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# A RECURRENCE RELATION FOR BERNOULLI NUMBERS

Ömer Küçüksakallı*

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**Abstract**

Inspired by a result of Saalschütz, we prove a recurrence relation for Bernoulli numbers. This recurrence relation has an interesting connection with real cyclotomic fields.

**Keywords:** Bernoulli numbers, cyclotomic fields, Newton's identities.

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## 1. Introduction

The Bernoulli numbers $B_n$, which can be defined by the Laurent series expansion

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j,$$

have several applications in different fields of mathematics, such as number theory, combinatorics, numerical analysis.

A classical problem in elementary number theory is to find formulas for summing the $m$-th powers of the first $n-1$ integers. The following result is obtained by Bernoulli [7, Chap. 15] and it is one of the most historical formula including Bernoulli numbers

$$\sum_{j=1}^{n-1} j^m = \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} B_j n^{m+1-j}. \tag{1.1}$$

Another historical formula including Bernoulli numbers is the Euler-Maclaurin summation formula which was found by Euler and Maclaurin independently in 1730s and used for computations in numerical analysis [2], [9]. A breakthrough in algebraic number theory is the Kummer's criterion, proved in 1850, which relates the numerator of Bernoulli numbers to the existence of integer solutions of Fermat's equation [8].

It is a powerful method to investigate Bernoulli numbers by recurrence relations. For example, in order to prove von Staudt-Clausen theorem, concerning the denominator of the Bernoulli numbers, it is important to write $B_j$ in terms of previous Bernoulli numbers

*Mathematics Department, Middle East Technical University, 06531, Ankara, Turkey Email: komer@metu.edu.tr

[12, pp. 56]. There is a vast literature concerning the recurrence relations of Bernoulli numbers. For a brief historical discussion, see Gould [5].

In this paper we prove a recurrence relation, inspired by a result of Saalschütz [10]. We achieve this by adapting his computations for the trigonometric function $\csc(x) = 1/\sin(x)$ to the function

$$f(x) = \frac{e^x}{(e^x-1)^2}.$$

As an application, we consider the relation between $f(x)$ and the real cyclotomic fields and prove a formula for odd integers $n$ which gives the sum $\sum_{j=1}^{n-1} f(2j\pi i/n)^m$ as a polynomial in $n$. This formula is an analogue of (1.1) and produces interesting recurrence relations for Bernoulli numbers via Newton's identities. Moreover it enables us to write some discrete sums in terms of residues.

## 2. A recurrence relation for Bernoulli Numbers

The trigonometric function $\csc(x) = 1/\sin(x)$ has a Laurent expansion at $x = 0$ whose coefficients are multiples of Bernoulli numbers. Moreover $\csc(x)$ satisfies the differential equation

$$\csc(x)^{m+2} = \frac{D_x^2(\csc(x)^m) + m^2\csc(x)^m}{m(m+1)}.$$

Saalschütz [10] uses this identity and obtains a recurrence relation for Bernoulli numbers. This is done by comparing the coefficients in the Laurent expansions of each function. See Agoh and Dilcher [1] for a modern revision of Saalschütz's paper.

Adapting computations of Saalschütz to our case require an identity for $f(x)$ that is similar to the differential equation of $\csc(x)$ above.

**2.1. Lemma.** *For any integer $m \geq 1$,*

$$f^{m+1} = \frac{D_x^2(f^m) - m^2 f^m}{(2m)(2m+1)}.$$

*Proof.* The chain and product rules of derivative give

$$\begin{aligned} D_x^2(f^m) &= D_x(mf^{m-1}D_x(f)) \\ &= m(m-1)f^{m-2}D_x(f)^2 + mf^{m-1}D_x^2(f). \end{aligned}$$

In order to simplify our computations, we define

$$g(x) = \frac{e^x+1}{2(e^x-1)}.$$

Note that $D_x(g) = -f$, $D_x(f) = -2fg$ and $g^2 = f + 1/4$. Using these identities, it is easy to establish $D_x(f)^2 = 4f^3 + f^2$ and $D_x^2(f) = 6f^2 + f$. Taking these polynomial expressions into account, we obtain

$$\begin{aligned} D_x^2(f^m) &= m(m-1)f^{m-2}(4f^3 + f^2) + mf^{k-1}(6f^2 + f) \\ &= (4m(m-1) + 6m)f^{m+1} + (m(m-1) + m)f^m \\ &= (2m)(2m+1)f^{m+1} + m^2 f^m. \end{aligned}$$

