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GREEN'S FUNCTIONAL CONCEPT FOR A NONLOCAL PROBLEM

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Abstract

In this work, by Green's functional concept, in order to obtain Green's solution we concentrate on a new constructive technique by which a linear completely nonhomogeneous nonlocal problem for a second-order loaded differential equation with generally variable coefficients satisfying some general properties such as p-integrability and boundedness is transformed into one and only one integral equation. A system of three integro-algebraic equations called the special adjoint system is obtained for this problem. A solution of this special adjoint system is Green's functional which enables us to determine Green's function and Green's solution for the problem. Two illustrative applications are provided.

Keywords: Green's function; loaded differential equation; nonlocal condition; adjoint problem.

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1. Introduction

Some boundary value problems with loaded equations involving local and nonlocal conditions arise in the various areas of mechanics frequently. However, the studies on the ones with nonlocal conditions are fewer than the studies with local conditions in literature. In [5], priori bounds for the stability of solutions to boundary value problems with some loaded equations are obtained. In [8], a boundary value problem for loaded equation involving nonlocal condition is considered in order to obtain the sufficient conditions for Fredholm property.

Green's functions of linear boundary value problems for ordinary differential equations with sufficiently smooth coefficients have been investigated in detail in several studies [11,

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12, 13, 14, 15]. In this work, a linear, generally nonlocal problem is studied for a second-order loaded differential equation. The coefficients of the equation are assumed to be generally nonsmooth functions satisfying some general properties such as p-integrability and boundedness. The operator of this equation, in general, does not have a formal adjoint operator, or any extension of the traditional type for this operator exists only on a space of distributions [9, 13]. In addition, the considered problem does not have a meaningful traditional type adjoint problem, even for simple cases of a differential equation and nonlocal conditions. Due to these facts, some serious difficulties arise in application of the classical methods for such a problem. As can be seen from [11], similar difficulties arise even for classical type boundary value problems if the coefficients of the differential equation are, for example, continuous nonsmooth functions. For this reason, a Green's functional approach is introduced for the investigation of the considered problem. This approach is based on [1, 2, 3, 4] and on some methods of functional analysis. The main idea of this approach is related to the usage of a new concept of the adjoint problem named adjoint system. Such an adjoint system includes three integro-algebraic equations with an unknown element $(f_2(\xi), f_1, f_0)$ in which $f_2(\xi)$ is a function, and f_j for $j = 0, 1$ are real numbers. One of these equations is an integral equation with respect to $f_2(\xi)$ and generally includes f_j as parameters. The other two equation can be considered as a system of algebraic equations with respect to f_0 and f_1, and they may include some integral functionals defined on $f_2(\xi)$. The form of the adjoint system depends on the operators of the equation and the conditions. The role of the adjoint system is similar to that of the adjoint operator equation in the general theory of the linear operator equations in Banach spaces [6, 10, 11]. The integral representation of the solution is obtained by Green's functional which is introduced as a special solution $f(x) = (f_2(\xi, x), f_1(x), f_0(x))$ of the corresponding adjoint system having a special free term depending on x as a parameter. The first component $f_2(\xi, x)$ of Green's functional $f(x)$ is corresponded to Green's function for the problem. The other two components $f_j(x)$ for $j = 0, 1$ correspond to the unit effects of the conditions. To summarize, this approach is principally different from the classical methods used for constructing Green's functions [14].

2. Statement Of The Problem

Let $\mathbb{R}$ be the set of real numbers. Let $G = (x_0, x_1)$ be a bounded open interval in $\mathbb{R}$. Let $L_p(G)$ with $1 \le p < \infty$ be the space of p-integrable functions on G. Let $L_\infty(G)$ be the space of measurable and essentially bounded functions on G, and let $W_p^{(2)}(G)$ with $1 \le p \le \infty$ be the space of all functions $u = u(x) \in L_p(G)$ having derivatives $d^k u/dx^k \in L_p(G)$, where $k = 1, 2$. The norm on the space $W_p^{(2)}(G)$ is defined as

$$\|u\|_{W_p^{(2)}(G)} = \sum_{k=0}^{2} \left\| \frac{d^k u}{dx^k} \right\|_{L_p(G)}$$

