Hacettepe Journal of Mathematics and Statistics
Volume 42 (4) (2013), 359 – 372

A CLASS OF MULTIVARIABLE POLYNOMIALS ASSOCIATED WITH HUMBERT POLYNOMIALS

Rabia Aktaş * and Abdullah Altın †

Received 20 : 06 : 2012 : Accepted 08 : 08 : 2012

Abstract

In this paper, we present a generalization (and unification) of a class of Humbert polynomials which include well known families of Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials. We derive various families of multilateral and multilinear generating functions for these polynomials. We also obtain other miscellaneous properties of these polynomials. Furthermore, for some special cases of these polynomials, we present hypergeometric representations and give expansions of these polynomials in series of some orthogonal polynomials.

Keywords: Humbert polynomials; Chan-Chyan-Srivastava multivariable polynomials; Lagrange-Hermite multivariable polynomials; Erkus-Srivastava multivariable polynomials; Multilinear and multilateral generating function; Recurrence relation; Hypergeometric function

2000 AMS Classification: 33C45

1. Introduction

An interesting generalization of Humbert, Gegenbauer, Legendre, Tchebycheff, Pincherle and Kinney polynomials, which is called generalized Humbert polynomials, was presented by Gould [10] and it is generated by

$$(c - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n\,(m, x, y, p, c)\, t^n \tag{1.1}$$

*Ankara University, Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey. E-mail: (R. Aktaş) raktas@science.ankara.edu.tr

†Ankara University, Faculty of Science, Department of Mathematics, Tandoğan TR-06100, Ankara, Turkey. E-mail: (A. Altın) altin@science.ankara.edu.tr

where m is a positive integer and other parameters are unrestricted in general (see also [20, p. 77, 86] and [19, 21]). For $c=1,\ p=-\nu$ and $y=1$, (1.1) reduces to Humbert polynomials $h_{n,m}^{\nu}(x)$, generated by

$$(1-mxt+t^m)^{-\nu}=\sum_{n=0}^{\infty}h_{n,m}^{\nu}(x)\,t^n.$$

The polynomials $\left\{p_{n,m}^{\lambda}\right\}_{n=0}^{\infty}$ considered by Milovanović and Dordević [14, 15] are defined by

$$(1-2xt+t^m)^{-\lambda}=\sum_{n=0}^{\infty}p_{n,m}^{\lambda}(x)\,t^n \tag{1.2}$$

where m is a positive integer and $\lambda>-1/2$. For the special cases of (1.2), including Horadam polynomials, Horadam-Pethe polynomials and Gegenbauer polynomials, see [11, 12, 17]. Sinha [18] introduced the polynomials $S_n^{\nu}(x)$, generated by

$$\left[1-2xt+t^2(2x-1)\right]^{-\nu}=\sum_{n=0}^{\infty}S_n^{\nu}(x)\,t^n. \tag{1.3}$$

Pathan and Khan [16] considered a generalization of polynomials mentioned above, defined by

$$\left[c-axt+bt^m(2x-1)^d\right]^{-\nu}=\sum_{n=0}^{\infty}p_{n,m,a,b,c,d}^{\nu}(x)\,t^n. \tag{1.4}$$

In [1], Aktaş *et.al* presented a multivariable generalization of Humbert polynomials including Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials, generated by

$$\prod_{i=1}^{r}\left\{(c_i-m_ix_it+y_it^{m_i})^{-\alpha_i}\right\}=\sum_{n=0}^{\infty}P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c})t^n \tag{1.5}$$

$$\left(|m_ix_it-y_it^{m_i}|<|c_i|\ ,\ c_i\neq 0\ ,\ \alpha_i\in\mathbb{C}\ ;\ i=1,2,...,r\right)$$

where $\mathbf{x}=(x_1,...,x_r)$, $\mathbf{y}=(y_1,...,y_r)$, $\mathbf{c}=(c_1,...,c_r)$, $\mathbf{m}=(m_1,...,m_r)$. For $i=1,2,...,r$; m_i is a positive integer and other parameters are unrestricted. (1.5) yields the following explicit representation:

$$P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c})=\sum_{m_1k_1+\ldots+m_rk_r+n_1+\ldots+n_r=n}\frac{(\alpha_1)_{n_1+k_1}\ldots(\alpha_r)_{n_r+k_r}}{n_1!\ldots n_r!k_1!\ldots k_r!}c_1^{-\alpha_1-n_1-k_1}\ldots c_r^{-\alpha_r-n_r-k_r}$$

$$\times\, m_1^{n_1}\ldots m_r^{n_r}(-1)^{k_1+\ldots+k_r}x_1^{n_1}\ldots x_r^{n_r}y_1^{k_1}\ldots y_r^{k_r} \tag{1.6}$$

where, as usual, $(\lambda)_k$ denotes the Pochhammer symbol given by

$$(\lambda)_k:=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}\quad\left(k\in\mathbb{N}_0:=\{0,1,2,3,...\}\right).$$

Note that the case

$$c_i=1,\ m_i=1,\ y_i=0\ ,\ i=1,2,...,r$$

of the polynomials given by (1.5) is reduced to the Chan-Chyan-Srivastava multivariable polynomials which is a multivariable extension of the Lagrange polynomials (see [8, p.

