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ON EDELSTEIN TYPE MULTIVALUED RANDOM OPERATORS

Akbar Azam *, Muhammad Arshad † and Pasquale Vetro ‡

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Abstract

The purpose of this paper is to provide stochastic versions of several results on fixed point theorems in the literature.

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1. Introduction and Preliminaries

Random operator theory is needed for the study of various classes of random operator equations in probabilistic functional analysis. During the last three decades several results (e.g., see, [3, 4, 6, 8, 10, 11, 13, 14, 15] and references therein) regarding random fixed points of various types of random operators have been established and a number of their applications have been obtained after a survey article of Bharucha Reid [5]. In fact, random fixed point theorems are stochastic generalizations of deterministic/classical fixed point theorems and have important applications in random operator equations, random differential equations and differential inclusions [5, 6, 7, 10]. In the present paper we derive common random fixed point theorems for a sequence of multivalued random operators satisfying Edelstein type contractive condition. We give, also a result of a common random fixed point for a sequence of multivalued random operators that have a common deterministic fixed point. Our paper establish stochastic versions of many Banach type fixed point theorems e.g., see, [2] and references therein.

*Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad, 44000, Pakistan. E-mail: (A. Azam) akbarazam@yahoo.com

†Department of Mathematics, International Islamic University, H-10, Islamabad,44000, Pakistan. E-mail: (M. Arshad) marshad_zia@yahoo.com

‡Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi 34, 90123 Palermo, Italy E-mail: (P. Vetro) vetro@math.unipa.it

For a metric space (X, d), we denote by 2^X the family of all nonempty subsets of X, $CB(X)$ the family of all nonempty closed and bounded subsets of X, we define Hausdorff metric H on $CB(X)$ as follows:

$$H(A, B) = \max\left\{\sup_{\alpha \in A} d(a, B), \sup_{b \in B} d(A, b)\right\}$$

for $A, B \in CB(X)$, where

$$d(x, E) = \inf\{d(x, y) : y \in E\}.$$

Let (Ω, Σ) be a measurable space (i.e., Σ is a $\sigma-$algebra of subsets of Ω). A function $\xi : \Omega \longrightarrow X$ is said to measurable if for any open subset C of X, $\xi^{-1}(C) \in \Sigma$. A multivalued mapping $T : \Omega \longrightarrow 2^X$ is called measurable if for any open subset C of X.

$$T^{-1}(C) = \{w \in \Omega : T(w) \cap C \neq \emptyset\} \in \Sigma.$$

This type of measurability is usually called weakly measurability (cf. Himmelberg [9]), but as in this paper we always use this type of measurability, thus we omit the term "weakly" for simplicity. A mapping $\xi : \Omega \longrightarrow X$ is said to be measurable selector of a measurable mapping $T : \Omega \to 2^X$ if ξ is measurable and for each $w \in \Omega$, $\xi(w) \in T(w)$. A mapping $T : \Omega \times X \longrightarrow 2^X$ is called multivalued random operator if for any $x \in X$, $T(\cdot, x)$ is measurable.

A measurable mapping $\xi : \Omega \longrightarrow X$ is said to be a random fixed point of multivalued random operator $T : \Omega \times X \longrightarrow 2^X$ if for each $w \in \Omega$, $\xi(w) \in T(w, \xi(w))$. A mapping $\xi : \Omega \longrightarrow X$ is said to be a deterministic fixed point of multivalued random operator $T : \Omega \times X \longrightarrow 2^X$ if for each $w \in \Omega$, $\xi(w) \in T(w, \xi(w))$.

In [8] Fierro et al. introduced a condition, named condition $(\mathcal{P})$, and we prove some random fixed points theorems. A mapping $T : X \to 2^X$ is said to satisfy condition $(\mathcal{P})$ if, for every closed ball B of X with radius $r \geq 0$ and any sequence $\{x_n\} \subset X$ for which $d(x_n, B) \to 0$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists $x_0 \in B$ such that $x_0 \in Tx_0$. The operator $T : \Omega \times X \to 2^X$ satisfies condition $(\mathcal{P})$ if, for each $\omega \in \Omega$, the mapping $T(\omega, \cdot) : X \to 2^X$ satisfy condition $(\mathcal{P})$.

1.1. Lemma. ([1, Lemma 2]) *Let $\{A_n\}$ be a sequence in $CB(X)$ and there exists $A \in CB(X)$ such that*

$$\lim_{n\to\infty} H(A_n, A) \to 0.$$

If $x_n \in A_n$ $(n = 1, 2, 3, ..)$ and there exists $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) \to 0$$

then $x \in A$.

1.2. Lemma. [9, Theorems 3.2(i), 3.3] *Let (X, d) be a Polish space and $T : \Omega \longrightarrow 2^X$ is a closed valued mapping. Consider the following statements:*

(a) for any closed subset C of X

$$T^{-1}(C) = \{w \in \Omega : T(w) \cap C \neq \emptyset\} \in \Sigma.$$

(b) T is measurable.