This finishes the proof. □

Bernoulli numbers with odd index are zero except $B_1$. To see this observe that the function $x/(e^x-1)+x/2$ is even. Therefore it is more natural to write

$$\frac{x}{e^x-1}+\frac{x}{2}=\frac{x(e^x+1)}{2(e^x-1)}=\sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}x^{2j}.$$

In order to compute the Laurent expansion of $f(x)$ at $x=0$, we first consider

$$g(x)=\frac{e^x+1}{2(e^x-1)}=\sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}x^{2j-1}.$$

Using the equality $D_x(g)=-f$, we obtain

$$f(x)=\frac{e^x}{(e^x-1)^2}=-\sum_{j=0}^{\infty}(2j-1)\frac{B_{2j}}{(2j)!}x^{2j-2}.$$

We will use the lemma above in order to find a recurrence relation for Bernoulli numbers. This will be done by comparing the coefficients in the Laurent expansions of each function. For this purpose, let us define $c_{2j}^m$ as follows

$$f^m=\sum_{j=0}^{\infty}c_{2j}^m x^{2j-2m}.$$

This definition is possible thanks to fact that $f^m$ is an even function with a pole of order $2m$ at $x=0$. Note that $(x^2f)^m=\sum_{j=0}^{\infty}c_{2j}^m x^{2j}$. Using the lemma above, we see that

$$(x^2f)^{m+1}=-\frac{x^2m^2(x^2f)^m}{(2m)(2m+1)}+\frac{x^{2m+2}D^2(f^m)}{(2m)(2m+1)}.$$

Comparing the coefficients of $x^{2j}$, we get

$$c_{2j}^{m+1}=-\frac{m^2}{(2m)(2m+1)}c_{2j-2}^m+\frac{(2j-2m)(2j-2m-1)}{(2m)(2m+1)}c_{2j}^m \tag{2.1}$$

for any integer $m\geq 1$. In particular setting $m=j$ cancels the last term in this relation and we are left with

$$c_{2m}^{m+1}=-\frac{m^2}{(2m)(2m+1)}c_{2m-2}^m.$$

We have $c_0^1=1$. Thus $c_2^2=-1/3!$ and in general

**2.2. Lemma.** *For any integer $m\geq 0$,*

$$c_{2m}^{m+1}=\frac{(-1)^m m!^2}{(2m+1)!}.$$

In order work with the recurrence relation (2.1) above, it is extremely helpful to use a normalization. Following Agoh and Dilcher [1], we define

$$C(m,k)=\frac{(2m-1)!}{(2k-1)!}c_{2m-2k}^m \tag{2.2}$$

for $m\geq k\geq 1$. We set $C(m,k)=0$ if $m<k$ or $k<1$. Observe that $C(1,1)=1$. If we put $2j=2m-2k$, then this normalization transforms (2.1) as follows

$$C(m+1,k+1)=-m^2C(m,k+1)+C(m,k). \tag{2.3}$$

Using the initial values $C(1,1)=1$ and $C(1,0)=0$, one can compute $C(m,k)$ for all $m\geq k\geq 1$. On the other hand the numbers $C(m,k)$ can be easily generated by a product as well.

**2.3. Lemma.** *For any integer $m \geq 1$,*

$$\sum_{k=0}^{m} C(m,k)x^k = \prod_{j=1}^{m}(x-(j-1)^2).$$

*Proof.* This proof is adapted from its $\csc(x)$ analogue [1, Lemma 2.1]. Note that the statement is true for $m=1$ since $C(1,1)=1$ and $C(1,0)=0$. Applying the recurrence relation (2.3) and doing further manipulations, we obtain

$$\begin{aligned}\sum_{k=0}^{m} C(m,k)x^k &= -(m-1)^2\sum_{k=0}^{m-1} C(m-1,k)x^k + \sum_{k=1}^{m} C(m-1,k-1)x^k\\ &= -(m-1)^2\sum_{k=0}^{m-1} C(m-1,k)x^k + x\sum_{k=0}^{m-1} C(m-1,k)x^k\\ &= (x-(m-1)^2)\sum_{k=0}^{m-1} C(m-1,k)x^k.\end{aligned}$$

The lemma follows by induction on $m$. □

If $n = m(m-1)/2 + k$, then $C(m,k)$ is the $n$-th term of the sequence A204579 in the On-Line Encyclopedia of Integer Sequences [11]. Before we prove our main result, we need one more fact. The following lemma enables us to write a certain coefficient in the Laurent expansion of $f^m$ as a sum of multiples of Bernoulli numbers.

