Hacettepe Journal of Mathematics and Statistics
Volume 42 (4) (2013), 373 – 379

A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Received 25 : 06 : 2012 : Accepted 18 : 02 : 2013

Abstract

In the present paper, we obtain coefficient estimates and distortion and growth theorems for certain subclass of close-to-convex functions. The results presented here contain those given in earlier works as in some special cases.

Keywords: Close-to-convex functions, Starlike functions, Distortion theorems, Growth theorems, Subordinations.

2000 AMS Classification: 30C45

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{S}^*$ denote the usual subclasses of $\mathcal{A}$ whose members are univalent, close-to-convex and starlike in $\mathbb{U}$, respectively. By $\mathcal{S}^*(\alpha)$, we also denote the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$.

For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write as:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 ,$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

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In particular, if the function g is univalent in $\mathbb{U}$, then $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$(cf. [1]) if and only if

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Recently, Kowalczyk et al. [4] discussed a class $K_s(\gamma)$ of analytic functions related to the starlike functions: A function $f(z) \in \mathcal{A}$ is said to be in the class $K_s(\gamma)$ if it satisfies the inequality:

$$\operatorname{Re}\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma \qquad (0 \le \gamma < 1;\ z \in \mathbb{U}),$$

where $g(z) \in \mathcal{S}^*(1/2)$.

By simple calculations, we see that the above inequality is equivalent to

$$\left|\frac{z^2 f'(z)}{g(z)g(-z)} + 1\right| < \left|\frac{z^2 f'(z)}{g(z)g(-z)} - 1 + 2\gamma\right| \qquad (0 \le \gamma < 1;\ z \in \mathbb{U}).$$

Motivated by the class $K_s(\gamma)$, we introduce a new class $K_s^{(k)}(\gamma, \alpha, \beta)$ of analytic functions related to starlike functions as follows:

1.1. Definition. Let $K_s^{(k)}(\gamma, \alpha, \beta)$ denote the class of functions in $\mathcal{A}$ satisfying the inequality:

$$\left|\frac{z^k f'(z)}{g_k(z)} - 1\right| < \beta \left|\frac{\alpha z^k f'(z)}{g_k(z)} + 1 - (1+\alpha)\gamma\right| \tag{1.2}$$

$$(0 \le \alpha \le 1;\ 0 < \beta \le 1;\ 0 \le \gamma < 1;\ z \in \mathbb{U}),$$

where $g_k(z)$ is defined by

$$g_k(z) = \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^{\nu} z) \qquad \left(\varepsilon^k = 1;\ g(z) \in \mathcal{S}^*\left(\frac{k-1}{k}\right);\ k \ge 1\right). \tag{1.3}$$

We note that $K_s^{(2)}(0,1,1) = K_s$, where K_s is the class of functions which was defined by Gao and Zhou [2]. Moreover, $K_s^{(2)}(\gamma,1,1) = K_s(\gamma)$ and $K_s^{(k)}(\gamma,1,1) = K_s^{(k)}(\gamma)$ which were studied by Kowalczyk *et al.* [4] and Seker [6], respectively so the class $K_s^{(k)}(\gamma,\alpha,\beta)$ are generalizations of $K_s(\gamma)$ and $K_s^{(k)}(\gamma)$.

In the present paper, we investigate characterization theorems, coefficient inequalities, growth and distortion theorems for functions belonging to the class $K_s^{(k)}(\gamma, \alpha, \beta)$.

2. Coefficient Estimates

First of all, we show in which way our class is associated with the appropriate subordination.

2.1. Theorem. *A function $f(z) \in K_s^{(k)}(\gamma,\alpha,\beta)$ if and only if there exits $g_k(z)$ satisfying the condition* (1.3) *such that*

$$\frac{z^k f'(z)}{g_k(z)} \prec \frac{1 + \beta[1-(1+\alpha)\gamma]z}{1 - \alpha\beta z} \qquad (z \in \mathbb{U}). \tag{2.1}$$

Proof. Let $f(z) \in K_s^{(k)}(\gamma,\alpha,\beta)$. Then, for $\alpha \ne 1$ and $\beta \ne 1$, squaring and expanding both sides of (1.2), we see that the region of $G(z) = z^k f'(z)/g_k(z)$ for $z \in \mathbb{U}$ is contained in the disk $\mathbf{C}$ whose center is $\{1 + \alpha\beta^2[1-(1+\alpha)\gamma]\}/(1-\alpha^2\beta^2)$ and radius is $\beta(1+\alpha)(1-\gamma)/(1-\alpha^2\beta^2)$. Since $q(z) = \{1+\beta[1-(1+\alpha)\gamma]z\}/(1-\alpha\beta z)$ maps the unit disk $\mathbb{U}$ to the disk $\mathbf{C}$ and $q(z)$ is univalent in $\mathbb{U}$, we obtain the relation (2.1). ☐

Conversely, assume that the relation (2.1) holds true. Then we have

$$\frac{z^k f'(z)}{g_k(z)} \prec \frac{1+\beta[1-(1+\alpha)\gamma]w(z)}{1-\alpha\beta w(z)},$$
$$(0 \le \alpha \le 1;\ 0 < \beta \le 1;\ 0 \le \gamma < 1;\ z \in \mathbb{U}),$$

where $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. Therefore from the above equation, we obtain the inequality (1.2), that is, $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$.