267]), generated by [4]

$$\prod_{i=1}^{r}\left\{(1-x_it)^{-\alpha_i}\right\}=\sum_{n=0}^{\infty}\mathrm{g}_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)\,t^n$$

$$\left(\alpha_i\in\mathbb{C}\ \ (i=1,2,\dots,r)\ ;\ |t|<\min\left\{|x_1|^{-1},\dots,|x_r|^{-1}\right\}\right). \tag{1.7}$$

Getting $c_i=1,\ m_i=i,\ x_i=0,\ y_i=-x_i\ \ ,\ i=1,2,\dots,r$ in (1.5) gives the multivariable Lagrange-Hermite polynomials presented by Altın *et al.* [3]

$$\prod_{i=1}^{r}\left\{\left(1-x_it^i\right)^{-\alpha_i}\right\}=\sum_{n=0}^{\infty}h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)\,t^n$$

$$\left(\alpha_i\in\mathbb{C}\ \ (i=1,2,\dots,r)\ ;\ |t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},\dots,|x_r|^{-1/r}\right\}\right) \tag{1.8}$$

which is a multivariable generalization of the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [5, 6].

Moreover, the special case

$$c_i=1,\ x_i=0,\ y_i=-x_i\ \ ,\ i=1,2,\dots,r$$

is reduced to the Erkus-Srivastava multivariable polynomials generated by [9]

$$\prod_{i=1}^{r}\left\{(1-x_it^{m_i})^{-\alpha_i}\right\}=\sum_{n=0}^{\infty}u_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)\,t^n$$

$$\left(\alpha_i\in\mathbb{C}\ \ (i=1,2,\dots,r)\ ;\ |t|<\min\left\{|x_1|^{-1/m_1},|x_2|^{-1/m_2},\dots,|x_r|^{-1/m_r}\right\}\right), \tag{1.9}$$

where $m_i\ \ (i=1,2,\dots,r)$ are positive integers. A generalization (and unification) of various polynomials given above is provided by the definition

$$\prod_{i=1}^{r}\left\{\left(c_i-a_ix_it^{p_i}+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}\right\}=\sum_{n=0}^{\infty}P_{n,\mathbf{p},\mathbf{m},\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n$$
$$=\sum_{n=0}^{\infty}\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n$$

$$\left(\left|a_ix_it^{p_i}-b_iy_i^{d_i}t^{m_i}\right|<|c_i|\ \ ,\ c_i\neq 0\ ,\ \alpha_i\in\mathbb{C}\ \ ;\ i=1,2,\dots,r\right) \tag{1.10}$$

where $\mathbf{x}=(x_1,\dots,x_r)\ \ ,\mathbf{y}=(y_1,\dots,y_r)\,,\ \mathbf{a}=(a_1,\dots,a_r)\,,\ \mathbf{b}=(b_1,\dots,b_r)\,,\ \mathbf{c}=(c_1,\dots,c_r)\,,$ $\mathbf{d}=(d_1,\dots,d_r)\,,\ \mathbf{p}=(p_1,\dots,p_r)\,,\ \mathbf{m}=(m_1,\dots,m_r)\,.$ For $i=1,2,\dots,r;\ m_i,\ p_i$ and d_i are positive integers and other parameters are unrestricted. (1.10) yields

$$\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$=\sum_{m_1k_1+\dots+m_rk_r+p_1n_1+\dots+p_rn_r=n}\frac{(\alpha_1)_{n_1+k_1}\dots(\alpha_r)_{n_r+k_r}}{n_1!\dots n_r!k_1!\dots k_r!}c_1^{-\alpha_1-n_1-k_1}\dots c_r^{-\alpha_r-n_r-k_r}$$

$$\times\,a_1^{n_1}\dots a_r^{n_r}b_1^{k_1}\dots b_r^{k_r}\,(-1)^{k_1+\dots+k_r}\,x_1^{n_1}\dots x_r^{n_r}\,y_1^{d_1k_1}\dots y_r^{d_rk_r}. \tag{1.11}$$

In this paper, we give some basic relations for the generalized (unified) multivariable polynomials given explicitly by (1.11). We derive various families of multilinear and multilateral generating functions for $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ similar to method given in [1, 2, 3, 7, 9] and obtain several recurrence relations. We also give hypergeometric representations for some special cases of $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ and expansions of these polynomials in series of some orthogonal polynomials. Furthermore, we present some special cases of our results and give some new results for Chan-Chyan-Srivastava, Lagrange-Hermite, Erkus-Srivastava multivariable polynomials and Humbert multivariable polynomials given by (1.6).

2. Bilinear and Bilateral Generating Functions

In this section, we consider many general families of bilinear and bilateral generating functions for the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ which are generated by (1.10) and given explicitly by (1.11).

We begin by stating the following theorem.

2.1. Theorem. *For a non-vanishing function $\Omega_\mu(\mathbf{z})$ of s complex variables $z_1,\dots,z_s$ ($s\in\mathbb{N}$) and of complex order μ, let*

$$\Lambda_{\mu,\nu}(\mathbf{z};w):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(\mathbf{z})w^k \tag{2.1}$$

where $(a_k\neq 0,\ \ \mu,\nu\in\mathbb{C})$ *;* $\mathbf{z}=(z_1,\dots,z_s)$ *and*

$$\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\zeta):=\sum_{k=0}^{[n/p]}a_k\Phi_{n-pk}^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})\ \Omega_{\mu+\nu k}(\mathbf{z})\zeta^k \tag{2.2}$$

where $n,p\in\mathbb{N}$*;* $\mathbf{x}=(x_1,\dots,x_r)$*;* $\mathbf{y}=(y_1,\dots,y_r)$*. Then we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\frac{\eta}{t^p})t^n=\prod_{i=1}^{r}\left\{\left(c_i-a_ix_it^{p_i}+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}\right\}\Lambda_{\mu,\nu}(\mathbf{z};\eta) \tag{2.3}$$

provided that each member of (2.3) exists.