(c) $w \longmapsto d(x, T(w))$ is measurable function of w for each $x \in X$.

Then

$$(a) \Rightarrow (b) \Leftrightarrow (c)$$

2. Main Results

Let (X,d) be a metric space, $\varepsilon > 0$ and $x,y \in X$. An ε-chain from x to y is a finite set of points $x_1, x_2, x_3, \dots, x_n$ such that $x = x_1$, $x_n = y$, and $d(x_{j-1}, x_j) < \varepsilon$ for all $j = 2,3,\dots,n$. A metric space (X,d) is said to be ε-chainable if and only if given $x,y \in X$, there exists an ε-chain from x to y. For every $\varepsilon \in (0,\infty]$, let $\mathcal{K}_\varepsilon$ the family of mappings $K : [0,\varepsilon) \to [0,1)$ that satisfy the condition: for $t > 0$, there exist $\delta(t) > 0$ and $s(t) < 1$ such that

$$0 \le r - t < \delta(t) \Longrightarrow K(r) \le s(t).$$

The following theorem is proved in [2].

2.1. Theorem. *Let (X,d) be a complete ε-chainable metric space and $\{T_n\}_{n=1}^{\infty}$ be a sequence of multivalued mapping from X to $CB(X)$ satisfying the following condition: $x,y \in X$ and $0 < d(x,y) < \varepsilon$ implies*

$$H(T_n x, T_m y) \le K(d(x,y))d(x,y)$$

for $n,m = 1,2...$, where $K \in \mathcal{K}_\epsilon$. Then there exists a point $y^* \in X$ such that $y^* \in \bigcap_{n=1}^{\infty} T_n y^*$.

The following theorem is the stochastic versions of the above result.

2.2. Theorem. *Let (X,d) be an ε-chainable Polish space and $\{T_n\}_{n=1}^{\infty}$ be a sequence of random operators from $\Omega \times X$ to $CB(X)$ satisfying the following condition: $x,y \in X$ and $0 < d(x,y) < \varepsilon$ implies*

$$H(T_n(w,x), T_m(w,y)) \le K(w, d(x,y))d(x,y)$$

for $n,m = 1,2...$, where $K : \Omega \times [0,\varepsilon) \to [0,1)$ is a mapping such that $K(w,\cdot) \in \mathcal{K}_\varepsilon$ and it is measurable for each $w \in \Omega$. If T_j enjoys condition $(\mathcal{P})$ for every $j \in \mathbb{N}$, then there exists a common random fixed point of $\{T_n\}_{n=1}^{\infty}$, that is, there exists a measurable mapping $\xi : \Omega \longrightarrow X$ such that for all $w \in \Omega$

$$\xi(w) \in \bigcap_{n=1}^{\infty} T_n(w, \xi(w)).$$

Proof. We note, that for every $w \in \Omega$, the sequence $\{T_n(w,\cdot)\}_{n=1}^{\infty}$ of multivalued mappings on X satisfy the hypothesis of Theorem 2.1, so there exists a point $x^* \in X$ such that

$$x^* \in \bigcap_{n=1}^{\infty} T_n(w, x^*).$$

Now we see that each $T_n(w,\cdot)$ is continuous, for all $w \in \Omega$. Let $\beta > 0$ and assume that $x_m \to x^*$, then there exists an integer $M_\beta > 0$ such that $m \ge M_\beta$ implies $d(x_m, x^*) < \min\{\beta, \varepsilon\}$. From inequality (2), we deduce that

$$H(T_n(w,x_m), T_n(w,x^*)) \le K(w, d(x_m,x^*))d(x_m,x^*) = K_w(d(x_m,x^*))d(x_m,x^*) < d(x_m,x^*) < \min\{\beta,\varepsilon\} \le \beta,$$

whenever, $m \ge M_\beta$. Consequently, $T_n(w,\cdot)$ is continuous.

We consider the multivalued mapping $F : \Omega \to 2^X$ defined by

$$F(w) = \left\{ x \in X : x \in \bigcap_{n=1}^{\infty} T_n(w,x) \right\}.$$

In the view of inequality (3), it follows that $F(w)$ is nonempty for each $w \in \Omega$.

To see that $F(\cdot)$ is closed valued, let u be a limit point of $F(w)$, this implies that there exists a sequence $\{u_1, u_2, u_3, ...\} \subset F(w)$ such that $u_i \to u$. Then $u_i \in \bigcap_{n=1}^{\infty} T_n(w, u_i)$, for every $i = 1, 2, 3, \ldots$. Since $u_i \in T_n(w, u_i)$, for every n, the continuity of $T_n(w, \cdot)$ implies that $T_n(w, u_i) \to T_n(w, u)$. By Lemma 1.1, it follows that $u \in \bigcap_{n=1}^{\infty} T_n(w, u)$, hence $F(\cdot)$ is closed valued.