**2.4. Lemma.** *For any integer $m \geq 1$,*

$$c_{2m}^m = -\frac{1}{(2m-1)!}\sum_{j=1}^{m} C(m,j)\frac{B_{2j}}{2j}.$$

*Proof.* Recall that $D_x(f) = -2gf$. In general $D_x(f^m) = -2mgf^m$ for all integers $m \geq 1$. Thus the coefficient of $x^{-1}$ in the product

$$gf^m = \left(\sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}x^{2j-1}\right)\left(\sum_{j=0}^{\infty} c_{2j}^m x^{2j-2m}\right)$$

is zero. It follows that $\sum_{j=0}^{m}\frac{B_{2j}}{(2j)!}c_{2m-2j}^m = 0$ and therefore

$$c_{2m}^m = -\sum_{j=1}^{m}\frac{B_{2j}}{(2j)!}c_{2m-2j}^m.$$

Recall that $c_{2m-2j}^m = \frac{(2j-1)!}{(2m-1)!}C(m,j)$ for $m \geq j \geq 1$ by (2.2). This finishes the proof. □

We are ready to prove our main result. This is an analogue of Saalschütz's result [10]. Note that the terms in the sum is either all positive or all negative since the sign of Bernoulli numbers $B_{2j}$ and $C(m,j)$ are both alternating.

**2.5. Theorem.** *For any integer $m \geq 1$,*

$$\sum_{j=1}^{m} C(m,j)B_{2j} = \frac{(-1)^{m+1}m!^2}{(2m+1)(2m)}.$$

*Proof.* Using the product $f \cdot f^m$, we can write

$$c_{2m}^{m+1} = \sum_{j=0}^{m} c_{2j}^1 c_{2m-2j}^m.$$

Separating the term with $j=0$ and using the fact $c_{2j}^1 = -(2j-1)\frac{B_{2j}}{(2j)!}$, we obtain

$$c_{2m}^{m+1} = c_{2m}^m - \frac{1}{(2m-1)!}\sum_{j=1}^{m}(2j-1)\frac{B_{2j}}{2j}C(m,j).$$

Using Lemma 2.4, we surprisingly see that

$$c_{2m}^{m+1} = -\frac{1}{(2m-1)!}\sum_{j=1}^{m} B_{2j}C(m,j).$$

Now our theorem follows easily from Lemma 2.2. □

Applying the recurrence relation (2.3) for $C(m,j)$, we will obtain variations of our theorem. For this purpose set

$$h_k(m) = \sum_{j=1}^{m} C(m,j)B_{2j+2k}.$$

It is easy to see that $h_0(m) = (-1)^{m+1}m!^2/((2m+1)(2m))$ by the theorem above. Now we will find a recurrence relation satisfied by $h_k(m)$'s. Since $C(m+1,m+1)$ is equal to 1 for all integer $m \geq 0$, we see that

$$B_{2m+2k+2} = h_k(m+1) - \sum_{j=1}^{m} C(m+1,j)B_{2j+2k}.$$

Applying the recurrence relation (2.3), we obtain

$$B_{2m+2k+2} = h_k(m+1) + m^2\sum_{j=1}^{m} C(m,j)B_{2j+2k} - \sum_{j=1}^{m} C(m,j-1)B_{2j+2k}.$$

The first sum is equal to $m^2h_k(m)$ and the second sum is equal to $h_{k+1}(m) - B_{2m+2k+2}$. Therefore for all integer $k \geq 0$, we have

$$h_{k+1}(m) = h_k(m+1) + m^2h_k(m). \qquad (2.4)$$

Using this new recurrence relation, we can compute

$$h_1(m) = \frac{(-1)^m(m+1)!^2}{(2m+3)(2m+2)} + m^2\frac{(-1)^{m+1}m!^2}{(2m+1)(2m)} = \frac{(-1)^m m!^2}{2(2m+1)(2m+3)}.$$