Proof. For convenience, let S denote the first member of the assertion (2.3) of Theorem 2.1. Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\frac{\eta}{t^p})$$

from the definition (2.2) into the left-hand side of (2.3), we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\Phi_{n-pk}^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})\Omega_{\mu+\nu k}(\mathbf{z})\eta^kt^{n-pk}. \tag{2.4}$$

Using the equality

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}A\left(k,n\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A\left(k,n+pk\right),$$

we may write

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})\Omega_{\mu+\nu k}(\mathbf{z})\eta^kt^n\\&=\sum_{n=0}^{\infty}\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(\mathbf{z})\eta^k\\&=\prod_{i=1}^{r}\left\{\left(c_i-a_ix_it^{p_i}+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}\right\}\Lambda_{\mu,\nu}(\mathbf{z};\eta),\end{aligned}$$

which completes the proof. □

In a similar manner, we can easily prove the following result.

2.2. Theorem. *Corresponding to an identically non-vanishing function* $\Omega_\mu(\mathbf{z})$ *of* s *complex variables* $z_1,...,z_s$ $(s\in\mathbb{N})$ *and for* $p\in\mathbb{N}$, $\mu,\nu\in\mathbb{C}$, $\mathbf{z}=(z_1,...,z_s)$, $\alpha:=(\alpha_1,...,\alpha_r)$, $\beta:=(\beta_1,...,\beta_r)$, *let*

$$\Lambda^{n,p}_{\mu,\nu,\alpha,\beta}(\mathbf{x},\mathbf{y};\mathbf{z};w):=\sum_{k=0}^{[n/p]}a_k\Phi^{(\alpha_1+\beta_1,...,\alpha_r+\beta_r)}_{n-pk}(\mathbf{x},\mathbf{y})\Omega_{\mu+\nu k}(\mathbf{z})w^k \tag{2.5}$$

where $a_k\neq 0$; $n,k\in\mathbb{N}_0$; $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. *Then*

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l\Phi^{(\alpha_1,...,\alpha_r)}_{n-k}(\mathbf{x},\mathbf{y})\Phi^{(\beta_1,...,\beta_r)}_{k-pl}(\mathbf{x},\mathbf{y})\ \Omega_{\mu+\nu l}(\mathbf{z})w^l\\ &=\Lambda^{n,p}_{\mu,\nu,\alpha,\beta}(\mathbf{x},\mathbf{y};\mathbf{z};w)\end{aligned} \tag{2.6}$$

provided that each member of (2.6) exists.

3. Special cases and miscellaneous properties

When the multivariable function $\Omega_{\mu+\nu k}(\mathbf{z})$, $\mathbf{z}=(z_1,...,z_s)$, $k\in\mathbb{N}_0$, $s\in\mathbb{N}$ is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. For example, if we set

$$s=r\quad\text{and}\quad\Omega_{\mu+\nu k}(\mathbf{z})=u^{(\beta_1,...,\beta_r)}_{\mu+\nu k}(\mathbf{z})$$

in Theorem 2.1, where Erkus-Srivastava multivariable polynomials

$$u^{(\alpha_1,...,\alpha_r)}_n(\mathbf{x})$$

are generated by (1.9), then we obtain the following result which provides a class of bilateral generating functions for Erkus-Srivastava multivariable polynomials and the generalized (unified) multivariable polynomials $\Phi^{(\alpha_1,...,\alpha_r)}_n(\mathbf{x},\mathbf{y})$ given explicitly by (1.11).

3.1. Corollary. *If* $\Lambda_{\mu,\nu}(\mathbf{z};w):=\sum_{k=0}^{\infty}a_k u^{(\beta_1,...,\beta_r)}_{\mu+\nu k}(\mathbf{z})w^k$, $a_k\neq 0$, $\mu,\nu\in\mathbb{C}$, $\mathbf{z}=(z_1,...,z_r)$ *and*

$$\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\zeta):=\sum_{k=0}^{[n/p]}a_k\Phi^{(\alpha_1,...,\alpha_r)}_{n-pk}(\mathbf{x},\mathbf{y})u^{(\beta_1,...,\beta_r)}_{\mu+\nu k}(\mathbf{z})\zeta^k$$

where $n\in\mathbb{N}_0$; $p\in\mathbb{N}$; $\mathbf{x}=(x_1,...,x_r)$; $\mathbf{y}=(y_1,...,y_r)$, *then*

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}(\mathbf{x},\mathbf{y};\mathbf{z};\frac{\eta}{t^p})t^n=\prod_{i=1}^{r}\left\{\left(c_i-a_ix_it^{p_i}+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}\right\}\Lambda_{\mu,\nu}(\mathbf{z};\eta) \tag{3.1}$$

provided that each member of (3.1) exists.

3.2. Remark. Using the generating relation (1.9) for Erkus-Srivastava multivariable polynomials and getting $a_k=1$, $\mu=0$, $\nu=1$, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\Phi^{(\alpha_1,...,\alpha_r)}_{n-pk}(\mathbf{x},\mathbf{y})u^{(\beta_1,...,\beta_r)}_k(\mathbf{z})\,\eta^k t^{n-pk}\\ &=\prod_{i=1}^{r}\left\{\left(c_i-a_ix_it^{p_i}+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}(1-z_i\eta^{n_i})^{-\beta_i}\right\},\\ &\left(|\eta|<\min\left\{|z_1|^{-1/n_1},...,|z_r|^{-1/n_r}\right\}\ ;\ \left|a_ix_it^{p_i}-b_iy_i^{d_i}t^{m_i}\right|<|c_i|\quad;\ i=1,2,...,r\right)\end{aligned}$$

where n_i, m_i, p_i and d_i $(i = 1, 2, ..., r)$ are positive integers.

