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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS AND APPLICATIONS

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Abstract

In the paper, the authors offer some new inequalities for differentiable convex functions, which are connected with Hermite-Hadamard integral inequality, and apply these inequalities to special means of two positive numbers.

Keywords: Integral inequality, Hermite-Hadamard integral inequality, Convex function, Mean, Application

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1. Introduction

In [2], the following Hermite-Hadamard type inequalities for differentiable convex functions were proved.

1.1. Theorem ([2, Theorem 2.2]). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.1}$$

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1.2. Theorem ([2, Theorem 2.3]). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping, $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the new mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$, then*

$$(1.2)\qquad \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right]^{(p-1)/p}.$$

In [6], the above inequalities were generalized as follows.

1.3. Theorem ([6, Theorems 1 and 2]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and let $q \ge 1$. If $|f'(x)|^q$ is convex on $[a,b]$, then*

$$(1.3)\qquad \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}$$

and

$$(1.4)\qquad \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}.$$

In [4], the above inequalities were further generalized as follows.

1.4. Theorem ([4, Theorems 2.3 and 2.4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and let $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$, then*

$$(1.5)\qquad \begin{aligned} \left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| &\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ &\quad\times\Big\{\big[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\big]^{(p-1)/p} \\ &\quad+\big[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\big]^{(p-1)/p}\Big\}\end{aligned}$$

and

$$(1.6)\qquad \left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}(|f'(a)|+|f'(b)|).$$

In [3], an inequality similar to the above ones was given as follows.

1.5. Theorem ([3, Theorem 3]). *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping whose derivative belongs to $L_p[a,b]$. Then*

$$(1.7)\qquad \left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}(b-a)^{1/q}\|f'\|_p,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$.

For more information on this topic, please refer to [1, 8, 9, 10, 11, 12] and plenty of references cited therein.

In this paper, motivated by Theorems 1.1 to 1.5, we will establish some new Hermite-Hadamard type inequalities for differentiable functions and apply them to derive some inequalities of special means.

2. A Lemma

For establishing our Hermite-Hadamard type inequalities, we need the following lemma.

2.1. Lemma. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned} &\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\ &\quad=\frac{b-a}{4}\int_0^1\left(\frac{1}{2}-t\right)\left[f'\left(ta+(1-t)\frac{a+b}{2}\right)+f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t. \end{aligned} \tag{2.1}$$

Proof. Integrating by part and changing variables of definite integrals yield

$$\begin{aligned} &\int_0^1\left(\frac{1}{2}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \\ &=-\frac{2}{b-a}\left[\left(\frac{1}{2}-t\right)f\left(ta+(1-t)\frac{a+b}{2}\right)\bigg|_0^1+\int_0^1 f\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right] \\ &=\frac{1}{b-a}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]-\frac{4}{(b-a)^2}\int_a^{(a+b)/2} f(x)\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} &\int_0^1\left(\frac{1}{2}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}t \\ &=-\frac{2}{b-a}\left[\left(\frac{1}{2}-t\right)f\left(t\frac{a+b}{2}+(1-t)b\right)\bigg|_0^1+\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}t\right] \\ &=\frac{1}{b-a}\left[f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{4}{(b-a)^2}\int_{(a+b)/2}^{b} f(x)\,\mathrm{d}x. \end{aligned}$$

Combining the above two equations leads to Lemma 2.1. □

2.2. Remark. Lemma 2.1 is the key point of this paper, which will lead to the new Hermite-Hadamard type inequalities for differentiable convex functions in next section. We notice that choosing a new and suitable function $g(t)$ instead of $\frac{1}{2}-t$ in integrals

$$\int_0^1\left(\frac{1}{2}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \quad \text{and} \quad \int_0^1\left(\frac{1}{2}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}t$$

will generalize Lemma 2.1 and, by this generalization, some more general and new inequalities of Hermite-Hadamard type can be derived. Due to the limitation of length of this paper, we will study in this direction in subsequent papers.

