

ON SOME COMBINATORIAL IDENTITIES INVOLVING THE TERMS OF GENERALIZED FIBONACCI AND LUCAS SEQUENCES

Zeynep Akyuz *, Serpil Halici †

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Abstract

In this paper, we consider the Horadam sequence and some summation formulas involving the terms of the Horadam sequence. We derive combinatorial identities by using the trace, the determinant, and the n th power of a special matrix.

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1. Preliminaries

Generalized Fibonacci sequence $W_n = W_n(a, b; p, q)$ is defined as follows;

$$(1.1) \quad W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a, \quad W_1 = b.$$

Where $a, b, p,$ and q are arbitrary complex numbers, with $q \neq 0$. Since, these numbers have been studied firstly by Horadam(see, e.g., [1]) they are called as Horadam numbers. Some special cases of this sequence such as

$$(1.2) \quad U_n = W_n(0, 1; p, q), \quad V_n = W_n(2, p; p, q)$$

were investigated by Lucas[6]. Further and in detailed knowledge can be found in[1, 2, 3, 4, 5, 6]. If α, β assumed distinct, are the roots of

$$(1.3) \quad \lambda^2 - p\lambda + q = 0$$

then the sequence W_n has the Binet representation

$$(1.4) \quad W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

*Department of Mathematics, Faculty of Arts and Sciences
Sakarya University, 54187 Sakarya/TURKEY

†Department of Mathematics, Faculty of Arts and Sciences
Sakarya University, 54187 Sakarya/TURKEY

where $A = b - a\beta$ and $B = b - a\alpha$. For negative indices, this formula is given as

$$W_{-n} = \frac{pW_{-n+1} - W_{-n+2}}{q}.$$

So, for all integer numbers n , we can write

$$(1.5) \quad W_n = pW_{n-1} - qW_{n-2}; \quad W_0 = a, \quad W_1 = b.$$

In [16], the authors used the matrix in relation to the recurrence relation (1);

$$(1.6) \quad M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}.$$

Indeed, if $p = 1$ and $q = -1$, then the matrix M reduces to Fibonacci Q -matrix. The matrix M is a special case of the general $k \times k$, Q -matrix [11]. Now, we use the matrix M and its powers to prove and derive the some combinatorial identities involving the terms from the sequence $\{W_n\}$. Such identities are quite extensive on literature, but for this purpose we use only the trace and determinant of the matrix M^n . In [9], J. McLaughlin gave a new formula for the n th power of a 2×2 matrix. The author proved that if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary 2×2 matrix, then for $n \geq 1$, B^n is

$$(1.7) \quad B^n = \begin{pmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{pmatrix}; \quad y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i} (-D)^i$$

where T and D are the trace and determinant of matrix B , respectively. In [10], K. S. Williams gave a formula for the n th power of any 2×2 matrix C with eigenvalues α and β as follows;

$$(1.8) \quad C^n = \begin{cases} \frac{\alpha^n(C-\beta I) - \beta^n(C-\alpha I)}{\alpha - \beta}; & \alpha \neq \beta \\ \alpha^{n-1}(nC - (n-1)\alpha I); & \alpha = \beta \end{cases}$$

In [8], H. Belbachir extended this result to any matrix A of order m , $m \geq 2$. Also, he derived some identities concerning the Stirling numbers.

2. Some Combinatorial Identities involving the terms of Horadam Sequence

In this section, firstly we give a general formula for the generalized Lucas numbers. Then, we investigate the special cases of this sequence. And then, we give some formulae for generalized Fibonacci numbers.

2.1. Theorem. For $n \geq 1$, we have the following identity;

$$(2.1) \quad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} p^{n-2k} (-q)^k = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{n-2k} (p^2 - 4q)^k.$$

Where p and q are the trace and determinant of the matrix M , respectively.

Proof. Using the matrix M^n ,

$$M^n = \begin{pmatrix} y_n & -qy_{n-1} \\ y_{n-1} & y_n - py_{n-1} \end{pmatrix},$$

we can write

$$(2.2) \quad \text{tr}(M^n) = \lambda_1^n + \lambda_2^n = 2y_n - py_{n-1}$$

$$2y_n - py_{n-1} = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} T^{n-2k} (-D)^k - p \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} T^{n-1-2k} (-D)^k$$

$$2y_n - py_{n-1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k} (-q)^k \left(2 - \frac{n-2k}{n-k} \right).$$

Thus, we have the right side of the equation (10).The left side of the equation (10) is follows;

$$\lambda_1^n + \lambda_2^n = \frac{1}{2^n} \left(\sum_{k=0}^n \binom{n}{k} p^{n-k} (\sqrt{p^2 - 4q})^k + \sum_{k=0}^n \binom{n}{k} p^{n-k} (-\sqrt{p^2 - 4q})^k \right)$$

$$\lambda_1^n + \lambda_2^n = \frac{1}{2^{n-1}} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{n-2k} (p^2 - 4q)^k \right)$$

Thus, the proof is completed. □

In the following theorem, we give the n th term of the generalized Lucas sequence by using this method.

2.2. Theorem. For $n \geq 0$ we have the following identities;

$$(2.3) \quad i) V_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} p^{n-2k} (-q)^k$$

and

$$(2.4) \quad ii) V_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} p^{n-2k} (p^2 - 4q)^k$$

where p and q are the trace and determinant of the matrix M , respectively.

