Hacettepe Journal of Mathematics and Statistics
Volume 42 (4) (2013), 455 – 463

A SCHUR TYPE THEOREM FOR ALMOST COSYMPLECTIC MANIFOLDS WITH KAEHLERIAN LEAVES

Nesip Aktan*, Gülhan Ayar† and İmren Bektaş ‡ §

Received 05 : 07 : 2012 : Accepted 26 : 03 : 2013

Abstract

In this study, we give a Schur type theorem for almost cosymplectic manifolds with Keahlerian leaves.

Keywords: Contact Manifold, Cosymplectic Manifold, Sectional Curvature

2000 AMS Classification: 53D10, 53C15, 53C25, 53C35

1. Introduction

Let M be a Riemannian manifold with curvature tensor R. The sectional curvature of a 2-plane α in a tangent space T_PM is defined by $K(\alpha, P) = R(X, Y, Y, X)$, where $\{X, Y\}$ is an orthonormal basis of T_PM. The classical theorem of F. Schur says that if M is a connected manifold of dimension $n \geq 3$ and in any point $P \in M$ the curvature $K(\alpha, P)$ does not depend on $\alpha \in T_PM$ then it does not depend on the point P too, i.e. it is a global constant. Such a manifold is called a manifold of constant sectional curvature. The Shur's theorem has been studied by many authors for different structures [11]. In 1989, Nobuhiro improves the Shur's theorem and gets a new version for locally symmetric spaces [10]. In 2001, Kassabov considers connected $2n$-dimensional almost Hermitian manifold M to be of pointwise constant antiholomorphic sectional curvature $\nu(p)$, $p \in M$ and proves that ν is a global constant [6]. In 2006, Cho defines a contact strongly pseudo-convex CR space-form using the Tanaka-Webster connection in a way similar to the Sasakian space form and then he studies the geometry of such spaces. He presents a Schur type theorem for such structures [7]. The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969, [19]. The simplest examples of such manifolds are those being the products (possibly local) of almost Kaehlerian manifolds and the real line $\mathbb{R}$ or the circle S^1. Curvature properties of almost cosymplectic manifolds were studied mainly by Goldberg and Yano [12], Olszak [13], [14], Kirichenko [15] and Endo [16]. We

*Duzce University, Department of Mathematics. E-Mail: (N. Aktan) nesipaktan@gmail.com
†Duzce University, Department of Mathematics. E-Mail: (G. Ayar) gulhanayar@gmail.com
‡Duzce University, Department of Mathematics. E-mail: (İ. Bektaş) bektasimren@hotmail.com
§Corresponding author

relate some of them in a historical order. A cosymplectic manifold of constant curvature is necessarily locally flat [17]. The existence of locally flat cosymplectic manifolds is obvious. In fact, they are locally products of locally flat Kaehlerian manifolds and the real line (for instance, $C^n \times R$). If the curvature operator R of an almost cosymplectic manifold M commutes with the fundamental singular collineation φ, then M is normal, that is, it is a cosymplectic manifold [12]. In particular, an almost cosymplectic manifold of constant curvature is cosymplectic if and if it is locally flat. Generalizing this, it is proved in [13], [14]. that almost cosymplectic manifolds of non-zero constant curvature do not exist. For a conformally flat almost cosymplectic manifold of dimension ≥ 5, the scalar curvature r is non-positive and the manifold is cosymplectic if and only if it is locally flat [13], [14]. If M is an almost cosymplectic manifold of constant φ sectional curvature then the scalar curvature r and the φ sectional curvature H satisfy the inequality $n(n+1)H \geq r$. This equality holds if and only if the manifold is cosymplectic [13].

In this paper, we concentrate on almost cosymplectic manifolds with Kaehlerian leaves and considering Schur's lemma on spaces of constant curvature, we get a new version for almost cosymplectic manifolds with Kaehlerian leaves.

2. Almost Cosymplectic Manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold equipped with a triple (φ, ξ, η), where φ is a type of $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on M such that

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

which implies

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad rank(\varphi) = 2n. \tag{2.2}$$

If M admits a Riemannian metric g, such that

$$\begin{gathered} g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\,\eta(Y), \\ \eta(X) = g(X,\xi), \end{gathered} \tag{2.3}$$

then M is said to have an almost contact structure (φ, ξ, η, g). On such a manifold, the fundamental 2-form Φ of M is defined by

$$\Phi(X,Y) = g(\varphi X, Y),$$

for any vector fields X, Y on M. An almost contact manifold (M, φ, ξ, η) is said to be normal if the Nijenhuis torsion