3. New inequalities of Hermite-Hadamard type

Now we are in a position to establish some new Hermite-Hadamard type inequalities for differentiable convex functions.

3.1. Definition. A function $f(x)$ is said to be convex on an interval I if

$$f(\lambda x_1+(1-\lambda)x_2)\le \lambda f(x_1)+(1-\lambda)f(x_2) \tag{3.1}$$

holds for all $x_1, x_2 \in I$ and $0 < \lambda < 1$.

3.2. Theorem. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leq \frac{b-a}{16}\left(\frac{1}{q+1}\right)^{1/q}$$
$$\times\left\{\left[\frac{(2q+5)|f'(a)|^q+(2q+3)|f'(b)|^q}{4(q+2)}\right]^{1/q}+\left[\frac{|f'(a)|^q+(4q+7)|f'(b)|^q}{4(q+2)}\right]^{1/q}\right.$$
$$\left.+\left[\frac{(4q+7)|f'(a)|^q+|f'(b)|^q}{4(q+2)}\right]^{1/q}+\left[\frac{(2q+3)|f'(a)|^q+(2q+5)|f'(b)|^q}{4(q+2)}\right]^{1/q}\right\}. \tag{3.2}$$

Proof. When $q > 1$, since $|f'(x)|^q$ is convex on $[a, b]$, by Lemma 2.1 and Hölder integral inequality, we have

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq \frac{b-a}{4}\left\{\int_0^{1/2}\left(\frac{1}{2}-t\right)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right.$$
$$\left.+\int_{1/2}^1\left(t-\frac{1}{2}\right)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\}$$
$$\leq \frac{b-a}{4}\left\{\left(\frac{1}{2}\right)^{1-1/q}\left[\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)^q\left(\frac{(1+t)|f'(a)|^q}{2}+\frac{(1-t)|f'(b)|^q}{2}\right)\mathrm{d}t\right)^{1/q}\right.\right.$$
$$\left.+\left(\int_0^{1/2}\left(\frac{1}{2}-t\right)^q\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)\mathrm{d}t\right)^{1/q}\right]$$
$$+\left(\frac{1}{2}\right)^{1-1/q}\left[\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)^q\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)\mathrm{d}t\right)^{1/q}\right.$$
$$\left.\left.+\left(\int_{1/2}^1\left(t-\frac{1}{2}\right)^q\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)\mathrm{d}t\right)^{1/q}\right]\right\}.$$

A direct calculation gives

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)^q\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)\mathrm{d}t$$
$$=\frac{1}{2^{q+3}}\left[\left(\frac{3}{q+1}-\frac{1}{q+2}\right)|f'(a)|^q+\left(\frac{1}{q+2}+\frac{1}{q+1}\right)|f'(b)|^q\right]$$
$$=\frac{1}{2^{q+3}(q+1)(q+2)}\left[(2q+5)|f'(a)|^q+(2q+3)|f'(b)|^q\right],$$
$$\int_0^{1/2}\left(\frac{1}{2}-t\right)^q\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)\mathrm{d}t$$
$$=\frac{1}{2^{q+3}}\left[\left(\frac{1}{q+1}-\frac{1}{q+2}\right)|f'(a)|^q+\left(\frac{1}{q+2}+\frac{3}{q+1}\right)|f'(b)|^q\right]$$
$$=\frac{1}{2^{q+3}(q+1)(q+2)}\left[|f'(a)|^q+(4q+7)|f'(b)|^q\right],$$
$$\int_{1/2}^1\left(t-\frac{1}{2}\right)^q\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)\mathrm{d}t$$
$$=\frac{1}{2^{q+3}}\left[\left(\frac{1}{q+2}+\frac{3}{q+2}\right)|f'(a)|^q+\left(\frac{1}{q+1}-\frac{1}{q+2}\right)|f'(b)|^q\right]$$
$$=\frac{1}{2^{q+3}(q+1)(q+2)}\left[(4q+7)|f'(a)|^q+|f'(b)|^q\right],$$

$$\int_{1/2}^{1}\left(t-\frac{1}{2}\right)^{q}\left(\frac{t}{2}|f'(a)|^{q}+\frac{2-t}{2}|f'(b)|^{q}\right)\mathrm{d}t$$
$$=\frac{1}{2^{q+3}}\left[\left(\frac{1}{q+2}+\frac{1}{q+1}\right)|f'(a)|^{q}+|f'(b)|^{q}\left(\frac{3}{q+1}-\frac{1}{q+2}\right)\right]$$
$$=\frac{1}{2^{q+3}(q+1)(q+2)}\left[(2q+3)|f'(a)|^{q}+(2q+5)|f'(b)|^{q}\right].$$