We consider the following boundary value problem

$$(V_2 u)(x) \equiv u''(x) + A_0(x)u(x) + A_1(x)u(x_0) = z_2(x), \quad x \in G, \tag{2.1}$$

subject to the nonlocal conditions

$$V_1 u \equiv a_1 u(x_0) + b_1 u'(x_0) + \int_{x_0}^{x_1} g_1(\xi) u''(\xi) d\xi = z_1,$$

$$V_0 u \equiv a_0 u(x_0) + b_0 u'(x_0) + \int_{x_0}^{x_1} g_0(\xi) u''(\xi) d\xi = z_0, \tag{2.2}$$

which are more general conditions than the ones in [4]. We investigate a solution to the problem in the space $W_p = W_p^{(2)}(G)$. Furthermore, we assume that the following conditions are satisfied: $A_i \in L_p(G)$ and $g_i \in L_q(G)$ for $i = 0, 1$, where $\frac{1}{p} + \frac{1}{q} = 1$, are given functions; a_i, b_i for $i = 0, 1$ are given real numbers; $z_2 \in L_p(G)$ is a given function and z_i for $i = 0, 1$ are given real numbers.

Problem (2.1)-(2.2) is a linear completely nonhomogeneous problem which can be considered as an operator equation:

$$Vu = z, \tag{2.3}$$

with the linear operator $V = (V_2, V_1, V_0)$ and $z = (z_2(x), z_1, z_0)$.

The assumptions considered above guarantee that V is bounded from W_p into the Banach space $E_p \equiv L_p(G) \times \mathbb{R} \times \mathbb{R}$ consisting of element $z = (z_2(x), z_1, z_0)$ with

$$\|z\|_{E_p} = \|z_2\|_{L_p(G)} + |z_1| + |z_0|, \quad 1 \leq p \leq \infty.$$

If, for a given $z \in E_p$, the problem (2.1)-(2.2) has a unique solution $u \in W_p$ with $\|u\|_{W_p} \leq c_0 \|z\|_{E_p}$, then this problem is called a well-posed problem, where c_0 is a constant independent of z. Problem (2.1)-(2.2) is well-posed if and only if $V : W_p \to E_p$ is a (linear) homeomorphism.

3. Adjoint Of The Solution Space

The solution to problem (2.1)-(2.2) is sought by virtue of a new concept of the adjoint problem. This concept is introduced in the papers [2, 3] by the adjoint operator V^* of V. On the other hand, some isomorphic decompositions of the solution space W_p and its adjoint space W_p^* are employed. Any function $u \in W_p$ can be represented as

$$u(x) = u(\alpha) + u'(\alpha)(x - \alpha) + \int_\alpha^x (x - \xi) u''(\xi) d\xi \tag{3.1}$$

where α is a given point in $\overline{G}$ which is the set of closure points for G. Furthermore, the trace or value operators $D_0 u = u(\gamma), D_1 u = u'(\gamma)$ are bounded and surjective from W_p onto $\mathbb{R}$ for a given point γ of $\overline{G}$. In addition, the values $u(\alpha), u'(\alpha)$ and the derivative $u''(x)$ are unrelated elements of the function $u \in W_p$ such that for any real numbers ν_0, ν_1 and any function $\nu_2 \in L_p(G)$, there exists one and only one $u \in W_p$ such that $u(\alpha) = \nu_0, u'(\alpha) = \nu_1$ and $u''(x) = \nu_2(x)$. Therefore, there exists a linear homeomorphism between W_p and E_p. In other words, the space W_p has the isomorphic decomposition $W_p = L_p(G) \times \mathbb{R} \times \mathbb{R}$.