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y] + 2d\eta(X,Y)\xi,$$

vanishes for any vector fields X, Y on M. As it is known that an almost contact metric structure is almost cosymplectic if and only if both $\triangledown\eta$ and $\triangledown\Phi$ vanish. A normal almost cosymplectic manifold is called a cosymplectic manifold. Let M be an almost cosymplectic manifold with structure (φ, ξ, η, g) and $\mathcal{D}$ is the distribution of M defined by $\mathcal{D} = \ker\eta$. Since $d\eta = 0$, $\mathcal{D}$ is integrable and the $(2n)$-dimensional distribution is given by $\varphi(\mathcal{D}) = \mathcal{D}$. Also, we obviously have ξ is orthogonal to $\mathcal{D}$. Let N be an maximal integral submanifold of $\mathcal{D}$. So the vector field ξ restricted to integral submanifold N is the normal vector of N. Hence, there exists a Hermitian structure. Moreover, the tensor field φ induces an almost complex structure J $(J^2 = -I)$ on M by $J\widetilde{X} = \varphi\widetilde{X}$ for any vector field $\widetilde{X}$ tangent to N. Let G be the Riemannian metric induced on N defined by $G(\widetilde{X}, \widetilde{Y}) = g(\widetilde{X}, \widetilde{Y})$. Then (J, G) becomes an almost Hermitian structure on N such that $G(\widetilde{X}, \widetilde{Y}) = G(J\widetilde{X}, J\widetilde{Y})$ for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ tangent to N. The fundamental 2-form Ω, $\Omega(\widetilde{X}, \widetilde{Y}) = G(J\widetilde{X}, \widetilde{Y})$ of (J, G) induced on N. We also have $\Omega(\widetilde{X}, \widetilde{Y}) = \Phi(X, Y)$, that is, Ω is the pull-back of the tensor field φ from M to N. As a result, Ω is closed, i.e., $d\Omega = 0$. So the pair (J, G) is an almost Kaehlerian structure on N of $\mathcal{D}$. Therefore, when the structure J is complex, (J, G) becomes a Kaehlerian structure on N. If the structure (J, G) is Kaehlerian on every integral submanifold of the distribution $\mathcal{D}$, such manifold is said to be an almost cosymplectic manifold with Kaehlerian integral submanifold. Suppose that M is an almost cosymplectic manifold. Denote by A the $(1, 1)$-tensor field on M defined by

(2.4) $\quad A = -\nabla\xi,$

and by h the $(1, 1)$-tensor field given by the following relation

$$h = \tfrac{1}{2}\mathcal{L}_\xi\varphi,$$

where $\mathcal{L}$ is the Lie derivative of g. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$. Moreover, the tensor fields A and h are symmetric operators and satisfy the following relations

(2.5) $\quad \nabla_X\xi = -\varphi hX,$

(2.6) $\quad (\varphi \circ h)X + (h \circ \varphi)X = 0,$

(2.7) $\quad (\nabla_X\eta)Y = g(\varphi Y, hX),$

(2.8) $\quad \delta\eta = 0, \quad tr(h) = 0,$

(2.9) $\quad tr(A) = 0,$

(2.10) $\quad tr(\varphi A) = 0,$

(2.11) $\quad A\varphi + \varphi A = 0,$

(2.12) $\quad A\xi = 0,$

(2.13) $\quad (\nabla_X A)\,\xi = A^2X,$

(2.14) $\quad tr(A^2) = \left\|A^2\right\|,$

for any vector fields X, Y on M.We also remark that

(2.15) $\quad h = 0 \Leftrightarrow \nabla\xi = 0.$

2.1. Proposition. *Let M be an almost cosymplectic manifold. M has Kaehlerian leaves if and only if it satisfies the condition*

(2.16) $\quad (\nabla_X\varphi)Y = -g(\varphi AX, Y)\xi + \eta(Y)\varphi AX.$

for any vector fields X, Y on M [1].

3. Basic Curvature Relations

In this section, we will briefly give the basic curvature relations. Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold. We denote the curvature tensor and Ricci tensor of g by R and S, respectively. We define a self adjoint operator $l = R(., \xi)\xi$ (The Jacobi operator with respect to ξ). One easily see the followings.

3.1. Proposition. *Let M be an almost cosymplectic manifold. Then we have*

(3.1) $\quad R(X, Y)\xi = (\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y,$

(3.2) $\quad R(X, Y)\xi = -(\nabla_X A)Y + (\nabla_Y A)X,$

(3.3) $\quad R(X, \xi)\xi = -h^2X + \varphi(\nabla_\xi h)X,$

(3.4) $\quad (\nabla_\xi h)X = -\varphi R(X, \xi)\xi - \varphi h^2X,$

(3.5) $\quad R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi = -2\left[h^2X\right],$

(3.6) $\quad S(X, \xi) = -\sum_{i=1}^{2n+1} g((\nabla_{e_i}\varphi h)\,e_i, X),$

$$tr\,(l) = S(\xi,\xi) = -tr(h^2). \tag{3.7}$$

for any vector fields X, Y on M.