Substituting the above identities into the above inequality results in

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\frac{1}{2^{q+3}(q+1)(q+2)}\big((2q+5)|f'(a)|^{q}+(2q+3)|f'(b)|^{q}\big)\right]^{1/q}\right.$$
$$+\left[\frac{1}{2^{q+3}(q+1)(q+2)}\big(|f'(a)|^{q}+(4q+7)|f'(b)|^{q}\big)\right]^{1/q}$$
$$+\left[\frac{1}{2^{q+3}(q+1)(q+2)}\big((4q+7)|f'(a)|^{q}+|f'(b)|^{q}\big)\right]^{1/q}$$
$$\left.+\left[\frac{1}{2^{q+3}(q+1)(q+2)}\big((2q+3)|f'(a)|^{q}+(2q+5)|f'(b)|^{q}\big)\right]^{1/q}\right\}$$
$$=\frac{b-a}{16}\left[\frac{1}{4(q+1)(q+2)}\right]^{1/q}\Big[\big((2q+5)|f'(a)|^{q}+(2q+3)|f'(b)|^{q}\big)^{1/q}$$
$$+\big(|f'(a)|^{q}+(4q+7)|f'(b)|^{q}\big)^{1/q}+\big((4q+7)|f'(a)|^{q}+|f'(b)|^{q}\big)^{1/q}$$
$$+\big((2q+3)|f'(a)|^{q}+(2q+5)|f'(b)|^{q}\big)^{1/q}\Big].$$

Thus, the inequality (3.2) is valid for $q>1$.

When $q=1$, by Lemma 2.1, we have

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\,\mathrm{d}x\right|$$
$$\le\frac{b-a}{4}\left\{\int_{0}^{1/2}\left(\frac{1}{2}-t\right)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right.$$
$$\left.+\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\}$$
$$\le\frac{b-a}{4}\left[\int_{0}^{1/2}\left(\frac{1}{2}-t\right)\left(\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right)\mathrm{d}t\right.$$
$$+\int_{0}^{1/2}\left(\frac{1}{2}-t\right)\left(\frac{t}{2}|f'(a)|+\frac{2-t}{2}|f'(b)|\right)\mathrm{d}t$$
$$+\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left(\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right)\mathrm{d}t$$
$$\left.+\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left(\frac{t}{2}|f'(a)|+\frac{2-t}{2}|f'(b)|\right)\mathrm{d}t\right]$$
$$=\frac{b-a}{16}(|f'(a)|+|f'(b)|).$$

Thus, the inequality (3.2) holds for $q=1$. Theorem 3.2 is proved. □

From Theorem 3.2 we can derive the following two corollaries.

3.3. Corollary. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ for $q \ge 1$ and*

$$\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right), \tag{3.3}$$

then

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &= \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{b-a}{16}\left(\frac{1}{q+1}\right)^{1/q}\left\{\left[\frac{(2q+5)|f'(a)|^q+(2q+3)|f'(b)|^q}{4(q+2)}\right]^{1/q}\right. \\ &\quad + \left[\frac{|f'(a)|^q+(4q+7)|f'(b)|^q}{4(q+2)}\right]^{1/q} + \left[\frac{(4q+7)|f'(a)|^q+|f'(b)|^q}{4(q+2)}\right]^{1/q} \\ &\quad \left. + \left[\frac{(2q+3)|f'(a)|^q+(2q+5)|f'(b)|^q}{4(q+2)}\right]^{1/q}\right\}. \end{aligned} \tag{3.4}$$