3.1. Theorem. *If $1 \leq p < \infty$, then any linear bounded functional $F \in W_p^*$ can be represented as*

$$F(u) = \int_{x_0}^{x_1} u''(x)\varphi_2(x)dx + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{3.2}$$

with a unique element $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Any linear bounded functional $F \in W_\infty^*$ can be represented as

$$F(u) = \int_{x_0}^{x_1} u''(x)d\varphi_2 + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{3.3}$$

with a unique element $\varphi = (\varphi_2(e), \varphi_1, \varphi_0) \in \widehat{E_1} = (BA(\sum, \mu)) \times \mathbb{R} \times \mathbb{R}$ where μ is the Lebesgue measure on $\mathbb{R}$, $\sum$ is σ-algebra of the μ-measurable subsets $e \subset G$ and $BA(\sum, \mu)$ is the space of all bounded additive functions $\varphi_2(e)$ defined on $\sum$ with $\varphi_2(e) = 0$ when $\mu(e) = 0$ [10]. The inverse is also valid, that is, if $\varphi \in E_q$, then (3.2) is bounded on W_p

for $1 \le p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\varphi \in \widehat{E_1}$, *then* (3.3) *is bounded on* W_∞.

Proof. The operator $Nu \equiv (u''(x), u'(x_0), u(x_0)) : W_p \to E_p$ is bounded and has a bounded inverse N^{-1} represented by

$$u(x) = (N^{-1}h)(x) \equiv \int_{x_0}^{x} (x-\xi)h_2(\xi)d\xi + h_1(x-x_0) + h_0,$$
$$h = (h_2(x), h_1, h_0) \in E_p. \tag{3.4}$$

The kernel Ker N of N is trivial and the image Im N of N is equal to E_p. Hence, there exists a bounded adjoint operator $N^* : E_p^* \to W_p^*$ with Ker $N^* = \{0\}$ and Im $N^* = W_p^*$. In other words, for a given $F \in W_p^*$ there exists a unique $\psi \in E_p^*$ such that

$$F = N^*\psi \quad \text{or} \quad F(u) = \psi(Nu), \quad u \in W_p. \tag{3.5}$$

If $1 \le p < \infty$, then $E_p^* = E_q$ in the meaning of an isomorphism [10]. Therefore, the functional ψ can be represented by

$$\psi(h) = \int_{x_0}^{x_1} \varphi_2(x)h_2(x)dx + \varphi_1 h_1 + \varphi_0 h_0, \quad h \in E_p, \tag{3.6}$$

with a unique element $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$. By expressions (3.5) and (3.6), any $F \in W_p^*$ can uniquely be represented by (3.2). For a given $\varphi \in E_q$, the functional F represented by (3.2) is bounded on W_p. Hence, (3.2) is a general form for the functional $F \in W_p^*$.

The proof is complete due to that the case $p = \infty$ can also be shown [4]. □

Theorem 3.1 guarantees that $W_p^* = E_q$ for all $1 \le p < \infty$ [4].

4. Adjoint Operator And Adjoint System Of The Integro-algebraic Equations

An explicit form for the adjoint operator V^* of V is tried to investigate in this section. For this purpose, any $f = (f_2(x), f_1, f_0) \in E_q$ is taken as a linear bounded functional on E_p and also

$$f(Vu) \equiv \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + f_1(V_1u) + f_0(V_0u), \quad u \in W_p, \tag{4.1}$$

can be presumed. By substituting expressions (2.1) and (2.2), and expression (3.1) (for $\alpha = x_0$) of $u \in W_p$ into (4.1), we have

$$\begin{aligned} f(Vu) \equiv & \int_{x_0}^{x_1} f_2(x)[u''(x) + A_0(x)\{u(x_0) + u'(x_0)(x-x_0) \\ & + \int_{x_0}^{x} (x-\xi)u''(\xi)d\xi\} + A_1(x)u(x_0)]dx + f_1\{a_1u(x_0) + b_1u'(x_0) \\ & + \int_{x_0}^{x_1} g_1(\xi)u''(\xi)d\xi\} + f_0\{a_0u(x_0) + b_0u'(x_0) + \int_{x_0}^{x_1} g_0(\xi)u''(\xi)d\xi\}. \end{aligned} \tag{4.2}$$