By simple computations, we have the following proposition that will be used in the next important result.

3.2. Proposition. *For the curvature transformation of almost cosymplectic manifold with Kaehlerian leaves, we have*

$$\begin{aligned} &R(X,Y)\varphi Z - \varphi R(X,Y)Z = g(AX,\varphi Z)AY - g(AY,\varphi Z)AX - g(AX,Z)\varphi AY \\ &+g(AY,Z)\varphi AX - \eta(Z)\varphi(R(X,Y)\xi) - g(R(X,Y)\xi,\varphi Z)\xi, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &R(\varphi X,\varphi Y)Z - R(X,Y)Z = \eta(Y)R(\xi,X,Z) + g(AZ,\varphi X)A\varphi Y - g(AZ,\varphi Y)A\varphi X \\ &-g(AZ,X)AY + g(AZ,Y)AX - \eta(X)R(\xi,Y,Z) + \eta(X)\eta(Y)R(\xi,\xi). \end{aligned} \tag{3.9}$$

3.3. Proposition. *If we denote*

$$P_\varphi\,(X,Y) = (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y,$$

and

$$P\,(X,Y) = (\nabla_Y h)X - (\nabla_X h)Y.$$

Then, we satisfy following relations

$$\begin{aligned} &P_\varphi\,(X,Y) = \varphi P\,(X,Y)\,, \\ &\varphi P_\varphi\,(X,Y) = -P\,(X,Y) + 2g\,(hX,\varphi hY)\,\xi, \\ &P_\varphi\,(X,Y) = -P_\varphi\,(Y,X)\,. \end{aligned}$$

3.4. Proposition. *Let M be an almost cosymplectic manifold. The necessary and sufficient condition for M to have pointwise constant $\varphi-$holomorphic sectional curvature H is*

$$\begin{aligned} &4R(X,Y,Z,W) = H\,[g(X,W)g(Z,Y) - g(X,Z)g(W,Y)] \\ &-H\,[\eta\,(X)\,\eta\,(W)\,g(Z,Y) + \eta(Y)\eta\,(Z)\,g(X,W) \\ &+2g(X,\varphi Y)g(Z,\varphi W) - \eta\,(Y)\,\eta\,(W)\,g(X,Z) - \eta(X)\eta\,(Z)\,g(W,Y)] \\ &+H\,[g(X,\varphi Z)g(W,\varphi Y) - g(X,\varphi W)g(Z,\varphi Y] \\ &-g(AX,\varphi Z)g(AY,\varphi W) + g(AW,\varphi X)g(AZ,\varphi Y) \\ &-g(AZ,\varphi X)g(AW,\varphi Y) + g(AX,\varphi W)g(AY,\varphi Z) \\ &+2g(AX,Z)g(AW,Y) - 2g(AX,W)g(AZ,Y) \\ &+4\eta(X)P_\varphi(Z,W,Y) + 4\eta(Z)P_\varphi(X,Y,W) \\ &-4\eta(W)P_\varphi(X,Y,Z) - 4\eta(X)\eta\,(W)\,P_\varphi(Z,\xi,Y) \\ &-4\eta(X)\eta\,(Z)\,P_\varphi(\xi,W,Y) - 4\eta(X)\eta\,(Y)\,P_\varphi(Z,W,\xi) \\ &-4\eta(Y)P_\varphi(Z,W,X) + 4\eta(Y)\eta\,(W)\,P_\varphi(Z,\xi,X) \\ &+4\eta(Y)\eta\,(Z)\,P_\varphi(\xi,W,X) + 4\eta(X)\eta\,(Z)\,P_\varphi(\xi,Y,W) \\ &-4\eta(X)\eta(W)P_\varphi(\xi,Y,Z). \end{aligned} \tag{3.10}$$

for all vector fields X, Y, Z, W in M.

