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ANTI-INVARIANT $\xi^{\perp}$-RIEMANNIAN SUBMERSIONS FROM ALMOST CONTACT MANIFOLDS

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Abstract

We introduce anti-invariant $\xi^{\perp}$-Riemannian submersions from almost contact manifolds onto Riemannian manifolds. We give an example, investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion and check the harmonicity of such submersions. We also find necessary and sufficient conditions for a special anti-invariant $\xi^{\perp}$-Riemannian submersion to be totally geodesic. Moreover, we obtain decomposition theorems for the total manifold of such submersions.

Keywords: Riemannian submersion, Sasakian manifold, Anti-invariant $\xi^{\perp}$-Riemannain submersion

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1. Introduction

Riemannian submersions between Riemannian manifolds were studied by O'Neil [9] and Gray [7]. In [13], Waston defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber has the same kind of structure as the total space, in most cases. He also showed that the vertical and horizontal distributions are invariant. On the other hand, the geometry of anti-invariant Riemannian submersions is quite different from the geometry of almost Hermitian submersions. For example, since every holomorphic map between Kähler manifolds is harmonic [5], it follows that any holomorphic submersion between Kähler manifolds is harmonic. However, this result is not valid for anti-invariant Riemannian submersions, which was first studied by Sahin in [11]. Simlarly, Ianus and Pastore [8] shows ϕ-holomorphic maps between contact manifolds are harmonic. This implies that

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any contact submersion is harmonic. However, this result is not valid for anti-invariant $\xi^{\perp}$-Riemannian submersions.

We also note that Riemannian submersions have applications in Klauza-Klein thoery and the theory of robotics. Indeed, in Klauza-Klein theory, the general solution of a recent model can be expressed in harmonic maps which satisfies Einstein equations. However, a very general class of solution is given by Riemannian submersions from the extra dimensional space onto the space in which the scalar fields take values (see [6] for details). On the other hand, Altafini [1] used the Riemannian submersion in redundant robots, it means that the robotic chain has more than six joints, and showed that the forward kinematic map from joint space to the workspace of the end effector is a Riemannian submersion. He also showed that there is a close relationship between inverse kinematic in robotics and the horizontal lift of vector fields in Riemannian submersions.

In [4], Chinea defined almost contact Riemannian submersions between almost contact metric manifolds and examined the differential geometric properties of Riemannian submersions between almost contact metric manifolds. More precisely, let $(M, g_M, \varphi, \xi, \eta)$ and $(N, g_N, \varphi', \xi', \eta')$ be almost contact manifolds with $dim M = 2m+1$ and $dim N = 2n+1$. A Riemannian submersion $F : M \longrightarrow N$ is called th almost contact metric submersion if F is an almost contact mapping, i.e., $\varphi' F_* = F_* \varphi$. The main result of this notion is that the vertical and horizontal distributions are φ-invariant. Moreover, the characteristic vector field ξ is horizontal. We note that only φ-holomorphic submersions have been considered on almost contact manifolds [6].

In this paper, we consider a Riemannian submersion from an almost contact manifold under the assumption that the fibers are anti-invariant with respect to the tensor field of type $(1,1)$ of the almost contact manifold. This assumption implies that the horizontal distribution is not invariant under the action of the tensor field of the total manifold of such submersions. Roughly speaking, almost contact submersions are useful for describing the geometry of base manifolds, anti-invariant submersions are however served to determine the geometry of total manifolds.

The paper is organized as follows: In Section 2, we present the basic information, needed for this paper. In Section 3, we give the definition of anti-invariant $\xi^{\perp}$-Riemannian submersions, provide an example and investigate the geometry of leaves of the distributions. We also introduce a special anti-invariant $\xi^{\perp}$-Riemannian submersions and obtain necessary and sufficient conditions for such submersions to be totally geodesic or harmonic. In Section 4, we give decomposition theorems by using the existence of anti-invariant $\xi^{\perp}$-Riemannian submersions and observe that such submersions put some restrictions on the geometry of the total manifold.