After some calculations, we can obtain

$$\begin{aligned} f(Vu) \equiv & \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + \sum_{i=0}^{1} f_i(V_iu) \\ = & \int_{x_0}^{x_1} (w_2f)(\xi)u''(\xi)d\xi + (w_1f)u'(x_0) + (w_0f)u(x_0) \\ \equiv & (wf)(u), \quad \forall f \in E_q, \quad \forall u \in W_p, \quad 1 \le p \le \infty \end{aligned} \tag{4.3}$$

where

$$(w_2 f)(\xi) = f_2(\xi) + f_1 g_1(\xi) + f_0 g_0(\xi) + \int_{\xi}^{x_1} f_2(s) A_0(s)(s-\xi) ds,$$

$$w_1 f = b_1 f_1 + b_0 f_0 + \int_{x_0}^{x_1} f_2(s) A_0(s)(s-x_0) ds$$

$$w_0 f = a_1 f_1 + a_0 f_0 + \int_{x_0}^{x_1} f_2(s)[A_0(s) + A_1(s)] ds. \tag{4.4}$$

The operators w_2, w_1, w_0 are linear and bounded from the space E_q of the triples $f = (f_2(x), f_1, f_0)$ into the spaces $L_q(G), \mathbb{R}, \mathbb{R}$ respectively. Therefore, the operator $w = (w_2, w_1, w_0) : E_q \to E_q$ represented by $wf = (w_2 f, w_1 f, w_0 f)$ is linear and bounded. By (4.3) and Theorem 3.1, the operator w is an adjoint operator for the operator V when $1 \le p < \infty$, in other words, $V^* = w$. When $p = \infty$, $w : E_1 \to E_1$ is bounded; in this case, the operator w is the restriction of the adjoint operator $V^* : E_\infty^* \to W_\infty^*$ of V onto $E_1 \subset E_\infty^*$.

(2.3) can be transformed into the following equivalent equation

$$VSh = z, \tag{4.5}$$

with an unknown $h = (h_2, h_1, h_0) \in E_p$ by the transformation $u = Sh$ where $S = N^{-1}$. If $u = Sh$, then $u''(x) = h_2(x)$, $u'(x_0) = h_1$, $u(x_0) = h_0$. Hence, (4.3) can be rewritten as

$$f(VSh) \equiv \int_{x_0}^{x_1} f_2(x)(V_2 Sh)(x) dx + \sum_{i=0}^{1} f_i(V_i Sh)$$

$$= \int_{x_0}^{x_1} (w_2 f)(\xi) h_2(\xi) d\xi + (w_1 f) h_1 + (w_0 f) h_0$$

$$\equiv (wf)(h), \quad \forall f \in E_q, \quad \forall h \in E_p, \quad 1 \le p \le \infty. \tag{4.6}$$

Therefore, one of the operators VS and w becomes an adjoint operator for the other one. Consequently, the equation

$$wf = \varphi, \tag{4.7}$$

with an unknown function $f = (f_2(x), f_1, f_0) \in E_q$ and a given function $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ can be considered as an adjoint equation of (4.5)(or of (2.3)) for all $1 \le p \le \infty$. (16) can be written in the explicit form as the system of equations

$$(w_2 f)(\xi) = \varphi_2(\xi), \quad \xi \in G, \quad w_1 f = \varphi_1, \quad w_0 f = \varphi_0. \tag{4.8}$$

By the expressions (4.4), the first equation in (4.8) is an integral equation for $f_2(\xi)$ and includes f_1 and f_0 as parameters; on the other hand, the other equations in (4.8) constitute a system of two algebraic equations for the unknowns f_1 and f_0 and they include some integral functionals defined on $f_2(\xi)$. In other words, (4.8) is a system of three integro-algebraic equations. This system called the adjoint system for (4.5)(or (2.3)) is constructed by using (4.3) which is actually a formula of integration by parts in a nonclassical form. The traditional type of an adjoint problem is defined by the classical Green's formula of integration by parts [14], therefore, has a meaning only for some restricted class of problems.

