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# TOPOLOGICAL K-THEORY OF THE CLASSIFYING SPACES OF CYCLIC AND DIHEDRAL GROUPS

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**Abstract**

We make a little survey and also present some new results on the topological K-theory of the classfying spaces of cyclic and dihedral groups.

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## 1. Introduction

The $K$-ring of a $CW$-complex $X$, denoted by $K(X)$, is defined by the ring completion of the semi-ring of the isomorphism classes of complex vector bundles over $X$. The $KO$-ring of $X$, denoted by $KO(X)$, is defined similarly, by means of real vector bundles over $X$. Similar rings can be constructed for the other fields like the field of quaternions or the finite fields. One of the most interesting questions of the topological $K$-theory is to determine these rings when $X$ is the classifying space $BG$ of a group, in particular a finite group, $G$. See, for example, the description given in [5] for the $KO$-ring of the skeletons of the classifying space of the cyclic group of order $2^n$.

In this note, we will make a brief survey and also present some new results for the $K$-rings and $KO$-rings of the classifying space of the cyclic and the dihedral groups.

Before starting the presentation, we should mention two important theorems in topological K-theory. Firstly, there is the Atiyah-Segal completion theorem (ASCT) which states that $K(BG)$ is isomorphic to the completion of the complex representation ring of $G$ at the augmentation ideal, that is,

$$K(BG) = R(G)\hat{}_I$$

Basically, this theorem says that $K(BG)$ is another way of writing the elements of $R(G)$, as formal sums of their reductions and what we are doing here is not more than representation theory with a little geometry added. A similar theorem holds for $KO$-rings and the real representation ring $RO(G)$.

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Secondly, there is a spectral sequence called the Atiyah-Hirzebruch spectral sequence (AHSS) which is schematically described by

$$E_2^{k,-k} = H^k(BG; \widetilde{K}(S^k)) \Longrightarrow K(BG)$$

on the anti-diagonal of the second page. Thus, if the odd dimensional cohomology of the group is trivial then the spectral sequence collapses on the second page. But, if the odd dimensional cohomology is not trivial then on the third page or on a higher odd numbered page there may be a non-zero differential and the spectral sequence may not collapse on the second page. A similar spectral sequence exists for the $KO$-theory as well.

We should note here that we are attacking to the problem from the computational point of view and not doing hard-core topology. This could be quite messy. You may see for example [3] where the problem is studied for the dihedral groups and more topology (quotient spaces etc.) is involved.

## 2. Real Projective Spaces

We start with a well-known and the simplest example. The AHSS for $K(BZ_2)$ collapses on the second page. The $K$-ring of the infinite dimensional projective space is

$$K(BZ_2) = Z\,[v] \diagup (v^2 + 2v)$$

where $v = \eta - 1$ is the reduction of the tautological complex line bundle $\eta$ over $BZ_2$. Similarly, the AHSS for $KO(BZ_2)$ collapses on the second page and the $KO$-ring of the infinite dimensional projective space is

$$KO(BZ_2) = Z\,[\lambda] \diagup (\lambda^2 + 2\lambda)$$

where $\lambda = \xi - 1$ is the reduction of the tautological real line bundle $\xi$. Note that the tautological bundles are induced from the corresponding tautological complex and real one dimensional representations of $Z_2$ via ASCT. You should have noticed that $K(BZ_2)$ and $KO(BZ_2)$ are the same rings! They are algebraically isomorphic, but geometrically they are different. The difference can be demonstrated by comparing the filtrations of their AHSSs.

## 3. Lens Spaces

This is the generalization of projective spaces. For $n \geq 3$, $BZ_n$ is called an infinite dimensional standard lens space. The AHSS for $K(BZ_n)$ collapses on the second page and the integral cohomology of the group $Z_n$ completely determines the filtrations of the spectral sequence. We have

$$K(BZ_n) = Z\,[\mu] \diagup ((1 + \mu)^n - 1)$$

where $\mu = \eta - 1$ is the reduction of the tautological complex line bundle $\eta$ (the Hopf bundle) over $BZ_n$.