2. Preliminaries

In this section, we define almost contact manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let M be an almost contact metric manifold with structure tensors $(\varphi, \xi, \eta, g_M)$ where φ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g_M is the Riemannian metric on M. Then these tensors satisfy [2]

$$(2.1) \qquad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1$$

$$(2.2) \qquad \varphi^2 = -I + \eta \otimes \xi, \quad \text{and} \quad g_M(\varphi X, \varphi Y) = g_M(X, Y) - \eta(X)\eta(Y),$$

where I denotes the identity endomorphism of TM and X, Y are any vector fields on M. Moreover, if M is Sasakian [12], then we have

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X \quad \text{and} \quad \nabla_X\xi = \varphi X, \tag{2.3}$$

where ∇ is the connection of Levi-Civita covariant differentiation.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds, where $dimM = m$, $dimN = n$ and $m > n$. A Riemannian submersion $F : M \longrightarrow N$ is a map from M onto N satisfying the following axioms:

(S1) F has the maximal rank.
(S2) The differential F_* preserves the lengths of horizontal vectors.

For each $q \in N$, $F^{-1}(q)$ is an $(m-n)$-dimensional submanifold of M. The submanifolds $F^{-1}(q)$ are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X_* on N, i.e., $F_*X_p = X_{*F(p)}$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $kerF_*$ and $(kerF_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall the following lemma from O'Neil [9].

2.1. Lemma. *Let $F : M \longrightarrow N$ be a Riemannian submersion between Riemannian manifolds and X, Y be basic vector fields of M. Then*

(a) $g_M(X,Y) = g_N(X_*, Y_*) \circ F$.
(b) *the horizontal part $[X,Y]^{\mathcal{H}}$ of $[X,Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i.e., $F_*([X,Y]) = [X_*, Y_*]$.*
(c) *$[V, X]$ is vertical for any vector field V of $kerF_*$.*
(d) *$(\nabla^M_X Y)^{\mathcal{H}}$ is the basic vector field corresponding to $\nabla^N_{X_*}Y_*$.*

The geometry of Riemannian submersions is characterized by O'Neil's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{2.4}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.5}$$

where ∇ is the Levi-Civita connection of g_M. It is easy to see that a Riemannian submersion $F : M \longrightarrow N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_C = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_U W = \mathcal{T}_W U, \qquad U, W \in \Gamma(kerF_*) \tag{2.6}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y], \qquad X, Y \in \Gamma((kerF_*)^{\perp}). \tag{2.7}$$

On the other hand, from (2.4) and (2.5), we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \tag{2.8}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X \tag{2.9}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V \tag{2.10}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.11}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$.

Finally, we recall the notion of harmonic maps betwwen Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and supposed that $\varphi : M \longrightarrow N$ is a smooth map. Then the differential φ_* of φ can be viewed a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibers $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection ∇^φ. Then the second fundamental form of φ is given by

$$(2.12) \qquad (\nabla\varphi_*)(X, Y) = \nabla^\varphi_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tensor field of φ is the section$\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$(2.13) \qquad \tau(\varphi) = div\varphi_* = \sum_{i=1}^{m} (\nabla\varphi_*)(e_i, e_i),$$

where $\{e_1, \cdots, e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$ (for details, see [3]).

3. Anti-invariant $\xi^\perp$-Rieamannian submersions

In this section, we define anti-invariant $\xi^\perp$-Riemannian submersion from an almost contact metric manifold onto a Riemannian manifold and investigate the integrabiltiy of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map. We also investigate the harmonicity of a special Riemannian submersions.

3.1. Definition. Let $(M, g_M, \varphi, \xi, \eta)$ be an almost contact metric manifold and (N, g_N) a Riemannian manifold. Suppose that there exists a Riemannian submersion $F : M \longrightarrow N$ such that ξ is normal to $kerF_*$ and $kerF_*$ is anti-invariant with respect to φ, i.e., $\varphi(kerF_*) \subset (kerF_*)^\perp$. Then we say that F is an anti-invariant $\xi^\perp$-Riemannian submersion.