3.4. Corollary. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and let $f' \in L[a, b]$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{16}(|f'(a)|+|f'(b)|). \tag{3.5}$$

Furthermore, if the equality (3.3) *is valid, then*

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &= \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{b-a}{16}(|f'(a)|+|f'(b)|). \end{aligned} \tag{3.6}$$

3.5. Remark. Somebody may ask that whether the condition (3.3) is significant. In other words, can one find an example satisfying conditions of Corollary 3.3? This question can be affirmatively answered by the functions

$$f(x) = \pm\frac{1}{3}x\left(x^2 - 9x + 27\right)$$

on the interval $[1, 5]$. Therefore, Corollary 3.3 and the inequality (3.6) in Corollary 3.4 are significant.

3.6. Definition. A function $f : I \subseteq \mathbb{R} \to (0, \infty)$ is said to be logarithmically convex if the function $\ln f$ is convex on I, that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le [f(x_1)]^\lambda [f(x_2)]^{1-\lambda} \tag{3.7}$$

for any two points x_1 and x_2 on I and any λ meeting $0 < \lambda < 1$.

3.7. Theorem. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is logarithmically convex on $[a, b]$ for $q \ge 1$, then*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{b-a}{4}\left(\frac{1}{8}\right)^{1-1/q}\left\{|f'(a)f'(b)|^{1/2}[g_1(\mu)]^{1/q} + |f'(b)|[g_1(\mu)]^{1/q}\right. \\ &\quad \left. + |f'(a)|[g_2(\mu)]^{1/q} + |f'(a)f'(b)|^{1/2}[g_2(\mu)]^{1/q}\right\}, \end{aligned} \tag{3.8}$$

where

$$\mu = \left|\frac{f'(a)}{f'(b)}\right|^{q/2},$$

(3.9)
$$g_1(u) = \begin{cases} \dfrac{u^{1/2}-1}{(\ln u)^2} - \dfrac{1}{2\ln u}, & u \neq 1, \\ \dfrac{1}{8}, & u = 1, \end{cases}$$

and

$$g_2(u) = \begin{cases} \dfrac{u^{-1/2}-1}{(\ln u)^2} + \dfrac{1}{2\ln u}, & u \neq 1, \\ \dfrac{1}{8}, & u = 1. \end{cases} \tag{3.10}$$

Proof. When $q > 1$, from the logarithmic convexity of $|f'(x)|^q$ on $[a, b]$ and (3.7), Lemma 2.1, and Hölder integral inequality, it follows that

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\leq \frac{b-a}{4}\left\{\int_0^{1/2}\left(\frac{1}{2}-t\right)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right| + \left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right. \\ &\quad \left. + \int_{1/2}^1\left(t-\frac{1}{2}\right)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right| + \left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\} \\ &\leq \frac{b-a}{4}\left\{\left(\frac{1}{8}\right)^{1-1/q}\left(\left[\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(a)|^{q(1+t)/2}|f'(b)|^{q(1-t)/2}\,\mathrm{d}t\right]^{1/q}\right.\right. \\ &\quad \left. + \left[\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(a)|^{qt/2}|f'(b)|^{q(2-t)/2}\,\mathrm{d}t\right]^{1/q}\right) \\ &\quad + \left(\frac{1}{8}\right)^{1-1/q}\left(\left[\int_{1/2}^1\left(t-\frac{1}{2}\right)|f'(a)|^{q(1+t)/2}|f'(b)|^{q(1-t)/2}\,\mathrm{d}t\right]^{1/q}\right. \\ &\quad \left.\left. + \left[\int_{1/2}^1\left(t-\frac{1}{2}\right)|f'(a)|^{qt/2}|f'(b)|^{q(2-t)/2}\,\mathrm{d}t\right]^{1/q}\right)\right\}. \end{aligned}$$