$KO$-rings are more complicated. But, again, the AHSS for $KO(BZ_n)$ collapses on the second page and and the integral cohomology of the group $Z_n$ together with the cohomology with $Z_2$ coefficients help us to detect the filtrations of the spectral sequence. We need to consider the odd case and the even case separately.

**3.1. The Odd Case.** If $n \geq 3$ is odd,

$$KO(BZ_n) = Z[w] \diagup (wf_n(w))$$

where

$$f_n(w) = n + \sum_{j=1}^{\frac{n-3}{2}} \frac{n(n^2-1^2)(n^2-3^2)...(n^2-(2j-1)^2)}{2^{2j}.(2j+1)!} w^j + w^{\frac{n-1}{2}}$$

and $w = r(\mu)$ is the realification of $\mu$, [5]. If $n = p$ is an odd prime number then $f_n(\cdot)$ is the minimal polynomial of the number $2\cos(\frac{2\pi}{p}) - 2$ which can be proved by using the Eisenstein's criterion and factorization of odd indexed Chebishev polynomials. We note that

$$wf_n(w) = \psi^{\frac{n+1}{2}}(w) - \psi^{\frac{n-1}{2}}(w)$$

where $\psi^k$ denotes the Adams operation of degree $k$. The relation $wf_n(w) = 0$ stems from the triviality of the complex conjugation on the real theory. We can think of this triviality as the equality $\psi^{-1} = \psi^1$ in the real theory. You may see [5] and [6] for the polynomials $\psi^k(w)$ and see more in [6] about the connection between $f_n(\cdot)$, the Adams operations and the Chebishev polynomials.

**3.2. The Even Case.** If $n$ is even, $n \geq 4$, the problem is twisted because of the effect of the prime number 2. In this case, we have the one dimensional tautological real line bundle $\xi$ together with the two dimensional tautological real plane bundle $r(\eta)$.

This time, the $KO$-ring can be described in the following way:

$$KO(BZ_n) = Z[\lambda, w] \diagup \begin{pmatrix} \lambda^2 + 2\lambda, \\ \lambda w - \psi^{\frac{n}{2}+1}(w) + \psi^{\frac{n}{2}}(w) + w, \\ \psi^{\frac{n}{2}+1}(w) - \psi^{\frac{n}{2}-1}(w) \\ 2\lambda - \psi^{\frac{n}{2}}(w) \end{pmatrix}$$

where $\lambda = \xi - 1$ is the reduction of $\xi$ and $w$ is defined as in the odd case. Here, again, $\psi^k$ is the Adams operation of degree $k$.

The third polynomial $\psi^{\frac{n}{2}+1}(w) - \psi^{\frac{n}{2}-1}(w)$ in the ideal that defines the ring is the main relation and it is related to the triviality of the complex conjugation on the real theory exactly as in the odd case. We can also find an explicit expression for this polynomial similar to the expression of the polynomial $f_n$ in the odd case. This will be important for the $K$-rings of dihedral spaces.

Finally, note that lens spaces has a connection with the infinite dimensional complex projective space $BS^1$ due to the group inclusions $Z_n < S^1$. Many works of İbrahim Dibağ relies on this connection. See for example, [1]. Note that the $K$-ring and $KO$-ring of $BS^1$ are generated by the tautological complex line bundle and its realification respectively.

## 4. Dihedral Spaces

Let $D_{2n}$ be the dihedral group with $2n$ elements. We will call the classifying space $BD_{2n}$ and its finite skeletons as dihedral spaces. Since, the representations of the dihedral groups are real, it follows that $KO(BD_{2n})$ is isomorphic to $K(BD_{2n})$. So, our main problem is to describe the rings $K(BD_{2n})$. We will treat odd and even cases of $n$ separately.