Now, we assume that $F : (M, g_M, \varphi, \xi, \eta) \longrightarrow (N, g_N)$ is an anti-invariant $\xi^\perp$-Riemannian submersion. First of all, from Definition 3.1, we have $(kerF_*)^\perp \cap kerF_* \neq \{0\}$. We denote the complementary orthogonal distribution to $\varphi(kerF_*)$ in $(kerF_*)^\perp$ by μ. Then we have

$$(3.1) \qquad (kerF_*)^\perp = \varphi(kerF_*) \oplus \mu,$$

where $\varphi(\mu) \subset \mu$. Hence μ contains ξ. Thus, for $X \in \Gamma((kerF_*)^\perp)$, we have

$$(3.2) \qquad \varphi X = BX + CX,$$

where $BX \in \Gamma(kerF_*)$ and $CX \in \Gamma(\mu)$. On the other hand, since $F_*((kerF_*)^\perp) = TN$ and F is a Riemannian submersion, using (3.2), we have $g_N(F_*\varphi V, F_*\varphi CX) = 0$ for any $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$, which implies that

$$TN = F_*(\varphi(kerF_*)) \oplus F_*(\mu)$$

3.2. Example. Let $(\mathbb{R}^5, g, \varphi_0, \xi, \eta)$ denote the manifold $\mathbb{R}^5$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}\left(dz - x^2 dx^1 - x^4 dx^3\right), \quad \xi = 2\frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{4}\left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 + dx^4 \otimes dx^4\right),$$

$$\varphi_0\left(X_1\frac{\partial}{\partial x^1}+X_2\frac{\partial}{\partial x^2}+X_3\frac{\partial}{\partial x^3}+X_4\frac{\partial}{\partial x^4}+Z\frac{\partial}{\partial z}\right)$$
$$=X_2\frac{\partial}{\partial x^1}-X_1\frac{\partial}{\partial x^2}+X_4\frac{\partial}{\partial x^3}-X_3\frac{\partial}{\partial x^4}+\left(X_2x^2+X_4x^4\right)\frac{\partial}{\partial z},$$

where $\left(x^1,x^2,x^3,x^4,z\right)$ are the Cartesian coordinates.
Let $F:(\mathbb{R}^5,g,\varphi_0,\xi,\eta)\longrightarrow\mathbb{R}^3$ be a map defined by $F\left(x^1,x^2,x^3,x^4,z\right)=\left(\frac{x^1+x^4}{\sqrt{2}},\frac{x^2+x^3}{\sqrt{2}},z\right)$.
Then, by the direct computations, we have

$$kerF_*=span\left\{V=\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^4},\ W=\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}\right\},$$

and

$$(kerF_*)^\perp=span\left\{X=\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^4}+x^4\frac{\partial}{\partial z},\ Y=\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^3}+x^2\frac{\partial}{\partial z},\ \xi=2\frac{\partial}{\partial z}\right\}.$$

Then it is easy to see that F is a Riemannian submersion. Moreover, since $\varphi_0(V)=-Y$ and $\varphi_0(W)=X$, $\varphi_0(kerF_*)\subset(kerF_*)^\perp$. As a result, F is an anti-invariant $\xi^\perp$-Riemannian submersion.

3.3. Lemma. *Let F be an anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold (M,g_M,φ,ξ,η) onto a Riemannian manifold (N,g_N). Then we have*

(3.3) $\quad g_M(CY,\varphi V)=0$

and

(3.4) $\quad g_M(\nabla_X CY,\varphi V)=-g_M(CY,\varphi\mathcal{A}_XV)$

for $X,Y\in\Gamma((kerF_*)^\perp)$ and $V\in\Gamma(kerF_*)$.

Proof. For $Y\in\Gamma((kerF_*)^\perp)$ and $V\in\Gamma(kerF_*)$, using (2.2), we have

$$g_M(CY,\varphi V)=g_M(\varphi Y-BY,\varphi V)=g_M(\varphi Y,\varphi V)=g_M(Y,V)+\eta(Y)\eta(V)=g_M(Y,V)=0$$

since $BY\in\Gamma(kerF_*)$ and $\varphi V,\xi\in\Gamma((kerF_*)^\perp)$. Differentiating (3.3) with respect to X, we get

$$\begin{aligned}g_M(\nabla_X CY,\varphi V)&=-g_M(CY,\nabla_X\varphi V)\\&=g_M(CY,(\nabla_X\varphi)V)-g_M(CY,\varphi(\nabla_XV))\\&=-g_M(CY,\varphi(\nabla_XV))\\&=-g_M(CY,\varphi\mathcal{A}_XV)-g_M(CY,\varphi\mathcal{V}\nabla_XV)\\&=-g_M(CY,\varphi\mathcal{A}_XV)\end{aligned}$$

due to $\varphi\mathcal{V}\nabla_XV\in\Gamma(\varphi(kerF_*))$. Our assertion is complete. □