Proof. For any vector fields X and $Y \in \mathcal{D}$,we have

$$g\,(R(X,\varphi X)X,\varphi X) = -Hg\,(X,X)^2 \tag{3.11}$$

By (3.8)we get

$$\begin{aligned} &R(X,\varphi Y,X,\varphi Y) = R(X,\varphi Y,Y,\varphi X) + g(AX,\varphi X)g(AY,\varphi Y) \\ &-g(A\varphi Y,\varphi X)g(AX,Y) + g(A\varphi Y,\varphi Y)g(AX,X) \\ &-g(A\varphi Y,X)g(AX,\varphi Y), \end{aligned} \tag{3.12}$$

$$R(X,\varphi X,Y,\varphi X)=R(X,\varphi X,X,\varphi Y), \tag{3.13}$$

for $X,Y\in\mathcal{D}$. Submitting $X+Y$ in (3.11),we see

$$\begin{aligned}&-H\left[2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right]\\&=\tfrac{1}{2}\left(gR(X+Y,\varphi X+\varphi Y)\left(X+Y\right),\varphi X+\varphi Y\right)+\tfrac{1}{2}H\left(g(X,X)^2+g(Y,Y)^2\right),\end{aligned}$$

and because of (3.8) and (3.13) and the Bianchi identity

$$\begin{aligned}&-H\left[2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right]\\&=2R(X,\varphi X,X,\varphi Y)+2R(X,\varphi Y,Y,\varphi X)+R(X,Y,\varphi X,\varphi Y)\\&+2R(Y,\varphi Y,Y,\varphi X)+R(X,\varphi Y,X,\varphi Y)\\&+\tfrac{1}{2}\left[g(AY,Y)g(A\varphi X,\varphi X)-g(AY,\varphi X)^2-g(AX,X)g(A\varphi Y,\varphi Y)\right.\\&\left.+g(AX,\varphi Y)^2\right],\end{aligned}$$

then because of (3.9) and (3.12), we get

$$\begin{aligned}&2R(X,\varphi X,X,\varphi Y)+2R(Y,\varphi X,Y,\varphi Y)+3R(X,\varphi Y,Y,\varphi X)\\&+R(X,Y,X,Y)+\tfrac{1}{2}\left[2g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)-2g\left(AX,\varphi Y\right)g\left(AY,\varphi X\right)\right.\\&\left.-2g\left(AX,X\right)g\left(AY,Y\right)\right]+4g\left(AX,Y\right)^2-g\left(AX,\varphi Y\right)^2\\&+2g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)+g\left(AY,Y\right)g\left(A\varphi X,\varphi X\right)\\&-g\left(AY,\varphi X\right)^2-g\left(AX,X\right)g\left(A\varphi Y,\varphi Y\right)\\&=-H\left[2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right]\end{aligned}\tag{3.14}$$

Replacing Y by $-Y$ in (3.14) and summing it to (3.14) we have

$$\begin{aligned}&3R\left(X,\varphi Y,Y,\varphi X\right)+R\left(X,Y,X,Y\right)=-H\left[2g\left(X,Y\right)^2+g\left(X,X\right)g\left(Y,Y\right)\right]\\&-2g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)+g\left(AX,\varphi Y\right)g\left(AY,\varphi X\right)+2g\left(AX,X\right)g\left(AY,Y\right)\\&+2g\left(AX,\varphi Y\right)^2-4g\left(AX,Y\right)^2-\tfrac{1}{2}\left[g\left(AY,Y\right)g\left(A\varphi X,\varphi X\right)\right.\\&\left.-g\left(AY,\varphi X\right)^2-g\left(AX,X\right)g\left(A\varphi Y,\varphi Y\right)\right],\end{aligned}\tag{3.15}$$

By virtue of (3.15) we see

$$\begin{aligned}&8R\left(X,Y,X,Y\right)=H\left[2g\left(X,\varphi Y\right)^2+g\left(X,X\right)g\left(\varphi Y,\varphi Y\right)\right.\\&\left.+2g\left(X,Y\right)^2+g\left(X,X\right)g\left(Y,Y\right)\right]\\&-4g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)+8g\left(AX,\varphi Y\right)g\left(AY,\varphi X\right)\\&+\tfrac{17}{2}g\left(AX,X\right)g\left(AY,Y\right)-\tfrac{11}{2}g\left(AX,Y\right)^2-\tfrac{7}{2}g\left(AX,\varphi Y\right)^2\\&+\tfrac{1}{2}g\left(AY,Y\right)g\left(A\varphi X,\varphi X\right)-\tfrac{1}{2}g\left(AY,\varphi X\right)^2+\tfrac{5}{2}g\left(AX,X\right)g\left(A\varphi Y,\varphi Y\right)\\&-\tfrac{3}{2}\left[g\left(A\varphi Y,\varphi Y\right)g\left(A\varphi X,\varphi X\right)-g\left(AY,\varphi X\right)^2\right]\end{aligned}\tag{3.16}$$