5. Solvability Conditions For Completely Nonhomogeneous Problem

The operator $Q = w - I_q$ is considered where I_q is the identity operator on E_q, i.e. $I_q f = f$ for all $f \in E_q$. This operator can also be defined as $Q = (Q_2, Q_1, Q_0)$ with

$$(Q_2 f)(\xi) = (w_2 f)(\xi) - f_2(\xi), \quad \xi \in G,$$

$$Q_i f = w_i f - f_i, \quad i = 0, 1. \tag{5.1}$$

By the expressions (4.4) and the conditions imposed on A_0 and g_i for $i = 0, 1$, $Q_m : E_q \to L_q(G)$ is a compact operator, and also $Q_i : E_q \to \mathbb{R}$ for $i = 0, 1$ are compact operators where $1 < p < \infty$. That is, $Q : E_q \to E_q$ is a compact operator, and therefore has a compact adjoint operator $Q^* : E_p \to E_p$. Since $w = Q + I_q$ and $VS = Q^* + I_p$, where $I_p = I_q^*$, (4.5) and (4.7) are canonical equations, and S is a right regularizer of (2.3) [11]. As a result, we can have the following theorem [4]:

5.1. Theorem. *If $1 < p < \infty$, then $Vu = 0$ has either only the trivial solution or a finite number of linearly independent solutions in W_p:*
(1) *If $Vu = 0$ has only the trivial solution in W_p, then also $wf = 0$ has only the trivial solution in E_q. Then, the operators $V : W_p \to E_p$ and $w : E_q \to E_q$ become linear homeomorphisms.*
(2) *If $Vu = 0$ has m linearly independent solutions $u_1, u_2, ..., u_m$ in W_p, then $wf = 0$ has also m linearly independent solutions*

$$f^{\star 1\star} = (f_2^{\star 1\star}(x), f_1^{\star 1\star}, f_0^{\star 1\star}), ..., f^{\star m\star} = (f_2^{\star m\star}(x), f_1^{\star m\star}, f_0^{\star m\star})$$

in E_q. In this case, (2.3) and (4.7) have solutions $u \in W_p$ and $f \in E_q$ for given $z \in E_p$ and $\varphi \in E_q$ if and only if the conditions

$$\int_{x_0}^{x_1} f_2^{\star i\star}(\xi) z_2(\xi) d\xi + f_1^{\star i\star} z_1 + f_0^{\star i\star} z_0 = 0, \quad i = 1, 2, ..., m, \tag{5.2}$$

$$\int_{x_0}^{x_1} \varphi_2(\xi) u_i''(\xi) d\xi + \varphi_1 u_i'(x_0) + \varphi_0 u_i(x_0) = 0, \quad i = 1, 2, ..., m, \tag{5.3}$$

are satisfied, respectively.

6. Green's Functional and Special Adjoint System

Consider the following equation given in the form of a functional identity

$$(wf)(u) = u(x), \quad \forall u \in W_p, \tag{6.1}$$

where $f = (f_2(\xi), f_1, f_0) \in E_q$ is an unknown triple and $x \in \overline{G}$ is a parameter.

6.1. Definition. Suppose that $f(x) = (f_2(\xi, x), f_1(x), f_0(x)) \in E_q$ is a triple with parameter $x \in \overline{G}$. If $f = f(x)$ is a solution of (6.1) for a given $x \in \overline{G}$, then $f(x)$ is called a Green's functional of V (or of (2.3)).

Since the operator $I_{W_p,C}$ of the imbedding of W_p into the space $C(\overline{G})$ of continuous functions on $\overline{G}$ is bounded, the linear functional $\theta(x)$ defined by $\theta(x)(u) = u(x)$ is bounded on W_p for a given $x \in \overline{G}$. Moreover, $(wf)(u) = (V^* f)(u)$. Thus, (6.1) can also be written as [2, 3, 4]

$$(V^* f) = \theta(x).$$

In other words, (6.1) can be considered as a special case of the adjoint equation $V^* f = \psi$ when $\psi = \theta(x)$.