**4.1. The Odd Case.** Let $n = 2k+1$ and $k \geq 1$. The representation ring $R(D_{2n})$ of $D_{2n}$ is generated by the one-dimensional tautological complex representation $\eta$ and the two-dimensional tautological representation $\rho$ by (direct) sums and (tensor) products. Let us denote the complex vector bundles induced from these representations over $BD_{2n}$ by the same letters. Due to the ASCT, it then follows that these bundles generate $K(BD_{2n})$.

The integral group cohomology of $D_{2n}$, for $n$ odd, is zero in odd dimensions, [2]. Thus, the AHSS collapses on the second page and we can read the filtrations of the ring $K(BD_{2n})$ from the integral cohomology.

We define the reductions $v = \eta - 1$ and $\phi = \rho - 2$ in the ring $K(BD_{2n})$. The relations $\eta^2 = 1$ and $\eta\rho = \rho$ immediately give the relations $v^2 + 2v = 0$ and $v\phi + 2v = 0$ in the $K$-ring.

But, the main relation of the ring is coming from the iterative relations between the two dimensional irreducible complex representations of the dihedral group $D_{2n}$. You may see [4] for the iterative relations and the structure of $R(D_{2n})$.

By iterating these relations for $\rho$ and by using the previous two relations and also by getting use of the induced homomorphisms induced by the group injections of the cyclic subgroups of the dihedral group, one can deduce a relation in the form

$$\phi f_n(\phi) = f_n(-2)v$$

where $f_n(\cdot)$ is the polynomial defined in the previous section.

Hence, since the AHSS collapses, we have

$$K(BD_{2n}) = Z[v, \phi] \diagup \begin{pmatrix} v^2 + 2v, \\ v\phi + 2v, \\ \phi f_n(\phi) - f_n(-2)v \end{pmatrix}$$

which completely describes the $K$-ring of the classifying space of the dihedral group of order $2n$ for $n$ odd. For details, see [4]. As we noted before, the representations of the dihedral groups are real and so the rings $K(BD_{2n})$ and $KO(BD_{2n})$ should be isomorphic as rings, but, geometrically they carry different information. Their spectral sequences are different and $KO(BD_{2n})$ is also related to the cohomology with $Z_2$ coefficients.

**4.2. The Even Case.** The even case is quite complicated with respect to the odd case as usual. We still don't have a complete description of the ring and we will give some examples to exhibit the complexity of the relations that generate the ring.

Let $n = 2k$ and $k \geq 1$. The representation ring $R(D_{2n})$ is generated by the one-dimensional tautological representations $\eta_1, \eta_2$ and the two-dimensional tautological representation $\rho$. Let us denote the complex vector bundles induced from these representations over $BD_{2n}$ by the same letters. Due to the ASCT, these bundles generate $K(BD_{2n})$.

We define the reductions $v_1 = \eta_1 - 1$, $v_2 = \eta_2 - 1$ and $\phi = \rho - 2$ in the ring $K(BD_{2n})$. The integral cohomology of the group $D_{2n}$ is given in [2] and we can read the filtrations of the second page of the AHSS from the cohomology described there. Note that the AHSS is not collapsing this time.

For $k = 1$, since $D_4 = Z_2 \times Z_2$ we have $BD_4 = BZ_2 \times BZ_2$ and so

$$K(BD_4) = Z[v_1, v_2] \diagup (v_1^2 + 2v_1, v_2^2 + 2v_2)$$

is just the $K$-ring of the product of two infinite dimensional real projective spaces. We believe that this result could also be derived from the Atiyah's Künneth theorem for $K$-cohomology. We have the filtrations of AHSS as fallows: $E_\infty^{2,-2} = (Z_2)^2$ which is generated by $v_1$ and $v_2$; $E_\infty^{2s,-2s} = (Z_2)^3$ for all $s \geq 2$ which is generated by $v_1^s, v_2^s$ and $2^{s-2}v_1v_2$.