2.2. Remark. From Theorem 2.1, we see that, if $f(z) \in K_s^{(k)}(\gamma, \alpha, \beta)$, then

$$\text{(2.2)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{g_k(z)/z^{k-1}}\right) > \gamma \qquad (z \in \mathbb{U}),$$

because of

$$\operatorname{Re}\left(\frac{1+\beta[1-(1+\alpha)\gamma]z}{1-\alpha\beta z}\right) > \gamma \qquad (z \in \mathbb{U}).$$

In order to give the coefficient estimate of functions belonging to the class $K_s^k(\gamma, \alpha, \beta)$, we shall require the following lemma.

2.3. Lemma. [7] *Let*

$$\text{(2.3)} \qquad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*\left(\frac{k-1}{k}\right),$$

then

$$\text{(2.4)} \qquad G_k(z) = \frac{g_k(z)}{z^{k-1}} = z + \sum_{n=2}^{\infty} B_n z^n \in \mathcal{S}^* \subset \mathcal{S},$$

where $g_k(z)$ is given by (1.3).

2.4. Remark. (i) In particular, for $k = 2$, the coefficients B_n in (2.4) is expressed as follows:

$$B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + \ldots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1} b_n^2.$$

(ii) If $g(z) \in \mathcal{S}^*((k-1)/k)$, then from Lemma 2.3., $G_k(z)$ given by (2.4) belongs to $\mathcal{S}^*$. Then by (2.2), we see that the class $K_s^k(\gamma, \alpha, \beta)$ is a subclass of the class $\mathcal{K}$ of close-to-convex functions.

Next, we prove the sufficient condition for functions to belong to the class $K_s^k(\gamma, \alpha, \beta)$.

2.5. Theorem. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathbb{U}$. If*

$$\text{(2.5)} \qquad \sum_{n=2}^{\infty}(1+\alpha\beta)n\,|a_n| + \sum_{n=2}^{\infty}[1+\beta\,|1-(1+\alpha)\gamma|]\,|B_n| \le \beta(1+\alpha)(1-\gamma)$$
$$(0 \le \alpha \le 1;\ 0 < \beta \le 1;\ 0 \le \gamma < 1)$$

where the coefficients B_n $(n = 2, 3, \cdots)$ are given by (2.4)*, then $f(z) \in K_s^k(\gamma, \alpha, \beta)$.*

Proof. Let the functions $f(z)$ and $g_k(z)$ be given by (1.1) and (1.3), respectively. Now, we obtain

$$\Delta = \left|zf'(z) - \frac{g_k(z)}{z^{k-1}}\right| - \beta\left|\alpha z f'(z) + \frac{[1-(1+\alpha)\gamma]g_k(z)}{z^{k-1}}\right|$$
$$= \left|\sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} B_n z^n\right| -$$

$$-\beta\left|(1+\alpha)(1-\gamma)z+\alpha\sum_{n=2}^{\infty}na_nz^n+[1-(1+\alpha)\gamma]\sum_{n=2}^{\infty}B_nz^n\right|.$$

Thus, for $|z|=r(0\leq r<1)$, we have, from (2.5),

$$\begin{aligned}\Delta &\leq \sum_{n=2}^{\infty}n\,|a_n|\,|z|^n+\sum_{n=2}^{\infty}|B_n|\,|z|^n\\ &-\beta\left((1+\alpha)(1-\gamma)\,|z|-\alpha\sum_{n=2}^{\infty}n\,|a_n|\,|z|^n-|1-(1+\alpha)\gamma|\sum_{n=2}^{\infty}|B_n|\,|z|^n\right)\\ &=-\beta(1+\alpha)(1-\gamma)\,|z|+\sum_{n=2}^{\infty}(1+\alpha\beta)n\,|a_n|\,|z|^n+\\ &\sum_{n=2}^{\infty}[1+\beta\,|1-(1+\alpha)\gamma|\,]\,|B_n|\,|z|^n\\ &<\left(-\beta(1+\alpha)(1-\gamma)+\sum_{n=2}^{\infty}(1+\alpha\beta)n\,|a_n|+\sum_{n=2}^{\infty}[1+\beta\,|1-(1+\alpha)\gamma|\,]\,|B_n|\right)\\ &\leq 0.\end{aligned}$$

Thus we have

$$\left|\frac{z^kf'(z)}{g_k(z)}-1\right|<\beta\left|\frac{\alpha z^kf'(z)}{g_k(z)}+1-(1+\alpha)\gamma\right|,$$

that is, $f(z)\in K_s^{(k)}(\gamma,\alpha,\beta)$. This completes the proof of Theorem 2.5. □

In the following theorem, we give the coefficient estimates of functions belonging to the class $K_s^{(k)}(\gamma,\alpha,\beta)$.