We study the ingetgrability of the distribution $(kerF_*)^\perp$ and then we investigate the geometry of leaves of $kerF_*$ and $(kerF_*)^\perp$. We note it is known that the distribution $kerF_*$ is integrable.

3.4. Theorem. *Let F be an anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold (M,g_M,φ,ξ,η) onto a Riemannian manifold (N,g_N). The followings are equivalent.*

(a) *$(kerF_*)^\perp$ is integrable*

(b)

$$\begin{aligned}g_N((\nabla F_*)(Y,BX),F_*\varphi V)&=g_N((\nabla F_*)(X,BY),F_*\varphi V)\\&+g_M(CY,\varphi\mathcal{A}_XV)-g_M(CX,\varphi\mathcal{A}_YV)\\&+\eta(Y)g_M(X,\varphi V)-\eta(X)g_M(Y,\varphi V)\end{aligned}$$

(c)

$$g_M(\mathcal{A}_X BY - \mathcal{A}_Y BX, \varphi V) = g_M(CY, \varphi \mathcal{A}_X V) - g_M(CX, \varphi \mathcal{A}_Y V) + \eta(Y) g_M(X, \varphi V) - \eta(X) g_M(Y, \varphi V)$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ *and* $V \in \Gamma(ker F_*)$.

Proof. For $Y \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$, from Definition 3.1, $\varphi V \in \Gamma((ker F_*)^\perp)$ and $\varphi Y \in \Gamma(ker F_* \oplus \mu)$. Using (2.2) and (2.3), we note that for $X \in \Gamma((ker F_*)^\perp)$,

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \varphi Y, \varphi V) - \eta(Y) g_M(X, \varphi V) \tag{3.5}$$

Therefore, from (3.5), we get

$$\begin{aligned} g_M([X,Y],V) &= g_M(\nabla_X \varphi Y, \varphi V) - g_M(\nabla_Y \varphi X, \varphi V) \\ &- \eta(Y) g_M(X, \varphi V) + \eta(X) g_M(Y, \varphi V) \\ &= g_M(\nabla_X BY, \varphi V) + g_M(\nabla_X CY, \varphi V) \\ &- g_M(\nabla_Y BX, \varphi V) - g_M(\nabla_Y CX, \varphi V) \\ &- \eta(Y) g_M(X, \varphi V) + \eta(X) g_M(Y, \varphi V). \end{aligned}$$

Since F is a Riemannian submersion, we obtain

$$\begin{aligned} g_M([X,Y],V) &= g_N(F_* \nabla_X BY, F_* \varphi V) + g_M(\nabla_X CY, \varphi V) \\ &- g_N(F_* \nabla_Y BX, F_* \varphi V) - g_M(\nabla_Y CX, \varphi V) \\ &- \eta(Y) g_M(X, \varphi V) + \eta(X) g_M(Y, \varphi V). \end{aligned}$$

Thus, from (2.13) and (3.4), we have

$$\begin{aligned} g_M([X,Y],V) &= g_N(-(\nabla F_*)(X, BY) + (\nabla F_*)(Y, BX), F_* \varphi V) \\ &- g_M(CY, \varphi \mathcal{A}_X V) + g_M(CX, \varphi \mathcal{A}_Y V) \\ &- \eta(Y) g_M(X, \varphi V) + \eta(X) g_M(Y, \varphi V), \end{aligned}$$

which proves $(a) \iff (b)$.