For $k \geq 2$, the generator $\phi$ emerges and we have some real difficulties for detecting the relations associated with $\phi$. For $k = 2$, we have the relation

$$\phi^2 + 4\phi = -v_1^2 - v_2^2 + v_1 v_2$$

by using the iterative relations on the representation ring. Since, there is only one irreducible complex representation of $D_8$, we have the relation $\eta_1 \rho = \eta_2 \rho = \rho$ and it then fallows that $v_1 \phi = v_1^2$ and $v_2 \phi = v_2^2$. Hence, the ring $K(BD_8)$ should probably be described as

$$Z[v_1, v_2, \phi] \Big/ \left( \begin{array}{c} v_1^2 + 2v_1, v_2^2 + 2v_2, \\ v_1\phi + 2v_1, v_2\phi + 2v_2, \\ \phi^2 + 4\phi - 2v_1 - 2v_2 - v_1 v_2 \end{array} \right).$$

But as stated in [4], we have some problems with this description because of the filtration incompatibility in AHSSs of this ring with respect to the $K$-rings of the projective and lens spaces induced from the subgroups of $D_8$; namely, because of the nonexistence of a place of the product $v_1 v_2$ in the filtration $E_\infty^{4,-4}$.

For $k \geq 3$, things become more twisted due to the fact that there are more than one irreducible complex representations of dimension 2 and multiplication of these by $\eta_1$ and $\eta_2$ are non-trivial. Because of that the products $v_1\phi$ and $v_2\phi$ are complicated. We believe that $K(BD_{12})$ looks like

$$Z[v_1, v_2, \phi] \Big/ \left( \begin{array}{c} v_1^2 + 2v_1, v_2^2 + 2v_2, \\ v_1\phi - \phi f_3(\phi) + 3v_1 + v_2 + v_1 v_2, \\ v_2\phi - \phi f_3(\phi) + v_1 + 3v_2 + v_1 v_2, \\ 3\phi + 13\phi^2 + 7\phi^3 + \phi^4 + 3v_1 + 3v_2 + 2v_1 v_2 \end{array} \right)$$

where $f_3(\cdot)$ is the polynomial introduced in Lens Spaces section. Unlike to the $k = 2$ case, we could be able to find a place for the product $\nu_1 v_2$ in a filtration of the AHSS and this description seems to be okay with respect to the cyclic subgroups of the dihedral group.

We are still working on the problem for $k \geq 4$ and hopefully we will find the general solution of the problem; in particular, the polynomial which will be the main relation of $\phi$. Before that, of course, we should make the computation of the $KO$-rings of lens spaces in the even case more explicit.

## 5. Other Finite Groups

One can consider the topological $K$-theory of other finite groups like the generalized quaternion groups $Q_{2^n}$, the symmetric groups $S_n$, the general linear groups $GL(n,p)$ over finite fields etc. Especially, the computation of $K(BS_n)$, where $S_n$ is the symmetric group on $n$ objects, is an open problem which is interesting and quite hard as well. One should study tensor products of the Specht modules for which we know that they could have very big dimensions when $n$ increases and things are very complicated.

## References


[1] Dibağ, İ. *J-approximation of Complex Projective Spaces by Lens Spaces*, Pacific Journal of Math. **191**(2), 223–242, 1999.

[2] Handel, D. *On Products in the Cohomology of the Dihedral Groups*, Tôhoku Math. J. **45**, 13–42, 1993.

[3] Imaoka, M. and Sugawara, M. *On the K-Ring of the Orbit Manifold* $\left(S^{2k+1} \times S^l\right) / D_n$ *by the Dihedral Group* $D_n$, Hiroshima Math. **4**, 53–70, 1974.

[4] Kırdar, M. *K-rings of the Classifying Space of the Dihedral Groups*, preprint.

[5] Kırdar, M. *KO-Rings of* $S^{2k+1}/Z_{2^n}$, K-theory **13**(1), 57–59, 1998.

[6] Kırdar, M. *Reduced K-theory Relations of the Hopf Bundle over Lens Spaces*, preprint.