By substituting $\alpha = x_0$ into (3.1) and using (4.3), we can rewrite (6.1) as

$$\int_{x_0}^{x_1} (w_2 f)(\xi)u''(\xi)d\xi + (w_1 f)u'(x_0) + (w_0 f)u(x_0) = \int_{x_0}^{x} (x-\xi)u''(\xi)d\xi$$
$$+u'(x_0)(x-x_0) + u(x_0), \quad \forall f \in E_q, \quad \forall u \in W_p. \tag{6.2}$$

The elements $u''(\xi) \in L_p(G), u'(x_0) \in \mathbb{R}$ and $u(x_0) \in \mathbb{R}$ of the function $u \in W_p$ are unrelated. Then, we can construct the following system

$$(w_2 f)(\xi) = (x-\xi)H(x-\xi), \quad \xi \in G, \quad w_1 f = x - x_0, \quad w_0 f = 1, \tag{6.3}$$

where $H(x-\xi)$ is a Heaviside function on $\mathbb{R}$.

(6.1) is equivalent to the system (6.3) which is a special case for the adjoint system (4.8) when $\varphi_2(\xi) = (x-\xi)H(x-\xi)$, $\varphi_1 = x - x_0$ and $\varphi_0 = 1$. Therefore, $f(x)$ is a Green's functional if and only if $f(x)$ is a solution of the system (6.3) for an arbitrary $x \in \overline{G}$. For a solution $u \in W_p$ of (2.3) and a Green's functional $f(x)$, we can rewrite (4.3) as

$$\int_{x_0}^{x_1} f_2(\xi, x)z_2(\xi)d\xi + f_1(x)z_1 + f_0(x)z_0 = \int_{x_0}^{x_1} (x-\xi)H(x-\xi)u''(\xi)d\xi$$
$$+ u'(x_0)(x-x_0) + u(x_0). \tag{6.4}$$

The right hand side of (6.4) is $u(x)$. Therefore, we have the following theorem:

6.2. Theorem. *If* (2.3) *has at least one Green's functional $f(x)$, then any solution $u \in W_p$ of* (2.3) *can be represented by*

$$u(x) = \int_{x_0}^{x_1} f_2(\xi, x)z_2(\xi)d\xi + f_1(x)z_1 + f_0(x)z_0. \tag{6.5}$$

Additionally, $Vu = 0$ has only the trivial solution.

Since one of the operators $V : W_p \to E_p$ and $w : E_q \to E_q$ is a homeomorphism, so is the other, and, there exists a unique Green's functional, where $1 \le p \le \infty$. Necessary and sufficient conditions for the existence of a Green's functional can be given in the following theorem for $1 < p < \infty$.

6.3. Theorem. *If there exists a Green's functional, then it is unique. Additionally, a Green's functional exists if and only if $Vu = 0$ has only the trivial solution.*

From Theorem 5.1 and Theorem 6.2, Theorem 6.3 can be shown easily.

6.4. Remark. If $Vu = 0$ has a nontrivial solution, then a Green's functional corresponding to $Vu = z$ does not exist due to Theorem 6.2. In this case, $Vu = z$ usually has no solution unless z is of a specific type. Therefore, a representation of the existing solution of $Vu = z$ can be constructed by a generalized Green's functional concept [3, 4].

It must be noticed that the proposed Green's functional approach can also be applied some classes of nonlinear equations involving linear nonlocal conditions to transform into the corresponding integral equations and then solve them. The corresponding integral equations will naturally become nonlinear type. These nonlinear integral equations can be solved approximately even if they can not be solved exactly.

7. Illustrative Applications

In this section, two simple problem involving nonlocal boundary condition are considered in order to emphasize the preferability of the presented approach.

7.1. Example. Firstly, we seek for a Green's function to the following problem

$$(V_2u)(x) \equiv u''(x) = 2 + x^4 - u^2(x) = z_2(x), \quad x \in G = (0,1),$$
$$V_1u \equiv u'(1) = 2 = z_1,$$
$$V_0u \equiv u(0) + 3\int_0^1 u(x)dx = 1 = z_0.$$

Thus, we have $A_0(x) = A_1(x) = 0$ and

$$a_1 = 0, \quad b_1 = 1, \quad g_1(\xi) = 1, a_0 = 4, \quad b_0 = \frac{3}{2}, \quad g_0(\xi) = \int_\xi^1 3(x-\xi)dx.$$