On the other hand, using (2.12), we obtain

$$(\nabla F_*)(Y, BX) - (\nabla F_*)(X, BY) = -F_*(\nabla_Y BX - \nabla_X BY) = -F_*(\mathcal{A}_Y BX - \mathcal{A}_X BY),$$

which shows that $(b) \iff (c)$. □

3.5. Corollary. *Let F be an anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(ker F_*)^\perp = \varphi(ker F_*) \oplus <\xi>$. Then the followings are equivalent.*

(a) $(ker F_*)^\perp$ *is integrable*
(b) $(\nabla F_*)(X, \varphi Y) + \eta(X) F_* Y = (\nabla F_*)(Y, \varphi X) + \eta(Y) F_* X$
(c) $\mathcal{A}_X \varphi Y + \eta(X) Y = \mathcal{A}_Y \varphi X + \eta(Y) X$, *for* $X, Y \in \Gamma((ker F_*)^\perp)$.

3.6. Theorem. *Let F be an anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). The followings are equivalent.*

(a) $(ker F_*)^\perp$ *defines a totally geodesic foliation on* M.
(b) $g_M(\mathcal{A}_X BY, \varphi V) = g_M(CY, \varphi \mathcal{A}_X V) - \eta(Y) g_M(X, \varphi V)$
(c) $g_N((\nabla F_*)(Y, \varphi X), F_* \varphi V) = g_M(CY, \varphi \mathcal{A}_X V) - \eta(Y) g_M(X, \varphi V)$, *for* $X, Y \in \Gamma((ker F_*)^\perp)$ *and* $V \in \Gamma(ker F_*)$.

Proof. For $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma(kerF_*)$, from (3.5), we have

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY, \varphi V) + g_M(\nabla_X CY, \varphi V) - \eta(Y) g_M(X, \varphi V)$$

Then from (3.4), we have

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X BY, \varphi V) - g_M(CY, \varphi \mathcal{A}_X V) - \eta(Y) g_M(X, \varphi V),$$

which shows $(a) \iff (b)$. On the other hand, from (2.10) and (2.12), we have $g_M(\mathcal{A}_X BY, \varphi V) = g_N(-(\nabla F_*)(X, BY), F_*\varphi V)$, which proves $(b) \iff (c)$. □

3.7. Corollary. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(kerF_*)^{\perp} = \varphi(kerF_*) \oplus < \xi >$. Then the followings are equivalent.*

(a) *$(kerF_*)^{\perp}$ defines a totally geodesic foliation on M.*
(b) $\mathcal{A}_X \varphi Y = \eta(Y) X$
(c) $(\nabla F_*)(Y, \varphi X) = \eta(Y) F_* X,$ *for $X, Y \in \Gamma((kerF_*)^{\perp})$.*

3.8. Theorem. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). The followings are equivalent.*

(a) *$kerF_*$ defines a totally geodesic foliation on M.*
(b) $g_N((\nabla F_*)(V, \varphi X), F_*\varphi W) = 0$
(c) $\mathcal{T}_V BX + \mathcal{A}_{CX} V \in \Gamma(\mu),$ *for $X \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$.*

Proof. For $X \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$, $g_M(W, \xi) = 0$ implies that from (2.3) $g_M(\nabla_V W, \xi) = -g_M(W, \nabla_V \xi) = -g(W, \varphi V) = 0$. Thus we have

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\varphi \nabla_V W, \varphi X) + \eta(\nabla_V W)\eta(X) \\ &= g_M(\varphi \nabla_V W, \varphi X) \\ &= g_M(\nabla_V \varphi W, \varphi X) - g_M((\nabla_V \varphi) W, \varphi X) \\ &= -g_M(\varphi W, \nabla_V \varphi X) \end{aligned}$$

Since F is a Riemannian submersion, we have

$$g_M(\nabla_V W, X) = -g_N(F_*\varphi W, F_*\nabla_V \varphi X) = g_N(F_*\varphi W, (\nabla F_*)(V, \varphi X)),$$

which proves $(a) \iff (b)$.
By direct calculation, we derive

$$\begin{aligned} g_N(F_*\varphi W, (\nabla F_*)(V, \varphi X)) &= -g_M(\varphi W, \nabla_V \varphi X) \\ &= -g_M(\varphi W, \nabla_V BX + \nabla_V CX) \\ &= -g_M(\varphi W, \nabla_V BX + [V, CX] + \nabla_{CX} V) \end{aligned}$$