In order to find a pattern for $h_k(m)$ in general, let us use the recurrence relation (2.4) one more time and write

$$h_2(m) = \frac{(-1)^{m+1}(m+1)!^2}{2(2m+3)(2m+5)} + m^2\frac{(-1)^m m!^2}{2(2m+1)(2m+3)}.$$

We see that each time we use the recurrence relation (2.4) from now on, there will be one new factor in the denominator of $h_k(m)$, namely $(2m+2k+1)$. Moreover the numerator of $h_k(m)$ can be obtained suitably as well. To formalize the numerator, we set $r_1(m) = 1$, a constant polynomial, and define $r_k(m)$ recursively for $k \geq 1$ by

$$r_{k+1}(m) = (m+1)^2(2m+1)r_k(m+1) - m^2(2m+2k+3)r_k(m). \qquad (2.5)$$

We now state our corollary.

**2.6. Corollary.** *For all integers $m, k \geq 1$,*

$$h_k(m) = \sum_{j=1}^{m} C(m,j)B_{2j+2k} = \frac{(-1)^{m+k+1}m!^2 r_k(m)}{2(2m+1)(2m+3)\cdots(2m+2k+1)}.$$

A few examples of the polynomials $r_k(m)$ are

$$\begin{aligned}
r_1(m) &= 1,\\
r_2(m) &= 4m + 1,\\
r_3(m) &= 34m^2 + 24m + 5,\\
r_4(m) &= 496m^3 + 672m^2 + 344m + 63,\\
r_5(m) &= 11056m^4 + 24256m^3 + 22046m^2 + 9476m + 1575.
\end{aligned}$$

As a special case, we can consider this corollary with $m = 1$, and obtain

$$B_{2k+2} = \frac{(-1)^k r_k(1)}{2 \cdot 3 \cdot 5 \cdots (2k+3)}. \tag{2.6}$$

In order to find the Bernoulli number $B_{2k+2}$, we might compute $r_k(1)$ by applying the recurrence relation (2.5) $(k^2 - k)/2$ times (without actually finding $r_k(m)$). For example, in order to find

$$B_{10} = \frac{5}{66} = \frac{(-1)^4 r_4(1)}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}$$

the recurrence relation (2.5) should be applied 6 times.

$$\begin{array}{ccccccc}
r_1(1) = 1 & & r_1(2) = 1 & & r_1(3) = 1 & & r_1(4) = 1\\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \\
r_2(1) = 5 & & r_2(2) = 9 & & r_2(3)13 & & \\
\downarrow & \swarrow & \downarrow & \swarrow & & & \\
r_3(1) = 63 & & r_3(2) = 189 & & & & \\
\downarrow & \swarrow & & & & & \\
r_4(1) = 1575 & & & & & &
\end{array}$$

Even though it is theoretically possible to compute all Bernoulli numbers described as above, this is not the fastest way. One of the biggest handicap of this recursion is that it accumulates a lot of unnecessary terms which could be canceled. If we write $B_{2j} = N_{2j}/D_{2j}$ as a rational number in its lowest terms, then

$$D_{2j} = \prod_{\substack{p \text{ prime} \\ p-1 \mid 2j}} p$$

by the theorem of von Staudt-Clausen [12, pp. 56]. Note that $D_{2j}$ has fewer terms than the denominator of (2.6).

The von Staudt-Clausen theorem makes it possible to use analytic methods efficiently in order to compute the Bernoulli numbers. For instance, the classical formula

$$\zeta(2j) = -\frac{1}{2} \frac{(2\pi i)^{2j}}{(2j)!} B_{2j}, \qquad \text{for } j \geq 1. \tag{2.7}$$

has been used extensively for this purpose. For a brief history of such zeta-function algorithms and a significantly better method, see [6].