If $\mu = 1$, then

$$\int_0^{1/2}\left(\frac{1}{2}-t\right)\mu^t\,\mathrm{d}t = \int_{1/2}^1\left(t-\frac{1}{2}\right)\mu^t\,\mathrm{d}t = \frac{1}{8}.$$

If $\mu \neq 1$, we have

$$\begin{aligned} &\int_0^{1/2}\left(\frac{1}{2}-t\right)\mu^t\,\mathrm{d}t = \frac{1}{\ln\mu}\left(\frac{\mu^{1/2}-1}{\ln\mu} - \frac{1}{2}\right) = g_1(\mu), \\ &\int_{1/2}^1\left(t-\frac{1}{2}\right)\mu^t\,\mathrm{d}t = \mu\left(\frac{1}{2\ln\mu} + \frac{\mu^{-1/2}-1}{(\ln\mu)^2}\right) = \mu g_2(\mu). \end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(a)|^{q(1+t)/2}|f'(b)|^{q(1-t)/2}\,\mathrm{d}t \quad &=|f'(a)f'(b)|^{q/2}\int_0^{1/2}\left(\frac{1}{2}-t\right)\mu^t\,\mathrm{d}t\\
&=|f'(a)f'(b)|^{q/2}g_1(\mu),\\
\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(a)|^{qt/2}|f'(b)|^{q(2-t)/2}\,\mathrm{d}t \quad &=|f'(b)|^{q}\int_0^{1/2}\left(\frac{1}{2}-t\right)\mu^t\,\mathrm{d}t\\
&=|f'(b)|^{q}g_1(\mu),\\
\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|f'(a)|^{q(1+t)/2}|f'(b)|^{q(1-t)/2}\,\mathrm{d}t \quad &=|f'(a)f'(b)|^{q/2}\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\mu^t\,\mathrm{d}t\\
&=|f'(a)|^{q}g_2(\mu),\\
\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|f'(a)|^{qt/2}|f'(b)|^{q(2-t)/2}\,\mathrm{d}t \quad &=|f'(b)|^{q}\int_{1/2}^{1}\left(t-\frac{1}{2}\right)\mu^t\,\mathrm{d}t\\
&=|f'(a)f'(b)|^{q/2}g_2(\mu).
\end{aligned}$$

Substituting these equalities into the first inequality above and rearranging yield the inequality (3.8).

If $q=1$, by the same argument as above, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{b-a}{4}\left[\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(a)|^{(1+t)/2}|f'(b)|^{(1-t)/2}\,\mathrm{d}t\right.\\
&\quad+\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|f'(a)|^{t/2}|f'(b)|^{(2-t)/2}\,\mathrm{d}t\\
&\quad+\int_0^{1/2}\left(\frac{1}{2}-t\right)|f'(a)|^{t/2}|f'(b)|^{(2-t)/2}\,\mathrm{d}t\\
&\quad\left.+\int_{1/2}^{1}\left(t-\frac{1}{2}\right)|f'(a)|^{(1+t)/2}|f'(b)|^{(1-t)/2}\,\mathrm{d}t\right]\\
&=\frac{b-a}{4}\left\{|f'(a)f'(b)|^{1/2}[g_1(\mu)+g_2(\mu)]+|f'(a)|g_2(\mu)+|f'(b)|g_1(\mu)\right\}.
\end{aligned}$$

Thus, Theorem 3.7 is proved. □

From Theorem 3.7 we can deduce the following corollaries.

3.8. Corollary. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I°, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'(x)|$ is logarithmically convex on $[a,b]$, then*

$$\begin{aligned}
\text{(3.11)}\quad &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\qquad\le\frac{b-a}{4}\left[|f'(a)f'(b)|^{1/2}[g_1(\mu)+g_2(\mu)]+|f'(b)|g_1(\mu)+|f'(a)|g_2(\mu)\right].
\end{aligned}$$

In particular, if the identity (3.3) is also valid, then

$$\begin{aligned}
\text{(3.12)}\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|=\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\qquad\le\frac{b-a}{4}\left[|f'(a)f'(b)|^{1/2}[g_1(\mu)+g_2(\mu)]+|f'(b)|g_1(\mu)+|f'(a)|g_2(\mu)\right].
\end{aligned}$$