Since $[V, CX] \in \Gamma(kerF_*)$, from (2.8) and (2.10), we obtain

$$g_N(F_*\varphi W, (\nabla F_*)(V, \varphi X)) = -g_M(\varphi W, \mathcal{T}_V BX + \mathcal{A}_{CX} V),$$

which proves $(b) \iff (c)$. □

As an analogue of a Lagrangian Riemannian submersion in [11], we have a similar result;

3.9. Corollary. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(kerF_*)^{\perp} = \varphi(kerF_*) \oplus < \xi >$. Then the followings are equivalent.*

(a) *$kerF_*$ defines a totally geodesic foliation on M.*
(b) $(\nabla F_*)(V, \varphi X) = 0$
(c) $\mathcal{T}_V \varphi W = 0,$ *for $X \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$.*

Proof. From Theorem 3.6, it is enough to show $(b) \iff (c)$. Using (2.12) and (2.9), we have

$$g_N(F_*\varphi W, (\nabla F_*)(V, \varphi X)) = g_M(\nabla_V \varphi W, \varphi X)$$
$$= g_M(\mathcal{T}_V \varphi W, \varphi X)$$

Since $\mathcal{T}_V\varphi W \in \Gamma(kerF_*)$, the proof is complete. □

We note that a differentiable map F between two Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$. For the special Riemannian submersion, we have the following characterization.

3.10. Theorem. *Let F be an anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(kerF_*)^\perp = \varphi(kerF_*) \oplus <\xi>$. Then F is a totally geodesic manp if and only if*

(3.6) $\mathcal{T}_V\varphi W = 0 \quad V, W \in \Gamma(kerF_*)$

and

(3.7) $\mathcal{A}_X\varphi W = 0 \quad X \in \Gamma((kerF_*)^\perp)$

Proof. First of all, we recall that the second fundamental form of a Riemannian submersion satisfies

(3.8) $(\nabla F_*)(X, Y) = 0 \quad \forall X, Y \in \Gamma((kerF_*)^\perp)$

For $V, W \in \Gamma(kerF_*)$, we get

(3.9) $(\nabla F_*)(V, W) = F_*(\varphi \mathcal{T}_V \varphi W)$.

On the other hand, from (2.1), (2.2) and (2.12), we get

(3.10) $(\nabla F_*)(X, W) = F_*(\varphi \mathcal{A}_X \varphi W), \quad X \in \Gamma((kerF_*)^\perp)$

Therefore, F is totally geodesic if and only if

(3.11) $\varphi(\mathcal{T}_V\varphi W) = 0, \quad \forall V, W \in \Gamma(kerF_*)$

and

(3.12) $\varphi(\mathcal{A}_X\varphi W) = 0, \quad \forall X \in \Gamma((kerF_*)^\perp)$

From (2.2), (2.4) and (2.5), we have

$$\mathcal{T}_V\varphi W = 0, \quad \forall V, W \in \Gamma(kerF_*)$$

and

$$\mathcal{A}_X\varphi W = 0, \quad \forall X \in \Gamma((kerF_*)^\perp)$$

From (2.3), F is totally geodesic if and only if the equations (3.6) and (3.7) hold. □

Finally, in this section, we give a necessary and sufficient condition for a special Riemannian submersion to be harmonic as an analogue of a Lagrangian Riemannian submersion in [11];

3.11. Theorem. *Let F be an anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(kerF_*)^\perp = \varphi(kerF_*) \perp <\xi>$. Then F is harmonic if and only if $Trace(\varphi \mathcal{T}_V) = 0$ for $V \in \Gamma(kerF_*)$.*