## 3. A connection with real cyclotomic fields

The function $f(x) = e^x/(e^x - 1)^2$ is closely related with real cyclotomic extensions. Let $n$ be a positive odd integer. Denote by $\zeta_n = e^{2\pi i/n}$, a primitive $n$-th root of unity. Set

$$\alpha_j = f(2j\pi i/n) = \frac{\zeta_n^j}{(\zeta_n^j - 1)^2}$$

for integers $j$ not divisible by $n$. Observe that $\alpha_j = (2\cos(2j\pi/n) - 2)^{-1}$ and it is an element of the $n$-th real cyclotomic field $\mathbf{Q}(\zeta_n) \cap \mathbf{R}$. Consider

$$p_m = \sum_{j=1}^{(n-1)/2} \alpha_j^m,$$

which can be regarded as a sum of *consecutive* powers. It is easy to see that $p_m \in \mathbf{Q}$ since it is invariant under conjugation. Note that it is the trace of $\alpha_1^m$ if $n$ is an odd prime. In order to compute $p_m$ explicitly as a polynomial in $n$, we use the following expansion

$$f(x) = \sum_{k \in \mathbf{Z}} \frac{1}{(x - 2k\pi i)^2}. \tag{3.1}$$

To justify this equality, we first observe that $f(x) = e^x/(e^x - 1)^2$ is a meromorphic function with double poles at points $x = 2k\pi i$ for integers $k$. Moreover we can use the Laurent expansion of $f(x)$ together with (2.7), and write

$$f(x) = \frac{1}{x^2} + 2\sum_{j=1}^{\infty}(2j-1)\frac{\zeta(2j)}{(2\pi i)^{2j}}x^{2j-2}$$

Putting $\zeta(2j) = \sum_{k=1}^{\infty} 1/(k^{2j})$, we see that

$$f(x) = \frac{1}{x^2} + 2\sum_{k=1}^{\infty}\frac{1}{(2k\pi i)^2}\sum_{j=1}^{\infty}(2j-1)\left(\frac{x}{2k\pi i}\right)^{2j-2}.$$

The sum on the right hand side is similar to the power series expansion of the function $(2k\pi i)^2/(x - 2k\pi i)^2$ but the odd terms are missing. Therefore we have

$$2\sum_{j=1}^{\infty}(2j-1)\left(\frac{x}{2k\pi i}\right)^{2j-2} = \frac{(2k\pi i)^2}{(x-2k\pi i)^2} + \frac{(2k\pi i)^2}{(-x-2k\pi i)^2}.$$

This finishes the proof of the equation (3.1). Now we are ready to prove our main result in this section.

**3.1. Theorem.** *For any integer $m \geq 1$,*

$$2p_m = -\frac{1}{(2m-1)!}\sum_{j=1}^{m} C(m,j)\frac{B_{2j}}{2j}(n^{2j} - 1).$$

*Proof.* Applying Lemma 2.1 recursively we see that

$$f^m(x) = \frac{1}{(2m-1)!}\sum_{j=1}^{m} C(m,j) f^{(2j-2)}(x).$$

for any integer $m \geq 1$. On the other hand, we have

$$f^{(2j-2)}(x) = \sum_{k \in \mathbf{Z}} \frac{(2j-1)!}{(x - 2k\pi i)^{2j}}$$

by equation (3.1). Observe that we can write $2p_m = \sum_{l=1}^{n-1} \alpha_l^m$ since $\alpha_l = \alpha_{n-l}$ for every integer $l$ in our range. The problem of computing $p_m$ is now reduced to evaluate the following sum explicitly

$$\sum_{l=1}^{n-1} f^{(2j-2)}(2l\pi i/n) = \sum_{l=1}^{n-1}\sum_{k \in \mathbf{Z}} \frac{(2j-1)!}{(2l\pi i/n - 2k\pi i)^{2j}}.$$

We claim that the double sum above is equal to $-\frac{B_{2j}}{2j}(n^{2j}-1)$. To see this, note that

$$2\zeta(2j)+\sum_{l=1}^{n-1}\sum_{k\in\mathbf{Z}}\frac{1}{(l/n-k)^{2j}}=2\sum_{k\in\mathbf{Z}-\{0\}}\frac{1}{(k/n)^{2j}}=2n^{2j}\zeta(2j).$$

As a result, the double sum is equal to

$$\frac{(2j-1)!}{(2\pi i)^{2j}}2\zeta(2j)(n^{2j}-1).$$

Applying (2.7), we see that our claim is true and this finishes the proof of the theorem. □