Now, for every $j = 1, 2, 3, ...,$ we consider the multivalued mapping $F_j : \Omega \to 2^X$ defined by

$$F_j(w) = \{x \in X : x \in T_j(w, x)\}.$$

To see that $F_j(\cdot)$ is a measurable mapping, let $B = B(z, r) := \{y \in X : d(z, y) \leq r\}$ be a closed ball of X and we prove that $F_j^{-1}(B) \in \Sigma$. Take a countable dense subset $S = \{x_1, x_2, ...\}$ of X and let

$$L(B) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in S_n} \left\{w \in \Omega : d(x_i, T_j(w, x_i)) < \frac{2\varepsilon}{n}\right\},$$

where

$$S_n = \left\{x \in S : d(x, B) < \frac{\varepsilon}{n}\right\}.$$

We show that $F_j^{-1}(B) = L(B)$. If $w \in F_j^{-1}(B)$ then $F_j(w) \cap B \neq \emptyset$. Let $x \in B$ such that $x \in T_j(w, x)$, then $S \cap \{z : d(x, z) < \frac{\varepsilon}{n}\} \neq \emptyset$. It follows that for each n there exists $x_{i(n)} \in S_n$ such that $d(x, x_{i(n)}) < \frac{\varepsilon}{n} \leq \varepsilon$. This implies that

$$H\left(T_j(w, x), T_j\left(w, x_{i(n)}\right)\right) \leq K(w, d(x, x_{i(n)}))d(x, x_{i(n)}) < d(x, x_{i(n)}) < \frac{\varepsilon}{n}.$$

We obtain

$$d\left(x_{i(n)}, T_j\left(w, x_{i(n)}\right)\right) \leq d\left(x_{i(n)}, x\right) + d\left(x, T_j\left(w, x_{i(n)}\right)\right) \leq \\ d\left(x_{i(n)}, x\right) + H\left(T_j(w, x), T_j\left(w, x_{i(n)}\right)\right) < \frac{\varepsilon}{n} + \frac{\varepsilon}{n}.$$

As $x_{i(n)} \in S_n$, it follows that $w \in L(B)$ and $F_j^{-1}(B) \subseteq L(B)$.

Conversely, if $\omega \in L(B)$, then for each n, we can take $x_{i(n)} \in S_n$ for which

$$d(x_{i(n)}, T_j(\omega, x_{i(n)})) < 2\varepsilon/n.$$

We have

$$d(x_{i(n)}, B) \to 0 \quad \text{i.e.} \quad d(x_{i(n)}, T_j(\omega, x_{i(n)}) \to 0,$$

since the mapping $T_j(\omega, \cdot)$ satisfies condition $(\mathcal{P})$, there exists $x \in B$ such that $x \in T_j(\omega, x)$ and hence $\omega \in F_j^{-1}(B)$. Therefore $L(B) = F_j^{-1}(B)$.

Now for any $x \in X$ define a mapping $G_{j(x)} : \Omega \to \mathbb{R}$ as $G_{j(x)}(\cdot) = d(x, T_j(\cdot, x))$, since (by hypotheses) $T_j(\cdot, x)$ is closed valued and measurable. By Lemma 1.2, the mapping $G_{j(x)}(\cdot)$ is measurable. It follows that

$$\left\{w \in \Omega : d(x_i, T_j(w, x_i)) < \frac{2\varepsilon}{n}\right\} \in \Sigma.$$

Hence $F_j^{-1}(B) = L(B) \in \Sigma$. To complete the proof, let G be an arbitrary open subset of X, by the separability of X there exists a sequence of closed ball $\{B_n\}$ such that

$$G = \bigcup_{n=1}^{\infty} B_n.$$

Since $F_j^{-1}(G) = \bigcup_{n=1}^{\infty} F_j^{-1}(B_n)$, we conclude that F_j is measurable.

Hence, $F(\cdot)$ is measurable. The Kuratowski-Ryll-Nardzewski theorem [12] further implies that there exists a measurable mapping $\xi : \Omega \longrightarrow X$ such that for all $w \in \Omega$

$$\xi(w) \in \bigcap_{n=1}^{\infty} T_n(w, \xi(w)).$$

This completes the proof. □

The following results are direct consequences of the above theorem.

2.3. Theorem. *Let (X, d) be an ε-chainable Polish space and $T : \Omega \times X \to CB(X)$ be a multivalued random operator satisfying the following condition: $x, y \in X$ and $0 < d(x, y) < \varepsilon$ implies*

$$H(T(w, x), T(w, y)) \leq K(w, d(x, y))d(x, y),$$

where $K : \Omega \times [0, \varepsilon) \to [0, 1)$ is a mapping such that $K(w, \cdot) \in \mathcal{K}_\varepsilon$ and it is measurable for each $w \in \Omega$. If T enjoys condition $(\mathcal{P})$, then T has a random fixed point.