By choosing $s = 2r$ and $\Omega_{\mu+\nu k}(\mathbf{z}) = \Phi_{\mu+\nu k}^{(\gamma_1,\dots,\gamma_r)}(\mathbf{t},\omega)$, $\mu,\nu \in \mathbb{N}_0$, $\mathbf{t} = (t_1, ..., t_r)$, $\omega = (\omega_1, ..., \omega_r)$ in Theorem 2.2, we obtain the following class of bilinear generating functions for the polynomials $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ given explicitly by (1.11).

3.3. Corollary. *Let*

$$\Xi^{n,p}_{\mu,\nu,\alpha,\beta,\gamma}(\mathbf{x},\mathbf{y};\mathbf{t},\omega;w) := \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha_1+\beta_1,\dots,\alpha_r+\beta_r)}(\mathbf{x},\mathbf{y})\Phi_{\mu+\nu k}^{(\gamma_1,\dots,\gamma_r)}(\mathbf{t},\omega)w^k$$

where $a_k \neq 0$; $p \in \mathbb{N}; n, k, \mu, \nu \in \mathbb{N}_0; \alpha := (\alpha_1, ..., \alpha_r)$; $\beta := (\beta_1, ..., \beta_r)$; $\gamma := (\gamma_1, ..., \gamma_r)$. *Then*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l \Phi_{n-k}^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})\Phi_{k-pl}^{(\beta_1,\dots,\beta_r)}(\mathbf{x},\mathbf{y})\Phi_{\mu+\nu l}^{(\gamma_1,\dots,\gamma_r)}(\mathbf{t},\omega)w^l = \Xi^{n,p}_{\mu,\nu,\alpha,\beta,\gamma}(\mathbf{x},\mathbf{y};\mathbf{t},\omega;w) \tag{3.2}$$

provided that each member of (3.2) exists.

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu k}(\mathbf{z})$, $\mathbf{z} = (z_1, ..., z_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.1 and Theorem 2.2 can be applied to derive various families of multilinear and multilateral generating functions for the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ given explicitly by (1.11).

We now give some further properties of the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ given by (1.11). The generating function (1.10) yields the following addition formula for these multivariable polynomials:

$$\Phi_n^{(\alpha_1+\beta_1,\dots,\alpha_r+\beta_r)}(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n} \Phi_{n-k}^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})\Phi_k^{(\beta_1,\dots,\beta_r)}(\mathbf{x},\mathbf{y}).$$

On the other hand, the polynomials $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$ satisfy the following differential equation:

$$\sum_{j=1}^{r}\left(p_j x_j \frac{\partial}{\partial x_j} + \frac{m_j}{d_j} y_j \frac{\partial}{\partial y_j}\right)\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y}) = n\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y}). \tag{3.3}$$

If we differentiate each member of the generating function (1.10) with respect to x_j and y_j $(j = 1, 2, ..., r)$, we arrive at the following (differential) recurrence relations for $\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y})$, respectively:

$$\frac{\partial}{\partial x_j}\Phi_n^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{\left[\frac{n-p_j}{p_j}\right]}\sum_{l=0}^{\left[\frac{n-(k+1)p_j}{m_j}\right]} \frac{(-1)^l\,(k+l)!\alpha_j b_j^l\,(a_j)^{k+1}}{k!l!c_j^{k+l+1}} x_j^k y_j^{ld_j} \times \Phi_{n-lm_j-(k+1)p_j}^{(\alpha_1,\dots,\alpha_r)}(\mathbf{x},\mathbf{y}) \tag{3.4}$$

for $n \geq p_j$, and

$$\frac{\partial}{\partial y_j}\Phi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y}) = -\sum_{l=0}^{\left[\frac{n-m_j}{m_j}\right]}\sum_{k=0}^{\left[\frac{n-(l+1)m_j}{p_j}\right]}\frac{(-1)^l\,(k+l)!\alpha_j d_j a_j^k b_j^{l+1}}{k!l!c_j^{k+l+1}}x_j^k y_j^{d_j(l+1)-1} \times \Phi_{n-(l+1)m_j-kp_j}^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y}) \tag{3.5}$$

where $n \geq m_j$, and m_j, p_j and d_j $(j = 1, 2, ..., r)$ are positive integers. By applying (3.3), (3.4) and (3.5), we can easily derive the following recurrence relation for the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$ given explicitly by (1.11):

$$\sum_{j=1}^{r}\sum_{k=0}^{\left[\frac{n-p_j}{p_j}\right]}\sum_{l=0}^{\left[\frac{n-(k+1)p_j}{m_j}\right]}\frac{(-1)^l\,(k+l)!p_j\alpha_j b_j^l a_j^{k+1}}{k!l!c_j^{k+l+1}}x_j^{k+1}y_j^{ld_j}\Phi_{n-lm_j-(k+1)p_j}^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$-\sum_{j=1}^{r}\sum_{l=0}^{\left[\frac{n-m_j}{m_j}\right]}\sum_{k=0}^{\left[\frac{n-(l+1)m_j}{p_j}\right]}\frac{(-1)^l\,(k+l)!\alpha_j m_j a_j^k b_j^{l+1}}{k!l!c_j^{k+l+1}}x_j^k y_j^{d_j(l+1)}\Phi_{n-(l+1)m_j-kp_j}^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$= n\Phi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$$

where $n \geq \max\{p_j, m_j\}$ $(j = 1, 2, ..., r)$ and, m_j and p_j $(j = 1, 2, ..., r)$ are positive integers.