**3.1. Symmetric polynomials and Newton's identities.** Now we will use Theorem 3.1 together with the $n$-th division polynomial of $\sin(\theta)$ to obtain other interesting relations of Bernoulli numbers. For an odd integer $n$, we have

$$\frac{\sin(n\theta)}{n}=\sin(\theta)-\frac{(n^2-1)}{3!}\sin(\theta)^3+\frac{(n^2-1)(n^2-9)}{5!}\sin(\theta)^5+\dots$$

where the sum on the right hand side has $(n+1)/2$ terms. A proof of this formula can be found in [3]. Observe that $\alpha_j=-1/(4\sin(j\pi/n)^2)$. Putting $\theta=j\pi/n$ in the equation above we see that both sides vanish for each integer $j$. Set $\tilde{n}=(n-1)/2$ to ease up the notation. We have

$$\prod_{j=1}^{\tilde{n}}(x-\alpha_j)=x^{\tilde{n}}+\frac{(n^2-1)}{3!4}x^{\tilde{n}-1}+\frac{(n^2-1)(n^2-9)}{5!4^2}x^{\tilde{n}-2}+\dots.$$

The coefficients in this polynomial are given by symmetric sums

$$\begin{aligned}
s_1&=\alpha_1+\alpha_2+\dots+\alpha_{\tilde{n}},\\
s_2&=\alpha_1\alpha_2+\alpha_1\alpha_3+\dots+\alpha_2\alpha_3+\alpha_2\alpha_4+\dots+\alpha_{\tilde{n}-1}\alpha_{\tilde{n}},\\
&\ \ \vdots\\
s_{\tilde{n}}&=\alpha_1\alpha_2\cdots\alpha_{\tilde{n}}.
\end{aligned}$$

For simplicity set $s_0=1$. Then $\prod_{j=1}^{\tilde{n}}(x-\alpha_j)=\sum_{j=0}^{\tilde{n}}(-1)^j s_j x^{\tilde{n}-j}$. Thus each $s_j$ can be written as a polynomial in $n$. Moreover Newton's identities [4, Chap. 7] enable us to obtain each $s_j$ recursively from $p_j$'s. We have

$$\begin{aligned}
s_1&=s_0p_1,\\
2s_2&=s_1p_1-s_0p_2,\\
3s_3&=s_2p_1-s_1p_2+s_0p_3,\\
&\ \ \vdots\\
ms_m&=s_{m-1}p_1-s_{m-2}p_2+\dots+(-1)^{m-1}s_0p_m.
\end{aligned}\tag{3.2}$$

Since each $p_j$ is a sum of multiples of Bernoulli numbers and each $s_j$ is a polynomial in $n$, we can obtain several recurrence relations for Bernoulli numbers by comparing the coefficients of $n^{2j}$ in the general equation (3.2) for $0\le j\le m$. For example if we compare the coefficients of $n^{2m}$, we obtain

$$\frac{m}{(2m+1)!4^m}=\frac{1}{2}\sum_{j=1}^{m}\frac{B_{2j}}{(2j)!(2m-2j+1)!4^{m-j}}$$

for all integer $m \geq 1$. From this equality, it is easy to establish that

$$\sum_{j=0}^{2m} \binom{2m+1}{j} 2^j B_j = 0$$

for any integer $m \geq 1$.

We can plug in values for $n$ to obtain other relations as well. For example regarding $p_j = p_j(n)$, a function of $n$, and putting $n = 3$ in the general equation (3.2) annihilates all terms but two and gives

$$p_m(3) = (-1/3)p_{m-1}(3)$$

for $m \geq 2$. It is easy to compute that $p_1(3) = -1/3$. Thus $p_m(3) = (-1/3)^m$. As a result, we get

$$\sum_{j=1}^{m} C(m,j)\frac{B_{2j}}{2j}(3^{2j}-1) = \frac{2(-1)^{m+1}(2m-1)!}{3^m}$$

for every integer $m \geq 1$.