Proof. From [5], we know that F is harmonic if and only if F has minimal fibers. Thus F is harmonic if and only if $\sum_{i=1}^{m_1} \mathcal{T}_{e_i} e_i = 0$. On the other hand, from (2.3), (2.9) and (2.8), we have

$$\mathcal{T}_V \varphi W = \varphi \mathcal{T}_V W \tag{3.13}$$

due to $\xi \in \Gamma((kerF_*)^{\perp})$ for any $V, W \in \Gamma(kerF_*)$. Using (3.13), we get

$$\sum_{i=1}^{m_1} g_M(\mathcal{T}_{e_i} \varphi e_i, V) = \sum_{i=1}^{m_1} g_M(\varphi \mathcal{T}_{e_i} e_i, V) = -\sum_{i=1}^{m_1} g_M(\mathcal{T}_{e_i} e_i, \varphi V)$$

for any $V \in \Gamma(kerF_*)$. Thus skew-symmetric $\mathcal{T}$ implies that

$$\sum_{i=1}^{m_1} g_M(\varphi e_i, \mathcal{T}_{e_i} V) = \sum_{i=1}^{m_1} g_M(\mathcal{T}_{e_i} e_i, \varphi V).$$

Using (2.6) and (2.2), we have

$$\sum_{i=1}^{m_1} g_M(e_i, \varphi \mathcal{T}_V e_i) = -\sum_{i=1}^{m_1} g_M(\varphi e_i, \mathcal{T}_V e_i) = -g_M(\sum_{i=1}^{m_1} \mathcal{T}_{e_i} e_i, \varphi V),$$

which shows our assertion. □

4. Decomposition theorems

In this section, we obtain decompostion theorems by using the existence of anti-invariant $\xi^{\perp}$-Riemannian submersions. First, we recall the following.

4.1. Theorem. [10]. *Let g be a Riemannian metric tensor on the manifold $B = M \times N$ and assume that the canonical foliations $\mathcal{D}_M$ and $\mathcal{D}_N$ intersect perpendicularly everywhere. Then g is the metric tensor of*

- (i) *a twisted product $M \times_f N$ if and olny if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_N$ is a totally umbilical foliation.*
- (ii) *a warped product $M \times_f N$ if and olny if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_N$ is a spheric foliation, i.e., it is umbilic and its mean curvature vector field is parallel.*
- (iii) *a usual product of Riemannian manifold if and only if $\mathcal{D}_M$ and $\mathcal{D}_N$ are totally geodesic foliations.*

Our first decomposition theorem for an anti-invariant $\xi^{\perp}$-Riemannian submersion comes from Theorem 3.4 and 3.6 in terms of the second fundamental forms of such submersions.

4.2. Theorem. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). Then M is a locally product manifold if and only if*

$$g_N((\nabla F_*)(Y, \varphi X), F_* \varphi V) = g_M(CY, \varphi \mathcal{A}_X V) - \eta(Y) g_M(X, \varphi V)$$

and

$$g_N((\nabla F_*)(V, \varphi X), F_* \varphi W) = 0$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$

From Corollary 3.5 and 3.7, we have the following decomposition theorem:

4.3. Theorem. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(kerF_*)^{\perp} = \varphi(kerF_*) \perp < \xi >$. Then M is a locally product manifold if and only if $\mathcal{A}_X \varphi Y = \eta(Y) X$ and $\mathcal{T}_V \varphi W = 0$, for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$.*

Next we obtain a decomposition theorem which is related to the notion of a twisted product manifold.

4.4. Theorem. *Let F be an anti-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, g_M, \varphi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(kerF_*)^{\perp} = \varphi(kerF_*)\perp <\xi>$. Then M is locally twisted product manifold of the form $M_{(kerF_*)^{\perp}} \times_f M_{kerF_*}$ if and only if*

$$\mathcal{T}_V \varphi X = -g_M(X, \mathcal{T}_V V)||V||^{-2}\varphi V$$

and

$$\mathcal{A}_X \varphi Y = \eta(Y)X$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma(kerF_*)$, where $M_{(kerF_*)^{\perp}}$ and M_{kerF_*} are integral manifolds of the distributions $(kerF_*)^{\perp}$ and $kerF_*$.

Proof. For $X \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma(kerF_*)$, from (2.3) and (2.9), we obtain

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\mathcal{T}_V \varphi W, \varphi X) \\ &= -g_M(\varphi W, \mathcal{T}_V \varphi X) \end{aligned}$$

since $\mathcal{T}_V$ is skew-symmetric. This implies that $kerF_*$ is totally umbilical if and only if

$$\mathcal{T}_V \varphi W = -X(\lambda)\varphi V,$$

where λ is a function on M. By the direct computation,

$$\mathcal{T}_V \varphi X = -g_M(X, \mathcal{T}_V V)||V||^{-2}\varphi V.$$

Then the proof follows from Corollary 3.5. □

However, in the sequel, we show that the notion of anti-invariant $\xi^{\perp}$-Riemannian submersion puts some restrictions on the source manifold.