2.6. Theorem. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}$, $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}$, and satisfy the inequality* (2.1)*. Then, for, $n\geq 2$, we have*

$$\begin{aligned}&|na_n-B_n|^2-[\beta(1+\alpha)(1-\gamma)]^2\\ &\leq(1+\beta\,|(1+\alpha)\gamma-1|)\sum_{k=2}^{n-1}\left\{2k\,|a_kB_k|+[1+\beta\,|(1+\alpha)\gamma-1|]\,|B_k|^2\right\},\end{aligned}\tag{2.6}$$

where B_n is given by (2.4)*.*

Proof. Suppose that the condition (1.2) is satisfied. Then, by using the a similar method as in the proof of (p. 30, [5]), we have

$$\frac{zf'(z)}{G_k(z)}=\frac{1+[(1+\alpha)\gamma-1]z\phi(z)}{1+\alpha z\phi(z)}\quad(z\in\mathbb{U}),\tag{2.7}$$

where ϕ is analytic in $\mathbb{U}$, $|\phi(z)|\leq\beta$ for $z\in\mathbb{U}$ and $G_k(z)$ is given by (2.4). Then from (2.7), we have

$$\left(\alpha zf'(z)-[(1+\alpha)\gamma-1]G_k(z)\right)z\phi(z)=G_k(z)-zf'(z)$$

Thus, putting

$$z\phi(z)=\sum_{n=1}^{\infty}t_nz^n,$$

we obtain

$$(2.8)\quad \begin{aligned}&\left((1+\alpha)(1-\gamma)z+\alpha\sum_{n=2}^{\infty}na_nz^n-[(1+\alpha)\gamma-1]\sum_{n=2}^{\infty}B_nz^n\right)\sum_{n=1}^{\infty}t_nz^n\\&=\sum_{n=2}^{\infty}B_nz^n-\sum_{n=2}^{\infty}na_nz^n.\end{aligned}$$

Equating the coefficient of z^n in (2.8), we have

$$\begin{aligned}B_n-na_n=&(1+\alpha)(1-\gamma)t_{n-1}+\{2\alpha a_2-[(1+\alpha)\gamma-1]\,B_2\}t_{n-2}\\&+...+\{(n-1)\alpha a_{n-1}-[(1+\alpha)\gamma-1]\,B_{n-1}\}t_1.\end{aligned}$$

Thus the coefficient combination on the right side of (2.8) depends only upon the coefficient combinations:

$$\{2\alpha a_2-[(1+\alpha)\gamma-1]\,B_2\},...,\{(n-1)\alpha a_{n-1}-[(1+\alpha)\gamma-1]\,B_{n-1}\}.$$

Hence for $n\geq 2$, the equation (2.8) can be written as

$$(2.9)\quad \begin{aligned}&\left[(1+\alpha)(1-\gamma)z+\sum_{k=2}^{n-1}\left(k\alpha a_k-[(1+\alpha)\gamma-1]\,B_k\right)z^k\right]z\phi(z)\\&=\sum_{k=2}^{n}\left(B_k-ka_k\right)z^k+\sum_{k=n+1}^{\infty}c_kz^k.\end{aligned}$$

Then, squaring the modulus of the both sides of (2.9) and integrating along $|z|=r<1$, so that by Parseval's identity (p. 192, [1]), we obtain

$$(2.10)\quad \begin{aligned}&\sum_{k=2}^{n}|ka_k-B_k|^2\,r^{2k}+\sum_{k=n+1}^{\infty}|c_k|^2\,r^{2k}\\&\leq\beta^2\left([(1+\alpha)(1-\gamma)]^2\,r^2+\sum_{k=2}^{n-1}|k\alpha a_k-[(1+\alpha)\gamma-1]\,B_k|^2\,r^{2k}\right).\end{aligned}$$

Letting $r\to 1$ on the left side of (2.10), we obtain

$$\sum_{k=2}^{n}|ka_k-B_k|^2\leq\beta^2\left([(1+\alpha)(1-\gamma)]^2+\sum_{k=2}^{n-1}|k\alpha a_k-[(1+\alpha)\gamma-1]\,B_k|^2\right).$$