3.9. Remark. Similar to Remark 3.5, we can give an example

$$f(x) = \ln(\sec x + \tan x)$$

on the symmetric interval $[-a, a]$, where $0 < a < \frac{\pi}{2}$, meeting conditions of the inequality (3.12). An alternative example is

$$f(x) = \pm\bigl(1 + x^{2k}\bigr)x$$

for $k \in \mathbb{N}$, which satisfy conditions of the inequality (3.12) on the interval $[-a, a]$ for $0 < a < 1$. These show us that the inequality (3.12) in Corollary 3.8 is significant.

3.10. Theorem. *Under conditions of Theorem 3.7, the inequality* (3.8) *is better than* (3.2). *Consequently, under conditions of Corollary 3.8, the inequality* (3.11) *is sharper than* (3.5)*; if the equality* (3.3) *is also valid, then the inequality* (3.12) *is stronger than* (3.6).

Proof. This follows from considering the following two facts in the proofs of Theorems 3.2 and 3.7.

(1) Any logarithmically convex function must be convex. See [7, Remarks 1.2 and 1.9] and related references therein.
(2) The inequality between the arithmetic and geometric weighted means (see [5, p. 49, Remark 1]) implies

$$|f'(a)|^{q(1+t)/2}|f'(b)|^{q(1-t)/2} \le \frac{1+t}{2}|f'(a)|^q + \frac{1-t}{2}|f'(b)|^q,$$

and

$$|f'(a)|^{qt/2}|f'(b)|^{q(2-t)/2} \le \frac{t}{2}|f'(a)|^q + \frac{2-t}{2}|f'(b)|^q.$$

Theorem 3.10 is thus proved. □

3.11. Remark. The term

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$

in (3.2) and (3.8) is different from the left hand sides of inequalities (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), and (1.7), so Theorems 3.2 and 3.7 can not be compared with Theorems 1.1, 1.2, 1.3, 1.4, and 1.5 mentioned in the first section.

4. Applications to means

For two positive numbers $a > 0$ and $b > 0$, define

$$A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}\,, \quad H(a, b) = \frac{2ab}{a+b},$$

and

$$I(a,b)=\begin{cases}\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a\ne b,\\ a, & a=b,\end{cases}$$

$$L_s(a,b)=\begin{cases}\left[\dfrac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{1/s}, & s\ne 0,-1 \text{ and } a\ne b,\\ \dfrac{b-a}{\ln b-\ln a}, & s=-1 \text{ and } a\ne b,\\ I(a,b), & s=0 \text{ and } a\ne b,\\ a, & a=b,\end{cases}$$

$$H_{\omega,s}(a,b)=\begin{cases}\left[\dfrac{a^s+\omega(ab)^{s/2}+b^s}{\omega+2}\right]^{1/s}, & s\ne 0,\\ \sqrt{ab}\,, & s=0\end{cases}$$

for $0\le\omega<\infty$. It is well known that A, G, H, $L=L_{-1}$, $I=L_0$, L_s, and $H_{\omega,s}$ are respectively called the arithmetic, geometric, harmonic, logarithmic, exponential, generalized logarithmic, and generalized Heronian means of two positive number a and b.

In what follows we will apply theorems and corollaries in the above section to establish inequalities for some special mean values.

4.1. Theorem. *Let $b>a>0$, $q\ge 1$, and $s\in\mathbb{R}$.*

(1) *If $s>1$ and $(s-1)q\ge 1$, then*

$$\left|\frac{A(a^s,b^s)+A^s(a,b)}{2}-[L_s(a,b)]^s\right|\le\frac{(b-a)s}{16}\left(\frac{1}{q+1}\right)^{1/q} \\ \times\left\{\left[\frac{(2q+5)a^{(s-1)q}+(2q+3)b^{(s-1)q}}{4(q+2)}\right]^{1/q}+\left[\frac{a^{(s-1)q}+(4q+7)b^{(s-1)q}}{4(q+2)}\right]^{1/q}\right. \\ \left.+\left[\frac{(4q+7)a^{(s-1)q}+b^{(s-1)q}}{4(q+2)}\right]^{1/q}+\left[\frac{(2q+3)a^{(s-1)q}+(2q+5)b^{(s-1)q}}{4(q+2)}\right]^{1/q}\right\}. \tag{4.1}$$