We verify (3.16), replacing Y by φY in (3.15), together with (3.9) and (3.12)

$$\begin{aligned}&-H\left[2g\left(X,\varphi Y\right)^2+g\left(X,X\right)g\left(\varphi Y,\varphi Y\right)\right]=3R\left(X,Y,X,Y\right)\\&+R\left(X,\varphi Y,Y,\varphi X\right)+2g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)\\&-3g\left(AX,\varphi Y\right)g\left(AY,\varphi X\right)-\tfrac{7}{2}g\left(AX,X\right)g\left(AY,Y\right)+\tfrac{5}{2}g\left(AX,Y\right)^2\\&-g\left(AX,X\right)g\left(A\varphi Y,\varphi Y\right)+g\left(AX,\varphi Y\right)^2\\&+\tfrac{1}{2}\left[g\left(A\varphi Y,\varphi Y\right)g\left(A\varphi X,\varphi X\right)-g\left(A\varphi Y,\varphi X\right)^2\right],\end{aligned}\tag{3.17}$$

and because of (3.15)

$$\begin{aligned}&-H\left[2g\left(X,\varphi Y\right)^2+g\left(X,X\right)g\left(\varphi Y,\varphi Y\right)\right]=3R\left(X,Y,X,Y\right)\\&-\tfrac{1}{3}R\left(X,Y,X,Y\right)-\tfrac{H}{3}\left[2g\left(X,Y\right)^2+g\left(X,X\right)g\left(Y,Y\right)\right]\\&+\tfrac{4}{3}g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)-\tfrac{8}{3}g\left(AX,\varphi Y\right)g\left(AY,\varphi X\right)\\&-\tfrac{17}{6}g\left(AX,X\right)g\left(AY,Y\right)+\tfrac{11}{6}g\left(AX,Y\right)^2+\tfrac{7}{6}g\left(AX,\varphi Y\right)^2\\&-\tfrac{1}{6}g\left(AY,Y\right)g\left(A\varphi X,\varphi X\right)+\tfrac{1}{6}g\left(AY,\varphi X\right)^2-\tfrac{5}{6}g\left(AX,X\right)g\left(A\varphi Y,\varphi Y\right)\\&+\tfrac{1}{2}\left[g\left(A\varphi Y,\varphi Y\right)g\left(A\varphi X,\varphi X\right)-g\left(A\varphi Y,\varphi X\right)^2\right]\end{aligned}$$

After simplification (3.16) follows. Therefore by a standard calculation we have

$$\begin{aligned}
&8R(X,Y,X,Y) = -3H\left[2g(X,\varphi Y)^2 + g(X,X)\,g(\varphi Y,\varphi Y)\right] \\
&+H\left[2g(X,Y)^2 + g(X,X)\,g(Y,Y)\right] - 4g(AX,\varphi X)\,g(AY,\varphi Y) \\
&+8g(AX,\varphi Y)\,g(AY,\varphi X) + \tfrac{17}{2}g(AX,X)\,g(AY,Y) \\
&-\tfrac{11}{2}g(AX,Y)^2 - \tfrac{7}{2}g(AX,\varphi Y)^2 + \tfrac{1}{2}g(AY,Y)\,g(A\varphi X,\varphi X) \\
&-\tfrac{1}{2}g(AY,\varphi X)^2 + \tfrac{5}{2}g(AX,X)\,g(A\varphi Y,\varphi Y) \\
&-\tfrac{3}{2}\left[g(A\varphi Y,\varphi Y)\,g(A\varphi X,\varphi X) - g(A\varphi Y,\varphi X)^2\right].
\end{aligned} \tag{3.18}$$

for any $X, Y \in \mathcal{D}$. Firstly, replacing $X = X + Z$ in (3.18) and then replacing $Y = Y + W$ in obtained result and by using Bianchi identity and (2.6) we get

$$\begin{aligned}
&48R(X,W,Z,Y) = H\,[12g(X,Y)\,g(Z,W) \\
&-12g(X,\varphi Y)\,g(Z,\varphi W) - 24g(X,\varphi W)\,g(Z,\varphi Y) \\
&-12g(X,Z)g(Y,W) + 12g(X,\varphi Z)g(Y,\varphi W)] \\
&+3g(AX,\varphi Z)\,g(AY,\varphi W) - 3g(AX,\varphi Y)\,g(AZ,\varphi W) \\
&-12g(AX,\varphi Z)\,g(AW,\varphi Y) + 12g(AX,\varphi Y)\,g(AW,\varphi Z) \\
&-12g(AZ,\varphi X)\,g(AY,\varphi W) + 12g(AY,\varphi X)\,g(AZ,\varphi W) \\
&-3g(AZ,\varphi X)\,g(AW,\varphi Y) + 3g(AY,\varphi X)\,g(AW,\varphi Z) \\
&+15g(AX,\varphi W)\,g(AY,\varphi Z) - 15g(AX,\varphi W)\,g(AZ,\varphi Y) \\
&+9g(AZ,\varphi Y)\,g(AW,\varphi X) - 9g(AY,\varphi Z)\,g(AW,\varphi X) \\
&+24g(AX,Z)\,g(AY,W) - 24g(AX,Y)\,g(AZ,W).
\end{aligned} \tag{3.19}$$

where $X, Y, Z, W \in \mathfrak{D}$.We now let X be an arbitrary vector field on M.Then we may write

$$X = X^T + \eta(X)\,\xi$$

where X^T denotes the horizontal part of X.Then we have all vector fields X, Y, Z, W in M.