4.5. Theorem. *Let $(M, g_M, \varphi, \xi, \eta)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Then there does not exist an anti-invariant $\xi^{\perp}$-Riemannian submersion from M to N with $(kerF_*)^{\perp} = \varphi(kerF_*)\perp <\xi>$ such that M is a locally proper twisted product manifold of the form $M_{kerF_*} \times_f M_{(kerF_*)^{\perp}}$.*

Proof. Suppose that $F : (M, g_M, \varphi, \eta, \xi) \longrightarrow (N, g_N)$ is an anti-invariant $\xi^{\perp}$-Riemannian submersion with $(kerF_*)^{\perp} = \varphi(kerF_*)\perp <\xi>$ and M is a locally twisted product of th form $M_{kerF_*} \times_f M_{(kerF_*)^{\perp}}$. Then M_{kerF_*} is a totally geodesic foliation and $M_{(kerF_*)^{\perp}}$ is a totally umbilical foliation. We denote the second fundamental form of $M_{(kerF_*)^{\perp}}$ by h. Then we have

$$g_M(\nabla_X Y, V) = g_M(h(X, Y), V) \quad X, Y \in \Gamma((kerF_*)^{\perp}), V \in \Gamma(kerF_*). \tag{4.1}$$

Since $M_{(kerF_*)^{\perp}}$ is a totally umbilical foliation, we have

$$g_M(\nabla_X Y, V) = g_M(H, V)g_M(X, Y),$$

where H is the mean curvature vector field of $M_{(kerF_*)^{\perp}}$.
On the other hand, from (3.5), we derive

$$g_M(\nabla_X Y, V) = -g_M(\varphi Y, \nabla_X \varphi V) - \eta(Y)g_M(X, \varphi V). \tag{4.2}$$

Using (2.11), we obtain

$$\begin{aligned} g_M(\nabla_X Y, V) &= -g_M(\varphi Y, \mathcal{A}_X \varphi V) - \eta(Y)g_M(X, \varphi V) \\ &= -g_M(Y, \varphi \mathcal{A}_X \varphi V + g_M(X, \varphi V)\xi) \end{aligned} \tag{4.3}$$

Therefore, from (4.1), (4.3) and (2.2), we have

$$\mathcal{A}_X\varphi V = g_M(H,V)\varphi X + \eta(\mathcal{A}_X\varphi V)\xi$$

Since $\mathcal{A}_X\varphi V$ is in $\Gamma(kerF_*)$, $\eta(\mathcal{A}_X\varphi V) = g_M(\mathcal{A}_X\varphi V,\xi) = 0$. Thus, we have

$$\mathcal{A}_X\varphi V = g_M(H,V)\varphi X.$$

Hence, we derive

$$g_M(\mathcal{A}_X\varphi V,\varphi X) = g_M(H,V)\{||X||^2 - \eta^2(X)\}$$

Then using (2.11) we have

$$g_M(\nabla_X\varphi V,\varphi X) = g_M(H,V)\{||X||^2 - \eta^2(X)\}$$

Thus (3.5) implies that

$$g_M(\nabla_X X,V) + \eta(X)g_M(X,\varphi V) = g_M(H,V)\{||X||^2 - \eta^2(X)\}$$

Then using (2.7), we have $\mathcal{A}_X X = 0$, which implies

$$\eta(X)g_M(X,\varphi V) = g_M(H,V)\{||X||^2-\eta^2(X)\}, \forall X \in \Gamma((kerF_*)^{\perp}), V \in \Gamma(kerF_*).$$

Choosing X which is orthogonal to ξ, $0 = g_M(H,V)|X||^2$. Since g_M is the Riemannian metric and $H \in \Gamma(kerF_*)$, we conclude that $H = 0$, which shows $(kerF_*)^{\perp}$ is totally geodesic, so M is usual product of Riemannian manifolds. □

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