Hence, the special adjoint system (6.3) corresponding to this problem can be constructed in the following form

$$f_2(\xi) + f_1 + f_0\int_\xi^1 3(x-\xi)dx = (x-\xi)H(x-\xi), \tag{7.1}$$
$$f_1 + \frac{3}{2}f_0 = x, \tag{7.2}$$
$$4f_0 = 1, \tag{7.3}$$

where $\xi \in (0,1)$. In order to solve (7.1)-(7.3), we firstly determine f_1 and f_0 by using only (7.2) and (7.3) owing to the condition $\Delta = 4 \neq 0$ where Δ is the determinant of coefficients matrix for (7.2) and (7.3). Thus, we have

$$f_1 = x - \frac{3}{8}, \quad f_0 = \frac{1}{4}.$$

After substituting f_1 and f_0 into equation (7.1), $f_2(\xi)$ becomes

$$f_2(\xi) = (x-\xi)H(x-\xi) - x + \frac{3}{8} - \frac{3(1-\xi)^2}{8}.$$

Thus, Green's functional $f(x) = (f_2(\xi,x), f_1(x), f_0(x))$ for the problem has been determined. The first component $f_2(\xi,x) = f_2(\xi)$ is Green's function for the problem. By (6.5) in Theorem 6.2, for the representation of the existing solution to the problem, the following equality

$$u(x) = \int_0^1 [(x-\xi)H(x-\xi) - x + \frac{3}{8} - \frac{3(1-\xi)^2}{8}][2 + \xi^4 - u^2(\xi)]d\xi + 2x - \frac{1}{2}$$

can be written easily. By the definition of Heaviside function, this last equality can also be written as

$$\begin{aligned} u(x) = &\int_0^x [-\xi + \frac{3}{8} - \frac{3(1-\xi)^2}{8}][2 + \xi^4 - u^2(\xi)]d\xi \\ &+ \int_x^1 [-x + \frac{3}{8} - \frac{3(1-\xi)^2}{8}][2 + \xi^4 - u^2(\xi)]d\xi + 2x - \frac{1}{2}. \end{aligned} \tag{7.4}$$

As can be seen, (7.4) is a nonlinear integral equation of Volterra type. In order to solve this equation exactly or approximately, the various analytical or numerical techniques can be utilized.

7.2. Example. Finally, we seek an integral equation yielding a Green's function to the following problem

$$(V_2u)(x) \equiv u''(x) + xu(0) = -\sin(x) = z_2(x), \quad x \in G = (0,1),$$
$$V_1u \equiv u'(1) + \int_0^1 u(x)dx = \frac{5}{2} = z_1,$$
$$V_0u \equiv u'(0) + \int_0^1 u'(x)dx = 3 + \sin(1) = z_0.$$

Thus, we have $A_0(x) = 0, A_1(x) = x$ and

$$a_1 = 1, \quad b_1 = \frac{3}{2}, \quad g_1(\xi) = \frac{3}{2} - \xi + \frac{\xi^2}{2}, a_0 = 0, \quad b_0 = 2, \quad g_0(\xi) = 1 - \xi.$$

Hence, the special adjoint system (6.3) corresponding to this problem can be constructed in the following form

$$f_2(\xi) + f_1[\frac{3}{2} - \xi + \frac{\xi^2}{2}] + f_0(1-\xi) = (x-\xi)H(x-\xi), \tag{7.5}$$

$$\frac{3}{2}f_1 + 2f_0 = x, \tag{7.6}$$

$$f_1 + \int_0^1 sf_2(s)ds = 1, \tag{7.7}$$

where $\xi \in (0,1)$. In order to solve (7.5)-(7.7), we firstly determine f_1 and f_0 by using only (7.6) and (7.7) owing to the condition $\Delta = -2 \neq 0$ where Δ is the determinant of coefficients matrix for (7.6) and (7.7). Thus, we have

$$f_1 = 1 - \int_0^1 sf_2(s)ds, \quad f_0 = \frac{x}{2} - \frac{3}{4} + \frac{3}{4}\int_0^1 sf_2(s)ds.$$