(2) *If $s<1$ and $s\ne 0,-1$, then*

$$\left|\frac{A(a^s,b^s)+A^s(a,b)}{2}-[L_s(a,b)]^s\right|\le\frac{(b-a)^{1-1/q}|s|}{16}\left[\frac{8}{(1-s)q}\right]^{1/q} \\ \times A\big(a^{(s-1)/2},b^{(s-1)/2}\big)[L(a,b)]^{1/q} \\ \times\Big\{b^{(s-1)/2}\Big[a^{(s-1)q/4}\big(L_{(1-s)q/4-1}(a,b)\big)^{(1-s)q/4-1}L(a,b)-1\Big]^{1/q} \\ +a^{(s-1)/2}\Big[1-b^{(s-1)q/4}\big(L_{(1-s)q/4-1}(a,b)\big)^{(1-s)q/4-1}L(a,b)\Big]^{1/q}\Big\} \\ \le\frac{b-a}{16}|s|\left(\frac{1}{q+1}\right)^{1/q}\left\{\left[\frac{(2q+5)a^{(s-1)q}+(2q+3)b^{(s-1)q}}{4(q+2)}\right]^{1/q}\right. \\ +\left[\frac{a^{(s-1)q}+(4q+7)b^{(s-1)q}}{4(q+2)}\right]^{1/q}+\left[\frac{(4q+7)a^{(s-1)q}+b^{(s-1)q}}{4(q+2)}\right]^{1/q} \\ \left.+\left[\frac{(2q+3)a^{(s-1)q}+(2q+5)b^{(s-1)q}}{4(q+2)}\right]^{1/q}\right\}.$$

(3) *If $s=-1$, then*

$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right| \le \frac{(b-a)^{1-1/q}}{16}\left(\frac{4}{q}\right)^{1/q}$$
$$\times \frac{[L(a,b)]^{1/q}}{H(a,b)}\left\{\frac{1}{b}\left[\frac{(L_{q/2-1}(a,b))^{q/2-1}L(a,b)}{a^{q/2}}-1\right]^{1/q}\right.$$
$$\left.+\frac{1}{a}\left[1-\frac{(L_{q/2-1}(a,b))^{q/2-1}L(a,b)}{b^{q/2}}\right]^{1/q}\right\}$$
$$\le \frac{b-a}{16}\left(\frac{1}{q+1}\right)^{1/q}\left\{\left[\frac{(2q+5)/a^{2q}+(2q+3)/b^{2q}}{4(q+2)}\right]^{1/q}\right.$$
$$+\left[\frac{1/a^{2q}+(4q+7)/b^{2q}}{4(q+2)}\right]^{1/q}+\left[\frac{(4q+7)/a^{2q}+1/b^{2q}}{4(q+2)}\right]^{1/q}$$
$$\left.+\left[\frac{(2q+3)/a^{2q}+(2q+5)/b^{2q}}{4(q+2)}\right]^{1/q}\right\}.$$

Proof. Let $f(x)=x^s$ for $x>0$ and $s\ne 0,1$. Then $f'(x)=sx^{s-1}$ and $|f'(x)|^q=|s|x^{(s-1)q}$. Further, it follows that

$$\big(|f'(x)|^q\big)''=|s|^q(s-1)q[(s-1)q-1]x^{(s-1)q-2}$$

and

$$\big(\ln|f'(x)|^q\big)''=\frac{(1-s)q}{x^2}.$$

If $s>1$ and $(s-1)q\ge 1$, the function $|f'(x)|^q=|s|^qx^{(s-1)q}$ is convex on $[a,b]$. By Theorem 3.2, the inequality (4.1) follows.