2.4. Corollary. *Let (X, d) be a Polish space and $T : \Omega \times X \to CB(X)$ be a multivalued random operator satisfying the following condition: $x, y \in X$*

$$H(T(w, x), T(w, y)) \leq K(w, d(x, y))d(x, y),$$

where $K : \Omega \times [0, \infty) \to [0, 1)$ is a mapping such that $K(w, \cdot) \in \mathcal{K}_\infty$ and it is measurable for each $w \in \Omega$. If T enjoys condition $(\mathcal{P})$, then T has a random fixed point.

3. A result for a sequence with a common deterministic fixed point

In this section we consider sequences of multivalued random operators with a common deterministic fixed point and we deduce the existence of a common random fixed point.

Let (X, d) be a metric space and $\varepsilon > 0$. A sequence $\{T_n\}_{n=1}^{\infty}$ of multivalued random operators from $\Omega \times X$ to $CB(X)$ is ε-locally nonexpansive if for every $x, y \in X$ such that $0 < d(x, y) < \varepsilon$ holds

$$H(T(\omega, x), T(\omega, y)) \leq d(x, y).$$

3.1. Theorem. *Let (X, d) be a Polish space and $\{T_n\}_{n=1}^{\infty}$ be a sequence of ε-locally nonexpansive multivalued random operators from $\Omega \times X$ to $CB(X)$. Assume that the T_j enjoys condition $(\mathcal{P})$ for every $j \in \mathbb{N}$. If the sequence $\{T_n\}_{n=1}^{\infty}$ of random operators, has a common deterministic fixed point, then there exists a common random fixed point of $\{T_n\}_{n=1}^{\infty}$, that is, there exists a measurable mapping $\xi : \Omega \longrightarrow X$ such that for all $w \in \Omega$*

$$\xi(w) \in \bigcap_{n=1}^{\infty} T_n(w, \xi(w)).$$

Proof. Let $F : \Omega \to 2^X$ be defined for every $\omega \in \Omega$ from

$$F(\omega) := \{x \in X : x \in \bigcap_{n=1}^{+\infty} T_n(\omega, x)\}.$$

Since the random operators T_n $(n = 1, 2, ...)$ have a common deterministic fixed point we deduce that $F(\omega) \neq \emptyset$ for all $\omega \in \Omega$. We note that $T_j(\omega, \cdot) : X \to CB(X)$ is continuous for all $\omega \in \Omega$ and $j \in \mathbb{N}$. In fact if $0 < d(x, y) < \varepsilon$, we have

$$H(T_j(\omega, x), T_j(\omega, y)) \leq d(x, y).$$

The set

$$A_j(\omega) := \{x \in X : x \in T_j(\omega, x)\}$$

is closed for all j and it implies that $F(\omega)$ is closed for all $\omega \in \Omega$. Proceeding as in the proof of Theorem 2.2, we obtain that there exists a common random fixed point of $\{T_n\}_{n=1}^{\infty}$. □

3.2. Proposition. *Let (X, d) be a locally compact metric space and T be a ε-locally nonexpansive multivalued random operator from $\Omega \times X$ to $CB(X)$. Then T enjoys condition $(\mathcal{P})$.*

Proof. Let B be a compact ball of X and $\{x_n\}$ a sequence such that $d(x_n, B) \to 0$, it is not restrictive to suppose that $x_n \to x_0 \in B$. If $0 < d(x_n, x_0) < \varepsilon$, then $H(T(\omega, x_0), T(\omega, x_n)) \leq d(x_0, x_n)$, consequently

$$H(T(\omega, x_0), T(\omega, x_n) \to 0.$$

By Lemma 1.1, we deduce that $x_0 \in T(\omega, x_0)$ and thus T satisfies condition $(\mathcal{P})$. □

The following results are direct consequences of the Theorem 3.1 and Proposition 3.2.

3.3. Corollary. *Let (X, d) be a locally compact separable complete metric space and $\{T_n\}_{n=1}^{\infty}$ be a sequence of ε-locally nonexpansive multivalued random operators from $\Omega \times X$ to $CB(X)$. If the random operators $\{T_n\}_{n=1}^{\infty}$ have a common deterministic fixed point, then there exists a common random fixed point of $\{T_n\}_{n=1}^{\infty}$.*

3.4. Corollary. *Let (X, d) be a locally compact separable complete metric space and T be a ε-locally nonexpansive multivalued random operator from $\Omega \times X$ to $CB(X)$. If the random operator T has a deterministic fixed point, then there exists a random fixed point of T.*

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