After substituting f_1 and f_0 into equation (7.5), an equation yielding $f_2(\xi)$ can be written as follows:

$$\begin{aligned} f_2(\xi) + [1 - \int_0^1 sf_2(s)ds][\frac{3}{2} - \xi + \frac{\xi^2}{2}] + [\frac{x}{2} - \frac{3}{4} + \frac{3}{4}\int_0^1 sf_2(s)ds](1-\xi) \\ = (x-\xi)H(x-\xi). \end{aligned} \tag{7.8}$$

As can be seen, (7.8) is a linear integral equation of Fredholm type. By solving this integral equation, firstly $f_2(\xi)$ and then f_1 and f_0 are identified. Thus, Green's functional $f(x) = (f_2(\xi,x), f_1(x), f_0(x))$ for the problem has been determined. The first component $f_2(\xi,x) = f_2(\xi)$ is Green's function for the problem.

8. Concluding Remarks

The proposed approach principally differs from the known classical methods used to construct of Green's function, it is based on the use of the structural properties of the solution space instead of the classical Green's formula of integration by parts, it decreases the difficulties emphasized in the introduction, and it has a natural and constructive property which can be easily applied to a very wide class of linear and some nonlinear boundary value problems involving nonlocal conditions. Owing to these properties, it is one of the useful methods which derive of a solution to such problems by reducing to an integral equation in general.

The introduced special adjoint system corresponding to the problem allows us to clarify about the existence and uniqueness of the solutions. A unique solution to the special adjoint system exists if and only if Green's function uniquely exists subject to the solvability conditions of the problem.

References

[1] Akhiev, S. S. *Representations of the solutions of some linear operator equations*, Soviet Math. Dokl., **21**(2), 555–558, 1980.

[2] Akhiev, S. S. *Fundamental solutions of functional differential equations and their representations*, Soviet Math. Dokl., **29**(2), 180–184, 1984.

[3] Akhiev, S. S. and Oruçoğlu, K. *Fundamental Solutions of Some Linear Operator Equations and Applications*, Acta Applicandae Mathematicae, **71**, 1-30, 2002.

[4] Akhiev, S. S. *Green and Generalized Green's Functionals of Linear Local and Nonlocal Problems for Ordinary Integro-differential Equations*, Acta Applicandae Mathematicae, **95**, 73–93, 2007.

[5] Alikhanov, A. A., Berezgov, A. M. and Shkhanukov-Lafishev, M. X. *Boundary Value Problems for Certain Classes of Loaded Differential Equations and Solving Them by Finite Difference Methods*, Computational Mathematics and Mathematical Physics, **48**(9), 1581–1590, 2008.

[6] Brown, A. L. and Page, A. *Elements of Functional Analysis*, New York, 1970.

[7] Denche, M. and Kourta, A. *Boundary Value Problem for Second-Order Differential Operators with Mixed Nonlocal Boundary Conditions*, Journal of Inequalities in Pure and Applied Mathematics, **5**(2), 1–16 2004.

[8] Fatemi, M. R. and Aliyev, N. A. *General Linear Boundary Value Problem for the Second-Order Integro-Differential Loaded Equation with Boundary Conditions Containing Both Nonlocal and Global Terms*, Abstract and Applied Analysis, **2010**, Article ID 547526, 1-12, 2010.

[9] Hörmander, L. *Linear Partial Differential Operators*, Springer-Verlag, New York, 1976.

[10] Kantorovich, L. V. and Akilov, G. P. *Functional Analysis* (2nd ed, translated by Howard L. Silcock), Pergamon Press, New York, 1982.

[11] Krein, S. G. *Linear Equations in Banach Space*, Nauka, Moscow, 1971 (in Russian).

[12] Naimark, M. A. *Linear Differential operators*, Nauka, Moscow, 1969 (in Russian).

[13] Shilov, G. E. *Mathematical Analysis: Second special course*, Nauka, Moscow, 1965; English transl., Generalized Functions and Partial Differential Equations, Gordon and Breach, New York, 1968.

[14] Stakgold, I. *Green's Functions and Boundary Value Problems*, Wiley-Interscience Publications, New York, 1998.

[15] Tikhonov, A. N., Vasil'eva, A. B. and Sveshnikov, A. G. *Differential Equations*, Nauka, Moscow, 1980 (in Russian).