4. Hypergeometric Representations for the special cases of $\Phi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$

In this section, we give a hypergeometric representation for the special case $p_i = 1$ $(i = 1, 2, ..., r)$ of the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$ generated by (1.10). In this special case, we denote the polynomials $\Phi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$ by $\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$. From (1.10), the polynomials $\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$ are generated by

$$\prod_{i=1}^{r}\left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i}\right)^{-\alpha_i} = \sum_{n=0}^{\infty}\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n$$
$$\left(\left|a_i x_i t - b_i y_i^{d_i} t^{m_i}\right| < |c_i|\ ,\ c_i \neq 0\ ,\ \alpha_i \in \mathbb{C}\ ;\ i = 1, 2, ..., r\right) \tag{4.1}$$

The generalized hypergeometric function ${}_pF_q$ is defined by [17]

$${}_pF_q\left[\begin{array}{ccc}\alpha_1, ..., \alpha_p; & & \\ & & z \\ \beta_1, ..., \beta_q; & & \end{array}\right] = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k}\frac{z^k}{k!}, \tag{4.2}$$

from which, we can give the following hypergeometric representation for $\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$.

4.1. Theorem. *The polynomials* $\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$ *have the following hypergeometric representation*

$$\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y}) = \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \frac{(\alpha_i)_{n_i} (a_i x_i)^{n_i}}{n_i! c_i^{n_i+\alpha_i}}$$

$$\times_{m_i} F_{m_i-1} \left[\begin{array}{l} -\frac{n_i}{m_i}, \frac{-n_i+1}{m_i}, \ldots, \frac{-n_i+m_i-1}{m_i}; \\ \frac{-\alpha_i-n_i+1}{m_i-1}, \frac{-\alpha_i-n_i+2}{m_i-1}, \ldots, \frac{-\alpha_i-n_i+m_i-1}{m_i-1}; \end{array} \frac{m_i^{m_i} b_i c_i^{m_i-1} y_i^{d_i}}{(a_i x_i)^{m_i} (m_i-1)^{m_i-1}} \right] \tag{4.3}$$

where $m_i \geq 2 \ (i = 1, 2, \ldots, r)$.

Proof. With the help of the result

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{n!},$$

from (4.1), we get

$$\prod_{i=1}^{r} \left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} =$$

$$\prod_{i=1}^{r} \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{n_i} \frac{c_i^{-\alpha_i-n_i} (\alpha_i)_{n_i}}{(n_i-k_i)! k_i!} (a_i x_i t)^{n_i-k_i} \left(-b_i y_i^{d_i} t^{m_i} \right)^{k_i} \right\}.$$

Replacing n_i by $n_i + k_i$ $(i = 1, 2, \ldots, r)$, we have

$$\prod_{i=1}^{r} \left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} =$$

$$= \prod_{i=1}^{r} \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{\infty} \frac{c_i^{-\alpha_i-n_i-k_i} (\alpha_i)_{n_i+k_i}}{n_i! k_i!} (a_i x_i t)^{n_i} \left(-b_i y_i^{d_i} t^{m_i} \right)^{k_i} \right\} \tag{4.4}$$

Getting $n_i - m_i k_i$ instead of n_i $(i = 1, 2, \ldots, r)$ in (4.4) gives

$$\prod_{i=1}^{r} \left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i}$$

$$= \prod_{i=1}^{r} \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{[n_i/m_i]} \frac{c_i^{-\alpha_i-n_i+(m_i-1)k_i} (\alpha_i)_{n_i-(m_i-1)k_i}}{(n_i-m_i k_i)! k_i!} (a_i x_i)^{n_i-m_i k_i} \left(-b_i y_i^{d_i} \right)^{k_i} t^{n_i} \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+\ldots+n_r=n} \left\{ \prod_{i=1}^{r} \sum_{k_i=0}^{[n_i/m_i]} \frac{c_i^{-\alpha_i-n_i+(m_i-1)k_i} (\alpha_i)_{n_i-(m_i-1)k_i}}{(n_i-m_i k_i)! k_i!} (a_i x_i)^{n_i-m_i k_i} \left(-b_i y_i^{d_i} \right)^{k_i} \right\} t^n.$$

Since it is known that [17, p.58(2)]

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \quad , \quad 0 \leq k \leq n,$$

using

$$(n-mk)! = \frac{(-1)^{mk} n!}{(-n)_{mk}} \quad , \quad 0 \leq mk \leq n,$$

$$(-n)_{mk} = m^{mk} \prod_{s=1}^{m} \left(\frac{-n+s-1}{m} \right)_k$$

and

$$(1-\alpha-n)_{(m-1)k}=(m-1)^{(m-1)k}\prod_{p=1}^{m-1}\left(\frac{-\alpha-n+p}{m-1}\right)_k \quad , \quad k=0,1,2,...,$$

we find

$$\sum_{n=0}^{\infty}\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n=\prod_{i=1}^{r}\left(c_i-a_ix_it+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+\ldots+n_r=n}\frac{(\alpha_1)_{n_1}\ldots(\alpha_r)_{n_r}(a_1x_1)^{n_1}\ldots(a_rx_r)^{n_r}}{c_1^{n_1+\alpha_1}\ldots c_r^{n_r+\alpha_r}n_1!\ldots n_r!}$$

$$\times\left\{\prod_{i=1}^{r}\sum_{k_i=0}^{[n_i/m_i]}\frac{\left(\frac{-n_i}{m_i}\right)_{k_i}\left(\frac{-n_i+1}{m_i}\right)_{k_i}\cdots\left(\frac{-n_i+m_i-1}{m_i}\right)_{k_i}}{\left(\frac{-\alpha_i-n_i+1}{m_i-1}\right)_{k_i}\left(\frac{-\alpha_i-n_i+2}{m_i-1}\right)_{k_i}\cdots\left(\frac{-\alpha_i-n_i+m_i-1}{m_i-1}\right)_{k_i}k_i!}\right.$$

$$\left.\times\left(\frac{m_i^{m_i}b_ic_i^{m_i-1}y_i^{d_i}}{(a_ix_i)^{m_i}(m_i-1)^{m_i-1}}\right)^{k_i}\right\}t^n.$$