$$\begin{aligned}
&R(X,Y,Z,W) = R(X^T,Y^T,Z^T,W^T) \\
&+\eta(X)R(\xi,Y^T,Z^T,W^T) + \eta(Y)R(X^T,\xi,Z^T,W^T) \\
&+\eta(Z)R(X^T,Y^T,\xi,W^T) + \eta(W)R(X^T,Y^T,Z^T,\xi) \\
&+\eta(X)\eta(Z)R(\xi,Y^T,\xi,W^T) + \eta(X)\eta(W)R(\xi,Y^T,Z^T,\xi) \\
&+\eta(Y)\eta(Z)R(X^T,\xi,\xi,W^T) + \eta(Y)\eta(W)R(X^T,\xi,Z^T,\xi).
\end{aligned} \tag{3.20}$$

If we use (3.19) and (3.20) the proof is completed. □

Moreover, from (3.10), we get

$$\begin{aligned}
&S(Y,Z) = \tfrac{1}{2}\left[(n+1)H\right]\{g(Y,Z) - \eta(Y)\,\eta(Z)\} \\
&+\eta(Z)\sum P_\varphi(E_i,Y,E_i) - \eta(Y)\sum P_\varphi(Z,E_i,E_i) \\
&+\eta(Y)\,\eta(Z)\sum P_\varphi(\xi,E_i,E_i) - 2P_\varphi(\xi,Y,Z).
\end{aligned} \tag{3.21}$$

for all vector fields X and Y in M where $\{E_i\}\,(i = 1, 2, ..., 2n+1)$ is an arbitrary local orthonormal frame field on M since the trace of h vanishes,from (3.21),we have for the scalar curvature

$$\tau = n(n+1)H - 2Tr(h^2). \tag{3.22}$$

4. A class of almost cosymplectic manifolds $\mathfrak{D}$

There are two typical examples of contact manifolds;one is formed by the principal circle bundles over symplectic manifolds of integral class (including the odd-dimensional spheres) and the other is given by the unit tangent sphere bundles. The former admit a Riemannian metric which is Sasakian. Concerning the latter, in [20], it was proved that the associated CR-structure of a unit tangent sphere bundle T_1M with standard contact Riemannian structure is integrable if and only if the base manifold is of constant curvature. Here,we note that the unit tangent sphere bundle of a space of constant curvature satisfies ([21])

$$g\left((\nabla_{X^T}h)Y^T, Z^T\right) = 0. \tag{4.1}$$

That is, h is $\eta-$parallel. Now, we consider a contact Riemannian manifold whose structure tensor h satisfies. (4.1) and (3.4) simultaneously. Then

$$\begin{aligned}
&0 = g\left((\nabla_{X^T}h)Y^T, Z^T\right) = g\left((\nabla_{X-\eta(X)\xi}h)(Y-\eta(Y)\xi, Z-\eta(Z)\xi\right)\\
&= g\left((\nabla_X h)Y, Z\right) - \eta(X)\, g\left((\nabla_\xi h)Y, Z\right) - \eta(Y)\, g\left((\nabla_X h)\xi, Z\right)\\
&-\eta(Z)\, g\left((\nabla_X h)Y, \xi\right) + \eta(X)\,\eta(Y)\, g\left((\nabla_\xi h)\xi, Z\right) + \eta(Y)\,\eta(Z)\, g\left((\nabla_X h)\xi, \xi\right)\\
&+\eta(Z)\,\eta(X)\, g\left((\nabla_\xi h)Y, \xi\right) - \eta(X)\,\eta(Y)\,\eta(Z)\, g\left((\nabla_\xi h)\xi, \xi\right).
\end{aligned}$$

From the above equation ,by using (2.6), (2.7) and using (3.4),we have

$$(\nabla_X h)Y = \eta(X)\left[-\varphi lY - \varphi h^2 Y\right] - \eta(Y)\left(\varphi h^2 X\right) - g\left(\varphi h^2 X, Y\right)\xi. \tag{4.2}$$

