q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right)$ are functions of a set of independent random variables $\chi^2_{n_i-1}$ and observed values $\bar{x}_i$ and v_i for $i = 1, 2, \cdots, k$. Thus, $\boldsymbol{T}$ does not depend on any unknown parameters. Thirdly, $\boldsymbol{T}$ tends to take large values for deviations from null hypothesis. Consequently, $\boldsymbol{T}$ becomes an actual GTV for testing the equality of IG means [11]. As $\boldsymbol{T}$ tends to take larger than 1 for deviations from null hypothesis, the generalized p-value could be obtained as shown in equation (2.13). Here, $G_{k-1}\left[\cdots\right]$ represents the cumulative distribution function of χ^2_{k-1} distribution and the expectation $E\left\{\cdots\right\}$ is taken with respect to k independent $\chi^2_{n_i-1}\left(i = 1, 2, \cdots, k\right)$ random variables [7, 11].

$$\begin{aligned} p - value &= \Pr ob\left(T \geq t_{obs} = 1 \,|H_0\right) \\ &= \Pr ob\left(Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right) \geq q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right) |H_0\right) \\ &= 1 - E\left\{G_{k-1}\left[q\left(R_{\lambda_1}, R_{\lambda_2}, \cdots, R_{\lambda_k}\right)\right]\right\} \end{aligned} \tag{2.13}$$

Based on the simulation results in Tian (2006), the proposed test based on the concept of GTV was strongly recommended being used instead of ANORE test for testing equality of IG means, owing to the fact that it has excellent type I risk control under both homogeneity and heterogeneity.

3. The Computational Approach Test for Testing Equality of Inverse Gaussian Means under Heterogeneity

In this section, a test procedure is given based on CAT for testing equality of several IG means under the unequal scale parameters. Firstly, before applying developed CAT procedure for testing the null hypothesis given in equation (1.2), the H_0 is expressed in terms of suitable scalar η. η is defined as shown in equation (3.1). Here, $\bar{\mu} = \sum_{i=1}^{k} n_i \mu_i / n$ and $n = \sum_{i=1}^{k} n_i$.

(3.1) $\eta = \eta\left(\mu_1, \cdots, \mu_k\right) = \sum_{i=1}^{k} n_i \left(\frac{1}{\mu_i} - \frac{1}{\mu}\right)$

It is clear that testing H_0 against H_1 is equivalent to testing $H_0^* : \eta = 0$ against $H_1^* : \eta > 0$. Thus, MLE of η can be used as a test statistic. If H_0 is true then η=0, otherwise η is getting greater than 0. In this study, $\eta = \sum_{i=1}^{k} n_i \left(\frac{1}{\mu_i} - \frac{1}{\mu}\right)$ is chosen. However, one can choose another suitable expression. The choice of η is the most important point for the success of the CAT method.

The test procedure of our proposed CAT could be given as shown below:

i) The MLE of the parameters are obtained as

$\hat{\mu}_{i(ML)} = \bar{X}_{i.}$ and $\hat{\lambda}_{i(ML)}^{-1} = \frac{1}{n_i} \sum_{j=1}^{n_i} \left(X_{ij}^{-1} - \bar{X}_{i.}^{-1}\right)$

Therefore, the test statistic is rewritten as $\hat{\eta}_{ML} = \sum_{i=1}^{k} n_i \left(\frac{1}{\bar{X}_{i.}} - \frac{1}{\bar{\bar{X}}}\right)$. The observed value of $\hat{\eta}_{ML}$is $\hat{\eta}_{ML0}$.

ii) If H_0 or H_0^* is true, then $X_{ij} \sim \mathrm{IG}\left(\mu, \lambda_i\right)$ $(1 \leq j \leq n_i, 1 \leq i \leq k)$ Here, μ and $(\lambda_1, \cdots, \lambda_k)$ are nuisance parameters. Hence, MLEs of μ and λ_i based on the assumption under null hypothesis $(\mu_1 = \mu_2 = \cdots = \mu_k = \mu)$ give the restricted maximum likelihood estimates (RMLEs) of these parameters. The procedure of obtaining RMLEs of the μ and λ_i parameters could be given as follows:

Firstly, under the restricted model, log - likelihood function of the sample x_{i1}, x_{i2}, $\cdots$, x_{in_i} is given as in equation (3.2).