Considering (4.2) and then comparing the coefficients of t^n from both sides, we obtain the desired hypergeometric representation. □

We can choose some special cases of this theorem here.

4.2. Corollary. *If we set $a_i=m_i,\ b_i=d_i=1\ (i=1,2,...,r)$ in (4.3), we get the following hypergeometric representation of the multivariable Humbert polynomials $P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c})$ given explicitly by (1.6)*

$$P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c})=\sum_{n_1+\ldots+n_r=n}\prod_{i=1}^{r}\frac{(\alpha_i)_{n_i}(m_ix_i)^{n_i}}{n_i!c_i^{n_i+\alpha_i}}$$

$$\times_{m_i}F_{m_i-1}\left[\begin{array}{c}-\frac{n_i}{m_i},\frac{-n_i+1}{m_i},...,\frac{-n_i+m_i-1}{m_i};\\ \\ \frac{-\alpha_i-n_i+1}{m_i-1},\frac{-\alpha_i-n_i+2}{m_i-1},...,\frac{-\alpha_i-n_i+m_i-1}{m_i-1};\end{array}\frac{c_i^{m_i-1}y_i}{x_i^{m_i}(m_i-1)^{m_i-1}}\right]. \tag{4.5}$$

4.3. Remark. For $r=1$ and $\alpha_1=-p$, (4.5) is reduced to the hypergeometric representation for the generalized Humbert polynomials $P_n(m,x,y,p,c)$

$$P_n(m,x,y,p,c)=\frac{(-p)_n(mx)^n}{n!c^{n-p}}$$

$$\times_{m}F_{m-1}\left[\begin{array}{c}-\frac{n}{m},\frac{-n+1}{m},...,\frac{-n+m-1}{m};\\ \\ \frac{p-n+1}{m-1},\frac{p-n+2}{m-1},...,\frac{p-n+m-1}{m-1};\end{array}\frac{c^{m-1}y}{x^m(m-1)^{m-1}}\right]. \tag{4.6}$$

4.4. Remark. If we get $c=1$, $y=1$ and $p=-\nu$ in (4.6), we have the following hypergeometric representation of Humbert polynomials $h_{n,m}^{\nu}(x)$

$$h_{n,m}^{\nu}(x)=\frac{(\nu)_n(mx)^n}{n!}$$

$$\times_{m}F_{m-1}\left[\begin{array}{c}-\frac{n}{m},\frac{-n+1}{m},...,\frac{-n+m-1}{m};\\ \\ \frac{-\nu-n+1}{m-1},\frac{-\nu-n+2}{m-1},...,\frac{-\nu-n+m-1}{m-1};\end{array}\frac{1}{x^m(m-1)^{m-1}}\right] \tag{4.7}$$

which is a known result (see [16]).

4.5. Remark. For $m = 2$, (4.7) gives the following hypergeometric representation for Gegenbauer polynomials [16]

$$C_n^{\nu}(x) = \frac{(\nu)_n (2x)^n}{n!} \; {}_2F_1 \left[\begin{array}{c} -\frac{n}{2}, \frac{-n+1}{2}; \\ 1-\nu-n; \end{array} \; \frac{1}{x^2} \right]$$

which is a generalization of a known result [17, p. 166(4)].

4.6. Remark. Setting $r = 1$, $y_1 = 2x - 1$ and $\alpha_1 = \nu$ in (4.3), we have the following known result [16] for the polynomials $p^{\nu}_{n,m,a,b,c,d}(x)$ given by (1.4)

$$p^{\nu}_{n,m,a,b,c,d}(x) = \frac{(\nu)_n (ax)^n}{n! c^{n+\nu}}$$
$$\times_m F_{m-1} \left[\begin{array}{c} -\frac{n}{m}, \frac{-n+1}{m}, ..., \frac{-n+m-1}{m}; \\ \frac{-\nu-n+1}{m-1}, \frac{-\nu-n+2}{m-1}, ..., \frac{-\nu-n+m-1}{m-1}; \end{array} \; \frac{m^m bc^{m-1}(2x-1)^d}{(ax)^m (m-1)^{m-1}} \right]$$

which gives the hypergeometric representation for $S_n^{\nu}(x)$ for $b = c = d = 1$ and $a = m = 2$ [18, p. 442(12)].

5. Series expansions for the special cases of $\Phi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$

We give expansions of $\Psi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$, which is the special case $p_i = 1$ $(i = 1, 2, ..., r)$ of $\Phi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$ given explicitly by (1.11), in series of some orthogonal polynomials.