Moreover from (3.22) we have

$$P(X,Y) = \eta(X)\,\varphi lY - \eta(Y)\,\varphi lX - 2g\left(\varphi h^2 X, Y\right)\xi, \tag{4.3}$$

$$P_\varphi(X,Y) = -\eta(X)\, lY + \eta(Y)\, lX. \tag{4.4}$$

for any vector fields X and Y Now we define a $(1,2)$- tensor field $Q_1(X,Y)$ by

$$\begin{aligned}
&Q_1(X,Y) = (\nabla_X h)Y - \eta(X)\left[-\varphi lY - \varphi h^2 Y\right]\\
&-\eta(Y)\left[\varphi h^2 X\right] + g\left(\varphi h^2 X, Y\right)\xi.
\end{aligned}$$

4.1. Definition. The class $\mathfrak{D}$ is given by the spaces of almost cosymplectic manifold with Kaehlerian leaves satisfying $Q_1 = 0$, that is

$$\mathfrak{D} = \{(M, \phi, \xi, \eta, g) : Q_1 = 0\}.$$

We can see that this class $\mathfrak{D}$ is invariant under D-homothetic deformations [21].

4.2. Lemma. *Let M be a space $\in \mathfrak{D}$ then the eigenvalues of h are constant.*

5. Shur Type Theorem

5.1. Theorem. *Let M be an almost cosymplectic manifold with Kaehlerian leaves belonging to the class $\mathfrak{D}$. If the φ-holomorphic sectional curvature at any point of M is independent of the choice of φ-holomorphic section, then it is constant on M and the curvature tensor is given by*

$$\begin{aligned}
&4R(X,Y,Z,W) = c\left[g(X,W)g(Z,Y) - g(X,Z)g(W,Y)\right]\\
&-c\left[\eta(X)\,\eta(W)\, g(Z,Y) + \eta(Y)\eta(Z)\, g(X,W)\right)\\
&+2g(X,\varphi Y)g(Z,\varphi W - \eta(Y)\,\eta(W)\, g(X,Z)\\
&-\eta(X)\eta(Z)\, g(W,Y]\\
&+H\left[g(X,\varphi Z)g(W,\varphi Y) - g(X,\varphi W)g(Z,\varphi Y)\right]\\
&-g(AX,\varphi Z)g(AY,\varphi W) + g(AW,\varphi X)g(AZ,\varphi Y)\\
&-g(AZ,\varphi X)g(AW,\varphi Y) + g(AX,\varphi W)g(AY,\varphi Z)\\
&+2g(AX,Z)g(AW,Y) - 2g(AX,W)g(AZ,Y).
\end{aligned} \tag{5.1}$$

for all vector fields X, Y, Z, W in M .

Proof. Suppose that M has pointwise constant φ-holomorphic sectional curvature H .Then, taking account of (4.2), (4.3) and (4.4),from (3.21) we obtain

$$\begin{aligned}
&S(Y,Z) = \tfrac{1}{2}\left[(n+1)\,H\right]\{g(Y,Z) - \eta(Y)\,\eta(Z)\}\\
&+Tr(l)\,\eta(Y)\,\eta(Z) + 2g(lY,Z),
\end{aligned} \tag{5.2}$$

$$\tau = n(n+1)\,H + 3Tr(l). \tag{5.3}$$

From (4.2) and by using (2.16) and Lemma 4.2, we have

$$\begin{aligned}
&2\left(\nabla_X S\right)(Y,Z) = \left[(n+1)\,X(H)\right]\left\{g(Y,Z)-\eta(Y)\,\eta(Z)\right\}\\
&+\left[2Tr(l)-(n+1)\,H\right]\left\{\eta(Z)\,g(Y,\nabla_X\xi)-\eta(Y)\,g(Z,\nabla_X\xi)\right\}\\
&+4g\left(\left(\nabla_X l\right)Y,Z\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\textstyle\sum 2\left(\nabla_{E_{\dot{I}}}S\right)(Y,E_{\dot{I}}) = \sum\left[(n+1)\,E_{\dot{I}}(H)\right]\left\{g(Y,E_{\dot{I}})-\eta(Y)\,\eta(E_{\dot{I}})\right\}\\
&\textstyle+\sum\left[2Tr(l)-(n+1)\,H\right]\left\{\eta(Y)\,g\left(E_{\dot{I}},\nabla_{E_{\dot{I}}}\xi\right)-\eta(E_{\dot{I}})\,g\left(Y,\nabla_{E_{\dot{I}}}\xi\right)\right\}\\
&\textstyle+\sum 4g\left(\left(\nabla_{E_{\dot{I}}}l\right)Y,E_{\dot{I}}\right)\\
&\textstyle=(n+1)\sum E_{\dot{I}}(H)\,g(Y,E_{\dot{I}})-(n+1)\,\xi(H)\,\eta(Y)+\sum 4g\left(\left(\nabla_{E_{\dot{I}}}l\right)Y,E_{\dot{I}}\right).
\end{aligned}\tag{5.4}$$