Hence we have

$$\begin{aligned}|na_n-B_n|^2<&[\beta(1+\alpha)(1-\gamma)]^2+\beta^2\sum_{k=2}^{n-1}|k\alpha a_k-[(1+\alpha)\gamma-1]\,B_k|^2-\\&-\sum_{k=2}^{n-1}|ka_k-B_k|^2=\end{aligned}$$

$$\begin{aligned}&=[\beta(1+\alpha)(1-\gamma)]^2+(\beta^2\alpha^2-1)\sum_{k=2}^{n-1}k^2\,|a_k|^2+\\&+\left\{(\beta\,[(1+\alpha)\gamma-1])^2-1\right\}\sum_{k=2}^{n-1}|B_k|^2+\left(\alpha\beta^2\,|(1+\alpha)\gamma-1|+1\right)\sum_{k=2}^{n-1}2k\,|a_k|\,|B_k|\leq\end{aligned}$$

$$\leq [\beta(1+\alpha)(1-\gamma)]^2 + (\beta\,|(1+\alpha)\gamma-1|+1)^2\sum_{k=2}^{n-1}|B_k|^2 +$$
$$+ (\beta\,|(1+\alpha)\gamma-1|+1)\sum_{k=2}^{n-1}2k\,|a_k|\,|B_k|\,,$$

which implies the inequality (2.6). Therefore, we complete the proof of Theorem 2.6. □

Finally, we provide the growth and the distortion theorems for functions belonging to the class $K_s^{(k)}(\gamma,\alpha,\beta)$.

2.7. Theorem. *If $f(z)\in K_s^{(k)}(\gamma,\alpha,\beta)$, then*

$$\frac{1-\beta\,[1-(1+\alpha)\gamma]\,r}{(1+\alpha\beta r)(1+r^2)} \leq \left|f'(z)\right| \leq \frac{1+\beta\,[1-(1+\alpha)\gamma]\,r}{(1-\alpha\beta r)(1-r^2)} \quad (|z|=r<1) \tag{2.11}$$

and

$$\begin{aligned} &\frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2}\ln\frac{1+\alpha\beta r}{1+r} + \frac{(1+\beta[1-(1+\alpha)\gamma])r}{(1-\alpha\beta)(1+r)} \leq |f(z)| \\ &\leq \frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2}\ln(1-\alpha\beta r)(1-r) - \frac{(1+\beta[1-(1+\alpha)\gamma])r}{(1-\alpha\beta)(1-r)} \quad (|z|=r<1), \end{aligned} \tag{2.12}$$

The results are sharp.

Proof. If $f(z)\in K_s^{(k)}(\gamma,\alpha,\beta)$, then there exists function $g_k(z)$ satisfying (1.2). Then it follows from the Lemma 2.3. that the function $G_k(z)$ given by (2.4) is a starlike function. Hence from (p. 70, [1]), we have

$$\frac{r}{1+r^2} \leq |G_k(z)| \leq \frac{r}{1-r^2} \quad (|z|=r<1). \tag{2.13}$$

Let us define $p(z)$ by

$$p(z) = \frac{zf'(z)}{G_k(z)} \quad (z\in\mathbb{U}).$$

Then by using a similar method as in (p. 105, [3]), we have

$$\frac{1-\beta\,[1-(1+\alpha)\gamma]\,r}{1+\alpha\beta r} \leq |p(z)| \leq \frac{1+\beta\,[1-(1+\alpha)\gamma]\,r}{1-\alpha\beta r} \quad (|z|=r<1). \tag{2.14}$$

Thus from (2.13) and (2.14), we have

$$\frac{1-\beta\,[1-(1+\alpha)\gamma]\,r}{(1+\alpha\beta r)(1+r^2)} \leq \left|f'(z)\right| \leq \frac{1+\beta\,[1-(1+\alpha)\gamma]\,r}{(1-\alpha\beta r)(1-r^2)} \quad (|z|=r<1)\,,$$

which gives us (2.11). Upon integrating (2.11) from 0 to r, we have the inequality (2.12). Moreover, the results are sharp for the functions given, respectively, by

$$f_1(z) = \frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2}\ln\frac{1+\alpha\beta z}{1+z} + \frac{(1+\beta[1-(1+\alpha)\gamma])z}{(1-\alpha\beta)(1+z)} \quad (z\in\mathbb{U})$$

and

$$f_2(z) = \frac{\beta(1+\alpha)(1-\gamma)}{(1-\alpha\beta)^2}\ln(1-\alpha\beta z)(1-z) - \frac{(1+\beta[1-(1+\alpha)\gamma])z}{(1-\alpha\beta)(1-z)} \quad (z\in\mathbb{U}).$$

□

Acknowledgements

The authors would like to express their thanks to the referee for many valuable advices regarding a previous version of this paper. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2011-0007037).

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