5.1. Theorem. *Some expansions of $\Psi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$ in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials are given by*

$$\Psi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$= \sum_{n_1+...+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]} \frac{(2n_i - 2m_ik_i - 4s_i + 1)(\alpha_i)_{n_i-(m_i-1)k_i}}{k_i! s_i! (3/2)_{n_i-m_ik_i-s_i}} \right.$$
$$\left. \times c_i^{-\alpha_i-n_i+(m_i-1)k_i} \left(-b_i y_i^{d_i}\right)^{k_i} P_{n_i-m_ik_i-2s_i}\left(\frac{a_ix_i}{2}\right) \right\},$$

$$\Psi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$= \sum_{n_1+...+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]} \frac{(\nu_i + n_i - m_ik_i - 2s_i)(\alpha_i)_{n_i-(m_i-1)k_i}}{k_i! s_i! (\nu_i)_{n_i-m_ik_i-s_i+1}} \right.$$
$$\left. \times c_i^{-\alpha_i-n_i+(m_i-1)k_i} \left(-b_i y_i^{d_i}\right)^{k_i} C^{\nu_i}_{n_i-m_ik_i-2s_i}\left(\frac{a_ix_i}{2}\right) \right\},$$

$$\Psi_n^{(\alpha_1,...,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$= \sum_{n_1+...+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]} \frac{(\alpha_i)_{n_i-(m_i-1)k_i}}{k_i! s_i! (n_i - m_ik_i - 2s_i)!} \right.$$
$$\left. \times c_i^{-\alpha_i-n_i+(m_i-1)k_i} \left(-b_i y_i^{d_i}\right)^{k_i} H_{n_i-m_ik_i-2s_i}\left(\frac{a_ix_i}{2}\right) \right\}$$

and

$$\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$$
$$=\sum_{n_1+\ldots+n_r=n}\prod_{i=1}^{r}\left\{\sum_{k_i=0}^{[n_i/m_i]}\sum_{s_i=0}^{n_i-m_ik_i}\frac{(\alpha_i)_{n_i-(m_i-1)k_i}(\beta_i+1)_{n_i-m_ik_i}}{k_i!\,(n_i-m_ik_i-s_i)!\,(\beta_i+1)_{s_i}}\right.$$
$$\left.\times 2^{n_i-m_ik_i}(-1)^{s_i}c_i^{-\alpha_i-n_i+(m_i-1)k_i}\left(-b_iy_i^{d_i}\right)^{k_i}L_{s_i}^{(\beta_i)}\left(\frac{a_ix_i}{2}\right)\right\}.$$

Proof. From (4.4), we have

$$\sum_{n=0}^{\infty}\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n$$
$$=\prod_{i=1}^{r}\left(c_i-a_ix_it+b_iy_i^{d_i}t^{m_i}\right)^{-\alpha_i}$$
$$=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\sum_{k_i=0}^{\infty}\frac{c_i^{-\alpha_i-n_i-k_i}(\alpha_i)_{n_i+k_i}}{n_i!k_i!}(a_ix_it)^{n_i}\left(-b_iy_i^{d_i}t^{m_i}\right)^{k_i}\right\}.$$

Using the result [17, p. 181]

$$\frac{(ax)^n}{n!}=\sum_{s=0}^{[n/2]}\frac{(2n-4s+1)\,P_{n-2s}(ax/2)}{s!\,(3/2)_{n-s}},$$

we can write

$$\sum_{n=0}^{\infty}\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n$$
$$=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\sum_{k_i=0}^{\infty}\sum_{s_i=0}^{[n_i/2]}\frac{(2n_i-4s_i+1)\,c_i^{-\alpha_i-n_i-k_i}(\alpha_i)_{n_i+k_i}}{s_i!k_i!\,(3/2)_{n_i-s_i}}\times\right.$$
$$\left.\times P_{n_i-2s_i}\left(\frac{a_ix_i}{2}\right)\left(-b_iy_i^{d_i}t^{m_i}\right)^{k_i}t^{n_i}\right\}.$$

Replacing n_i by $n_i-m_ik_i$ in the last equality, we get

$$\sum_{n=0}^{\infty}\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})t^n$$
$$=\prod_{i=1}^{r}\left\{\sum_{n_i=0}^{\infty}\sum_{k_i=0}^{[n_i/m_i]}\sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]}\frac{(2n_i-2m_ik_i-4s_i+1)\,c_i^{-\alpha_i-n_i+(m_i-1)k_i}}{s_i!k_i!\,(3/2)_{n_i-m_ik_i-s_i}}\right.$$
$$\left.\times(\alpha_i)_{n_i-(m_i-1)k_i}P_{n_i-m_ik_i-2s_i}\left(\frac{a_ix_i}{2}\right)\left(-b_iy_i^{d_i}\right)^{k_i}t^{n_i}\right\}$$
$$=\sum_{n=0}^{\infty}\sum_{n_1+\ldots+n_r=n}\left\{\prod_{i=1}^{r}\sum_{k_i=0}^{[n_i/m_i]}\sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]}\frac{(2n_i-2m_ik_i-4s_i+1)}{s_i!k_i!\,(3/2)_{n_i-m_ik_i-s_i}}\right.$$
$$\left.\times c_i^{-\alpha_i-n_i+(m_i-1)k_i}(\alpha_i)_{n_i-(m_i-1)k_i}P_{n_i-m_ik_i-2s_i}\left(\frac{a_ix_i}{2}\right)\left(-b_iy_i^{d_i}\right)^{k_i}\right\}t^n.$$

If we compare the coefficients of t^n from the both sides, we find the desired relation.