(3.2) $L = \frac{1}{2} \sum_{i=1}^{k} n_i \log \frac{\lambda_i}{2\pi} - \frac{3}{2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \log x_{ij} - \frac{1}{2\mu^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_i x_{ij} + \frac{1}{\mu} \sum_{i=1}^{k} n_i \lambda_i - \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_i}{x_{ij}}$

Differentiating the equation (3.2) with respect to μ and λ_i yields the following results:

$$\begin{aligned} &\frac{\partial L}{\partial \mu} = \frac{1}{\mu^3} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_i x_{ij} - \frac{1}{\mu^2} \sum_{i=1}^{k} n_i \lambda_i \\ &\frac{\partial L}{\partial \lambda_i} = \frac{n_i}{2\lambda_i} + \frac{n_i}{\mu} - \frac{1}{2} \sum_{j=1}^{n_i} \frac{1}{x_{ij}} - \frac{1}{2\mu^2} \sum_{j=1}^{n_i} x_{ij}, \qquad i = 1, ..., l \end{aligned} \tag{3.3}$$

As can be seen from equation (3.3), the RMLEs of the μ and λ parameters have no closed forms. Therefore, the RMLEs of these parameters could be obtained using the iterations given in equation (2.9). Here $\mu^{(l)}, \lambda_1^{(l)}, \cdots, \lambda_k^{(l)}$ converge to the RMLEs denoted as $\hat{\mu}_{RML}, \hat{\lambda}_{i(RML)}$.

iii) Generate artificial sample $X_{i1}, \ldots, X_{in_i}$, $i = 1, \ldots, k$ i.i.d. from $IG(\hat{\mu}_{RML}, \hat{\lambda}_{i(RML)})$ a large of number of times (say, m times). For each of these replicated samples, recalculate the values of $\hat{\eta}_{ML}^{(j)}$ $(j = 1, ..., m)$.

iv) Calculate the p-value as $p = \frac{\#\left(\hat{\eta}_{ML}^{(j)} > \hat{\eta}_{ML0}\right)}{m}$. In the case of p<α, H_0 is rejected.

4. A Simulation Study

In this section for testing equality of IG means under heterogeneity, the GTV, PB approaches and CAT are compared according to type I risks and powers for different combinations of parameters (μ, λ) and sample sizes. For this purpose, we consider some cases from smaller to larger sample sizes with different number of groups as k=3, k=4 and k=5. For specified nominal level of α=0.05, we used m=10000 times to calculate the simulated type I risks and powers of these tests.Firstly, we calculated simulated type I risks of tests under null hypothesis. The numerical results for estimated type I risks are given as in Table 1 to Table 3.

TABLE 1. Estimated type I risks of three tests for k=3.

μ	n	λ_1, λ_2, λ_3	PB	GTV	CAT
10	15	30 30 30	0.059	0.046	0.048
		30 35 40	0.063	0.048	0.049
		30 40 50	0.058	0.047	0.047
	20	30 30 30	0.058	0.049	0.048
		30 35 40	0.056	0.046	0.047
		30 40 50	0.059	0.051	0.050
	30	30 30 30	0.054	0.046	0.048
		30 35 40	0.049	0.045	0.047
		30 40 50	0.054	0.048	0.050
20	15	30 30 30	0.062	0.048	0.047
		30 35 40	0.056	0.043	0.044
		30 40 50	0.060	0.049	0.046
	20	30 30 30	0.059	0.050	0.049
		30 35 40	0.054	0.045	0.042
		30 40 50	0.058	0.048	0.048
	30	30 30 30	0.533	0.046	0.045
		30 35 40	0.053	0.048	0.047
		30 40 50	0.059	0.052	0.053

TABLE 2. Estimated type I risks of three tests for k=4.