by the well-known formula

$$\left(\nabla_X\tau\right)=2\sum\left(\nabla_{E_{\dot{I}}}S\right)(X,E_{\dot{I}}).$$

for any local orthonormal frame field $\{E_i\}\,(i=1,2,...,2n+1)$ and by using (5.3), (5.4) and Lemma 4.2, we have

$$(n+1)\left\{XH-(\xi H)\,\eta(X)\right\}=2n\,(n+1)\,XH.$$

This says that $\xi H=0$ and $(n-1)\,XH=0$ .Since $n>1$,we see that H is constant,say c. by applying (4.2), (4.3) and (4.4) in Proposition 3.4, we obtain (5.1) □

5.2. Definition. A complete and simply connected almost cosymplectic manifold of class $\mathfrak{D}$ with constant φ-holomorphic sectional curvature is said to be an almost cosymplectic space form.

So, from the proof of Proposition 3.4 and Theorem 5.1, we have,

5.3. Theorem. *Let M be a complete and simply connected almost cosymplectic space belonging to the class $\mathfrak{D}$. Then M is an almost cosymplectic space form if and only if the curvature tensor R is given by (5.1).*

References

[1] Kim, T. W. and Pak, H. K. *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. **4** (21), 841–846, 2005.

[2] Dileo, G. and Pastore, A. M. *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin **14**, 343–354, 2007.

[3] Boeck, E. and Cho, J. T. *η-parallel contact metric spaces, Differential geometry and its applications* **22**, 275–285, 2005.

[4] Blair, D. E. *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, **203**. (Birkhâuser Boston, Inc., Boston, MA, 2002.

[5] Vaisman, I. *Conformal changes of almost contact metric manifolds*, Lecture Notes in Math. (Berlin-Heidelberg-New York, 1980), 435–443.

[6] Kassabov, O. T. *Schur's theorem for almost Hermitian manifolds*, C. R. Acad. Bulg. Sci. **54** (3), 15–18, 2001.

[7] Cho, J. T. *Geometry of contact strongly pseudo-convex CR-manifolds*, J. Korean Math. **43** (5), 1019–1045, 2006.

[8] Kulkarni, R. S. *On a theorem of F. Shur*, Journal Diff. Geom. **4**, 453–456, 1970.

[9] Gabriel, E. V. *A Schur-type Theorem on Indefinite Quaternionic Keahler Manifolds*, Int. J. Contemp. Math. **11** (2), 529–536, 2007.

[10] Nobuhiro, I. *A theorem of Schur type for locally symmetric spaces*, Sci. Rep. Niigata Univ., Ser. A **25**, 1–4, 1989.

[11] Schur, F. *Ueber den Zusammenhang der Raume constanten Riemann'schen Kriimmungsmasses mit den projectiven Raumen.* Math. **27**, 537–567, 1886.

[12] Goldberg, S. I. and Yano, K. *Integrability of almost cosymplectic structures*, Pacific J. Math. **31**, 373–382, 1969.

[13] Olszak, Z. *On almost cosymplectic manìfolds*, Kodai Math. J. **4**, 239–250, 1981.

[14] Olszak, Z. *Almost cosymplectic manìfolds with Kàhlerian leaves*, Tensor N. S. **46**, 117–124, 1987.

[15] Kirichenko, V. F. *Almost cosymplectic manifolds satisfying the axiom of ϕ-Pholomorphic planes* (in Russian), Dokl. Akad. Nauk SSSR **273**, 280–284,1983.

[16] Endo, H. *On Ricci curvatures of almost cosymplectic manifolds*, An. Stiint. Univ. "Al. I. Cuza" Iasi, Mat. **40**, 75–83, 1994.

[17] Blair, D. E. *The theory of quasi-Sasakian structures*, J. Diff. Geometry, **1**, 331–345, 1967.

[18] Dacko, P. and Olszak, Z. *On conformally flat almost cosymplectic manifolds with Keahlerian leaves*, Rend. Sem. Mat. Univ. Pol. Torino, **56** (1), 89–103, 1998.

[19] Goldberg, S. I. and Yano, K. *Integrability of almost cosymplectic structure*, Pacific J. Math. **31** , 373–382, 1969

[20] Tanno, S. *The standard CR structure on the unit tangent bundle* Tohoku Math. J. **44** (2), 535–543, 1992.

[21] Blair, D. E. *Contact metric manifolds satisfying a nullity condition* Israel J.of Math. **91**, 1–3, 189-214, 1995.