Note that if we take $p = 1$ and $q = -1$ in the Theorem 2.1 and Theorem 2.2, then we obtain that

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 5^k.$$

The each sides of the last equation can be found in [13]. In addition to this, in the Theorem 2.1 and Theorem 2.2 if we take $p = 2$, $q = -1$ and $p = 1$, $q = -2$ then we obtain the identities for the Pell-Lucas and Jacobsthal-Lucas sequences, respectively, as follows;

$$Q_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} 2^{n-2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{k+1},$$

and

$$j_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \frac{n}{n-k} 2^k = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 9^k.$$

Similarly, for the certain values of p and q we can get the bivariate Lucas, Pell-Lucas and Jacobsthal-Lucas polynomials . The equation is given in the following theorem can be seen many studies, but we give this identity by using a different method.

2.3. Theorem. For $n \geq 0$ we have the following identity;

$$(2.5) \quad U_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} p^{n-1-2i} (-q)^i,$$

where $U_n = W_n(0, 1; p, q)$.

By using the Theorem 2.3 we can write the following identities;

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i}, \quad p = 1, \quad q = -1,$$

$$P_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^{n-1-2i}, \quad p = 2, \quad q = -1,$$

and

$$J_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^i, \quad p = 1, \quad q = -2.$$

Now, we give a formula for the generalized Fibonacci numbers by using the studies of Shannon and Laughlin.

2.4. Theorem. For $k \geq 1$, we have

$$(2.6) \quad U_{nk} = U_n \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} V_n^{k-1-2i} (-q^n)^i,$$

where $U_n = W_n(0, 1; p, q)$ and $V_n = W_n(2, p; p, q)$.

Finally, we give the most general formula for generalized Fibonacci numbers in the following theorem.

2.5. Theorem. For $n, k \geq 1$ and $r \neq 0$, we have

$$(2.7) \quad U_{nk+r} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} V_n^{k-2i} (-q^n)^i \delta,$$

where $\delta = q^r \frac{U_{n-r}}{V_n} \frac{k-2i}{k-i} + U_r$.

Proof. The proof can be seen by the powers of the matrix M as follows; Note that

$$M^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}$$

and

$$M^{nk+r} = \begin{pmatrix} U_{nk+r+1} & -qU_{nk+r} \\ U_{nk+r} & -qU_{nk+r-1} \end{pmatrix}.$$

Then, we write

$$(2.8) \quad U_{nk+r} = y_{k-1} (U_n U_{r+1} - U_r U_{n+1}) + U_r y_k$$

where $y_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} T^{k-2i} (-D)^i$, $T = U_{n+1} - qU_{n-1} = V_n$, $D = q^n$ and since

$$U_n U_{r+1} - U_r U_{n+1} = q^r U_{n-r},$$

we obtain

$$U_{nk+r} = q^r U_{n-r} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} T^{k-1-2i} (-D)^i + U_r \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} T^{k-2i} (-D)^i,$$

$$U_{nk+r} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} V_n^{k-2i} (-q^n)^i \left(q^r \frac{U_{n-r}}{V_n} \frac{k-2i}{k-i} + U_r \right).$$

Thus, the proof is completed. \square

References

- [1] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, The Fibonacci Quaterly, 3(2) (1965), 161-176.
- [2] A. F. Horadam, *Tschebyscheff and other functions associated with the sequence $W_n(a, b; p, q)$* , The Fibonacci Quaterly, 7(1)(1969), 14-22.
- [3] A. Nalli, P. Haukkanen, *On generalized Fibonacci and Lucas Polynomials*, Chaos, Solutions and Fractals, 42(2009), 3179-3186.
- [4] D. Jarden, *Recurring sequences*, Jerusalem; Rivion Lematematika, 1966.
- [5] E. Kilic, E. Tan, *On binomial sums for the general second order linear recurrence*, Integers 10(2010), 801-806.
- [6] E. Lucas, *Theorie des fonctions numeriques simplement periodiques*, American Journal of Math.(1978), 189-240.
- [7] G. E. Bergum, V. E. Hoggatt Jr., *Sums and products for recurring sequences*, The Fibonacci Quaterly, 13(2) (1975), 115-120.
- [8] H. Belbachir, *Linear recurrent sequences and Powers of a square matrix*, Integers: Electronic J. of Combinatorial Number Theory, 2006, 1-17.
- [9] J. Mc. Laughlin, *Combinatorial identities deriving from the n th power of a 2×2 matrix*, Integers: Electronic J. of Combinatorial Number Theory 4(2004), 1-15.
- [10] K. S. Williams, *The n th power of a 2×2 matrix* (in notes), Math. Magazine, 65(5), 336.
- [11] M. E. Waddill, L. Sacks, *Another generalized Fibonacci sequence*, The Fibonacci Quaterly, 5(3)(1967), 209-222.
- [12] M. S. El Naschie, *Notes on super string and the infinite sums of Fibonacci and Lucas numbers*, Chaos, Solitions and Fractals 10(8)(1999), 1303-1307.
- [13] N. Robbins, *A new formula for Lucas numbers*, The Fibonacci Quaterly, 29.4(1991), 362-363.
- [14] T. Mansour, *A formula for the generating functions of powers of Horadam's sequence*, Australasian J. of Combinatorics 30(2004), 207-212.
- [15] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience Publications, (2001).
- [16] R. S. Melham and A. G. Shannon, *Some summation identities using generalized Q -matrices*, The Fibonacci Quaterly, 33(1)(1995), 64-73.