μ	n	λ_1, λ_2, λ_3, λ_4	PB	GTV	CAT
10	15	30 30 30 30	0.055	0.053	0.042
		30 35 40 45	0.054	0.051	0.042
		30 40 50 60	0.053	0.049	0.044
	20	30 30 30 30	0.054	0.050	0.046
		30 35 40 45	0.051	0.049	0.044
		30 40 50 60	0.052	0.050	0.045
	30	30 30 30 30	0.052	0.050	0.046
		30 35 40 45	0.048	0.046	0.045
		30 40 50 60	0.049	0.048	0.047
20	15	30 30 30 30	0.053	0.048	0.043
		30 35 40 45	0.055	0.052	0.045
		30 40 50 60	0.055	0.052	0.044
	20	30 30 30 30	0.054	0.052	0.044
		30 35 40 45	0.054	0.052	0.048
		30 40 50 60	0.053	0.051	0.048
	30	30 30 30 30	0.053	0.051	0.048
		30 35 40 45	0.052	0.052	0.048
		30 40 50 60	0.052	0.050	0.047

As seen from Table 1-Table 3, while the type I risks of PB approach exceed the nominal level in the case of k=3, they close to the 0.05 for k=5. GTV performs contrast to the

TABLE 3. Estimated type I risks of three tests for k=5.

μ	n	λ_1, λ_2, λ_3, λ_4,λ_5	PB	GTV	CAT
10	15	30 30 30 30 30	0.055	0.059	0.044
		30 35 40 45 50	0.053	0.057	0.043
		30 40 50 60 70	0.051	0.055	0.044
	20	30 30 30 30 30	0.055	0.059	0.049
		30 35 40 45 50	0.052	0.055	0.046
		30 40 50 60 70	0.049	0.054	0.043
	30	30 30 30 30 30	0.053	0.054	0.050
		30 35 40 45 50	0.052	0.053	0.045
		30 40 50 60 70	0.049	0.051	0.044
20	15	30 30 30 30 30	0.049	0.054	0.040
		30 35 40 45 50	0.051	0.056	0.044
		30 40 50 60 70	0.051	0.055	0.042
	20	30 30 30 30 30	0.051	0.053	0.043
		30 35 40 45 50	0.049	0.052	0.043
		30 40 50 60 70	0.049	0.054	0.047
	30	30 30 30 30 30	0.053	0.056	0.049
		30 35 40 45 50	0.048	0.051	0.043
		30 40 50 60 70	0.049	0.050	0.043

PB approach that its type I risks close to 0.05 for k=3, whereas they exceed the 0.05 for k=5. However, the CAT method seems to have type I risks lower also almost close to the nominal level for k=3, k=4 and k=5.

After calculating the type I risks of three methods, we calculated the estimated power of the tests for different combinations of parameters and sample sizes.

TABLE 4. Estimated power of three tests for k=3.

n	λ_1, λ_2, λ_3	μ_1, μ_2, μ_3	PB	GTV	CAT
15	30 30 30	10 11 12	0.108	0.091	0.092
		10 12 14	0.235	0.205	0.209
		10 13 16	0.385	0.343	0.356
	30 35 40	10 11 12	0.118	0.098	0.104
		10 12 14	0.263	0.229	0.247
		10 13 16	0.429	0.391	0.420
	30 40 50	10 11 12	0.128	0.107	0.112
		10 12 14	0.276	0.245	0.266
		10 13 16	0.458	0.419	0.450
30	30 30 30	10 11 12	0.176	0.162	0.166
		10 12 14	0.431	0.409	0.422
		10 13 16	0.689	0.669	0.681
	30 35 40	10 11 12	0.179	0.166	0.174
		10 12 14	0.484	0.464	0.476
		10 13 16	0.734	0.717	0.743
	30 40 50	10 11 12	0.197	0.181	0.188
		10 12 14	0.517	0.495	0.516
		10 13 16	0.790	0.776	0.795

TABLE 5. Estimated power of three tests for k=4.