**3.2. Formulas for discrete sums via residues.** Consider the sum of $m$-th powers of the first $n-1$ integers

$$P_m(n) = 1^m + 2^m + \ldots + (n-1)^m.$$

Bernoulli discovered an explicit polynomial in $n$ of degree $m+1$ which is equal to $P_m(n)$. See equation (1.1). It turns out that this polynomial can be obtained by a generating function. For this purpose, consider the Bernoulli polynomials $B_n(x)$ defined by

$$\frac{xe^{nx}}{e^x - 1} = \sum_{j=0}^{\infty} B_j(n)\frac{x^j}{j!}.$$

It is a classical fact that $P_m(n) = (B_{m+1}(n) - B_{m+1})/(m+1)$. For a proof of this identity, see [7, pp. 231]. In order to express the analogy between the integer and real cyclotomic cases, we observe that $P_m(n)$ can be obtained alternatively as follows

$$P_m(n) = \text{Res}\left(\frac{(e^{nx}-1)m!}{(e^x-1)x^{m+1}}\right).$$

Similar to $P_m(n)$, the sum $2p_m = \sum_{j=1}^{n-1} \alpha_j^m$ consists of $n-1$ *consecutive* powers and equals to a polynomial in $n$, see Theorem 3.1. Another similarity is that the sum $2p_m$ can be obtained via the residue of a function.

**3.2. Corollary.** *For any integer $m \geq 1$,*

$$2p_m = \text{Res}(-g(x)f(x/n)^m).$$

*Proof.* Recall that $g(x) = \sum \frac{B_{2j}}{(2j)!}x^{2j-1}$ and $f(x)^m = \sum c_{2j}^m x^{2j-2m}$ where sums run from $j = 0$ to infinity. Thus

$$\text{Res}(-g(x)f(x/n)^m) = -\sum_{j=0}^{m} \frac{B_{2j}}{(2j)!}c_{2m-2j}^m n^{2j}.$$

In order to use the normalization (2.2), we separate the term with $j = 0$ and obtain

$$\text{Res}(-g(x)f(x/n)^m) = -c_{2m}^m - \frac{1}{(2m-1)!}\sum_{j=1}^{m} C(m,j)\frac{B_{2j}}{2j}n^{2j}.$$

Now the result follows from Lemma 2.4 and Theorem 3.1. □

This corollary (together with Theorem 3.1) enables us to see $\mathrm{Res}(-g(x)f(x/n)^m)$ as a polynomial in $n$. Differentiating with respect to $n$ would annihilate the $2j$-term appearing in the polynomial expression of $2p_m$. Using this idea, we prove a stronger version of Theorem 2.5 now.

**3.3. Corollary.** *Let $n$ be an odd positive integer. For any integer $m \geq 1$, we have*

$$\frac{1}{(2m-1)!}\sum_{j=1}^{m} C(m,j)B_{2j}n^{2j} = \mathrm{Res}((g(x) - xf(x))f(x/n)^m).$$

*Proof.* Using Theorem 3.1, we define the polynomial $Q(n) = 2p_m$ which is of degree $2m$. Differentiating the equation of Theorem 3.1 with respect to $n$, we obtain

$$D_n(Q(n)) = -\frac{1}{(2m-1)!}\sum_{j=1}^{m} C(m,j)B_{2j}n^{2j-1}.$$

Our purpose is to find a formula for $-nD_n(Q(n))$. Recall that we have

$$Q(n) = \mathrm{Res}(-g(x)f(x/n)^m)$$

by the corollary above. Since $D_x(f(x)) = -2f(x)g(x)$, we obtain

$$\mathrm{Res}(-nD_n(-g(x)f(x/n)^m)) = \mathrm{Res}(xg(x)2mg(x/n)f(x/n)^m/n).$$

Set $u = xg(x)$ and $v = f(x/n)^m$. Note that $-nD_n(Q(n)) = \mathrm{Res}(-uD_x(v))$ by the equation above. Now observe that both $u$ and $v$ are even functions of $x$ and it follows that $\mathrm{Res}(uD_x(v) + D_x(u)v) = 0$. Therefore

$$-nD_n(Q(n)) = \mathrm{Res}(D_x(u)v).$$

Since $D_x(u) = g(x) - xf(x)$, the corollary is proved. □

Observe that Theorem 2.5 is a corollary of the result above with $n = 1$. It easily follows from Lemma 2.2 together with the fact that $\mathrm{Res}(gf^m) = 0$ for every integer $m \geq 1$.

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