In a similar manner, in (4.4), using the following results respectively [17, p. 283 (36), p. 194 (4), p. 207 (2)]

$$\frac{(ax)^n}{n!} = \sum_{k=0}^{[n/2]} \frac{(\nu+n-2k)\, C_{n-2k}^{\nu}(ax/2)}{k!\,(\nu)_{n+1-k}},$$

$$\frac{(ax)^n}{n!} = \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(ax/2)}{k!\,(n-2k)!}$$

and

$$\frac{(ax)^n}{n!} = 2^n \sum_{k=0}^{n} \frac{(-1)^k (\alpha+1)_n\, L_k^{(\alpha)}(ax/2)}{(n-k)!\,(\alpha+1)_k},$$

we can easily give the other expansions of $\Psi_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x},\mathbf{y})$ in series of Gegenbauer, Hermite and Laguerre polynomials. □

We can give some special cases of this theorem.

5.2. Corollary. *Setting $a_i = m_i$, $b_i = d_i = 1$ $(i = 1,2,...,r)$ in Theorem 5.1, for the multivariable Humbert polynomials $P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c})$ given explicitly by (1.6), expansions in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials are given by*

$$P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c}) = \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]} \frac{(2n_i-2m_ik_i-4s_i+1)\,(\alpha_i)_{n_i-(m_i-1)k_i}}{k_i!s_i!\,(3/2)_{n_i-m_ik_i-s_i}} \times c_i^{-\alpha_i-n_i+(m_i-1)k_i} (-y_i)^{k_i} P_{n_i-m_ik_i-2s_i}\left(\frac{m_ix_i}{2}\right)\right\},$$

$$P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c}) = \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]} \frac{(\nu_i+n_i-m_ik_i-2s_i)\,(\alpha_i)_{n_i-(m_i-1)k_i}}{k_i!s_i!\,(\nu_i)_{n_i-m_ik_i-s_i+1}} \times c_i^{-\alpha_i-n_i+(m_i-1)k_i} (-y_i)^{k_i} C_{n_i-m_ik_i-2s_i}^{\nu_i}\left(\frac{m_ix_i}{2}\right)\right\},$$

$$P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c}) = \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{\left[\frac{n_i-m_ik_i}{2}\right]} \frac{(\alpha_i)_{n_i-(m_i-1)k_i}}{k_i!s_i!\,(n_i-m_ik_i-2s_i)!} \times c_i^{-\alpha_i-n_i+(m_i-1)k_i} (-y_i)^{k_i} H_{n_i-m_ik_i-2s_i}\left(\frac{m_ix_i}{2}\right)\right\}$$

and

$$P_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{m},\mathbf{x},\mathbf{y},\mathbf{c}) = \sum_{n_1+\ldots+n_r=n} \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{n_i-m_ik_i} \frac{(\alpha_i)_{n_i-(m_i-1)k_i}\,(\beta_i+1)_{n_i-m_ik_i}}{k_i!\,(n_i-m_ik_i-s_i)!\,(\beta_i+1)_{s_i}} \times 2^{n_i-m_ik_i} (-1)^{s_i} c_i^{-\alpha_i-n_i+(m_i-1)k_i} (-y_i)^{k_i} L_{s_i}^{(\beta_i)}\left(\frac{m_ix_i}{2}\right)\right\}.$$

5.3. Remark. If we get $r=1$ and $\alpha_1=-p$ in Corollary 5.2, we have similar results for the generalized Humbert polynomials $P_n\left(m,x,y,p,c\right)$.

5.4. Corollary. *Setting $r=1$, $y_1=2x-1$ and $\alpha_1=\nu$ in Theorem 5.1, then we get the known results for the polynomials $p^{\nu}_{n,m,a,b,c,d}\left(x\right)$ given by (1.4)* [16].

Similar to Theorem 5.1, we can give series expansions for Chan-Chan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials.

5.5. Theorem. *Some expansions of Erkus-Srivastava multivariable polynomials in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials are as follows*

$$u_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{m_1n_1+\ldots+m_rn_r=n}\prod_{i=1}^{r}\sum_{k_i=0}^{\left[\frac{n_i}{2}\right]}\frac{(2n_i-4k_i+1)\left(\alpha_i\right)_{n_i}}{k_i!\left(3/2\right)_{n_i-k_i}}P_{n_i-2k_i}\left(\frac{x_i}{2}\right),$$

$$u_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{m_1n_1+\ldots+m_rn_r=n}\prod_{i=1}^{r}\sum_{k_i=0}^{\left[\frac{n_i}{2}\right]}\frac{(\nu_i+n_i-2k_i)\left(\alpha_i\right)_{n_i}}{k_i!\left(\nu_i\right)_{n_i-k_i+1}}C^{\nu_i}_{n_i-2k_i}\left(\frac{x_i}{2}\right),$$

$$u_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{m_1n_1+\ldots+m_rn_r=n}\prod_{i=1}^{r}\sum_{k_i=0}^{\left[\frac{n_i}{2}\right]}\frac{\left(\alpha_i\right)_{n_i}}{k_i!\left(n_i-2k_i\right)!}H_{n_i-2k_i}\left(\frac{x_i}{2}\right)$$

and

$$u_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = \sum_{m_1n_1+\ldots+m_rn_r=n}\prod_{i=1}^{r}\sum_{k_i=0}^{n_i}\frac{(-1)^{k_i}\,2^{n_i}\left(\alpha_i\right)_{n_i}\left(\beta_i+1\right)_{n_i}}{\left(n_i-k_i\right)!\left(\beta_i+1\right)_{k_i}}L^{(\beta_i)}_{k_i}\left(\frac{x_i}{2}\right).$$

5.6. Corollary. *In Theorem 5.5, getting $m_i=i$ and $m_i=1$ $(i=1,2,...,r)$ respectively gives the similar results for Lagrange-Hermite and Chan-Chyan-Srivastava multivariable polynomials.*

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