n	λ_1, λ_2, λ_3, λ_4	μ_1, μ_2, μ_3, μ_4	PB	GTV	CAT
15	30 30 30 30	10 11 12 13	0.152	0.143	0.129
		10 12 14 16	0.346	0.337	0.330
		10 13 16 19	0.559	0.553	0.554
	30 35 40 45	10 11 12 13	0.158	0.153	0.148
		10 12 14 16	0.406	0.398	0.399
		10 13 16 19	0.625	0.621	0.643
	30 40 50 60	10 11 12 13	0.175	0.167	0.166
		10 12 14 16	0.449	0.442	0.445
		10 13 16 19	0.687	0.686	0.712
30	30 30 30 30	10 11 12 13	0.271	0.266	0.260
		10 12 14 16	0.673	0.669	0.670
		10 13 16 19	0.899	0.897	0.900
	30 35 40 45	10 11 12 13	0.323	0.317	0.319
		10 12 14 16	0.756	0.751	0.769
		10 13 16 19	0.943	0.944	0.956
	30 40 50 60	10 11 12 13	0.347	0.339	0.344
		10 12 14 16	0.798	0.795	0.810
		10 13 16 19	0.965	0.965	0.973

TABLE 6. Estimated power of three tests for k=5.

n	λ_1, λ_2, λ_3, λ_4,λ_5	μ_1, μ_2, μ_3, μ_4, μ_5	PB	GTV	CAT
15	30 30 30 30 30	10 11 12 13 14	0.201	0.209	0.184
		10 12 14 16 18	0.481	0.497	0.480
		10 13 16 19 21	0.671	0.693	0.678
	30 35 40 45 50	10 11 12 13 14	0.242	0.252	0.236
		10 12 14 16 18	0.583	0.599	0.597
		10 13 16 19 21	0.770	0.788	0.807
	30 40 50 60 70	10 11 12 13 14	0.276	0.291	0.270
		10 12 14 16 18	0.652	0.667	0.671
		10 13 16 19 21	0.824	0.842	0.860
30	30 30 30 30 30	10 11 12 13 14	0.408	0.413	0.400
		10 12 14 16 18	0.855	0.860	0.859
		10 13 16 19 21	0.967	0.969	0.972
	30 35 40 45 50	10 11 12 13 14	0.493	0.498	0.497
		10 12 14 16 18	0.925	0.928	0.937
		10 13 16 19 21	0.987	0.989	0.993
	30 40 50 60 70	10 11 12 13 14	0.565	0.572	0.569
		10 12 14 16 18	0.961	0.962	0.964
		10 13 16 19 21	0.995	0.995	0.997

The numerical results for estimated power of the three tests are presented as above in Table 4 to Table 6. Since the type I risks of PB approach exceed the nominal level for k=3, it appears to be more powerful than the other two tests. If both of the CAT and GTV are compared for k=3, the CAT appears to be more powerful than the GTV.

For k=4 and n=15, the PB approach appears to be more powerful than the other two tests. However, it could be emphasized that when the sample size increases i.e., n=30, the

CAT performs a bit more better than the other two tests in terms of power. Additionally, the CAT is positively affected from the increments in the differences between the scale parameters and the means of groups.

For k=5, the GTV appears to be more powerful than other tests because of its type I risks exceed the nominal level. When the differences between scale parameters and means of groups are increased, the CAT appears to be more powerful than the PB approach.

5. Conclusion

In this article, we have proposed the CAT for testing the equality of several inverse Gaussian means, especially, under heterogeneity. We have compared the CAT with some of the existing tests; the PB and GTV. For a different sample sizes and parameters, we have investigated the performance of these three tests using Monte Carlo simulation. It could be observed from the simulation results that for k=3 the PB approach seems to have type I risks exceeding the nominal level and for k=5 the GTV performs in a similar way. However, the type I risks of CAT are generally less than 0.05 for all the numbers of groups. Therefore, we could mention that the CAT is not affected from the changes in the number of groups. Furthermore, according to power comparison results, the CAT appears to be more powerful than the other tests, when the differences between scale parameters and means of groups are increased.

Consequently, in respect of our simulation study, even for comparing different number of groups (as k=3, 4 or 5), CAT could be suggested as a good alternative for testing the equality of inverse Gaussian means under heterogeneity of scale parameters.

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