If $s<1$ and $s\ne 0$, the function $|f'(x)|^q=|s|^qx^{(s-1)q}$ is logarithmically convex on $[a,b]$. Meanwhile, the formulas (3.9) and (3.10) become

$$\mu=\left(\frac{b}{a}\right)^{(1-s)q/2},$$
$$g_1(\mu)=\frac{L(a,b)}{q(1-s)(b-a)}\Big\{a^{(s-1)q/4}[L_{(1-s)q/4-1}(a,b)]^{(1-s)q/4-1}L(a,b)-1\Big\},$$
$$g_2(\mu)=\frac{L(a,b)}{q(1-s)(b-a)}\Big\{1-b^{(s-1)q/4}[L_{(1-s)q/4-1}(a,b)]^{(1-s)q/4-1}L(a,b)\Big\}.$$

Substituting these scalars into Theorem 3.10 yields the required results. □

4.2. Corollary. *Let $b>a>0$ and $s\in\mathbb{R}$.*

(1) *If $s\ge 2$, we have*

$$\left|\frac{A(a^s,b^s)+[A(a,b)]^s}{2}-[L_s(a,b)]^s\right|\le\frac{b-a}{8}|s|A\big(a^{s-1},b^{s-1}\big). \tag{4.2}$$

(2) *If $s<1$ and $s\ne 0,-1$, we have*

$$\left|\frac{A(a^s,b^s)+[A(a,b)]^s}{2}-[L_s(a,b)]^s\right|\le\frac{b-a}{8}|s|A\big(a^{(s-1)/2},b^{(s-1)/2}\big) \tag{4.3}$$
$$\times[G(a,b)]^{s-1}L(a,b)\left\{\frac{2}{[L_{-(1+s)/2}(a,b)]^{(1+s)/2}}-\frac{L(a,b)}{[L_{-(3+s)/2}(a,b)]^{(3+s)/2}}\right\}$$
$$\le\frac{b-a}{8}|s|A\big(a^{s-1},b^{s-1}\big).$$

(3) *If $s=-1$, we have*

$$\left|\frac{1}{2}\left[\frac{1}{H(a,b)}+\frac{1}{A(a,b)}\right]-\frac{1}{L(a,b)}\right| \le \frac{b-a}{8}\frac{L(a,b)}{H(a,b)[G(a,b)]^2} \le \frac{b-a}{8}\frac{1}{H\left(a^2,b^2\right)}.$$

4.3. Theorem. *For $b>a>0$ and $q\ge 1$, we have*

$$\begin{aligned}&\left|\frac{\ln G(a,b)+\ln A(a,b)}{2}-\ln I(a,b)\right| \le \frac{(b-a)^{1-1/q}}{16}\left(\frac{8}{q}\right)^{1/q}\frac{[L(a,b)]^{1/q}}{H(a^{1/2},b^{1/2})}\\
&\quad\times\left\{\frac{1}{b^{1/2}}\left[\frac{(L_{q/4-1}(a,b))^{q/4-1}L(a,b)}{a^{q/4}}-1\right]^{1/q}\right.\\
&\quad\left.+\frac{1}{a^{1/2}}\left[1-\frac{(L_{q/4-1}(a,b))^{q/4-1}L(a,b)}{b^{q/4}}+\right]^{1/q}\right\}\\
&\le \frac{b-a}{16}\frac{1}{(q+1)^{1/q}}\left\{\left[\frac{(2q+5)/a^q+(2q+3)/b^q}{4(q+2)}\right]^{1/q}+\left[\frac{1/a^q+(4q+7)/b^q}{4(q+2)}\right]^{1/q}\right.\\
&\quad\left.+\left[\frac{(4q+7)/a^q+1/b^q}{4(q+2)}\right]^{1/q}+\left[\frac{(2q+3)/a^q+(2q+5)/b^q}{4(q+2)}\right]^{1/q}\right\}.\end{aligned}$$

Proof. Let $f(x)=\ln x$ for $x>0$. It is clear that $f'(x)=\frac{1}{x}$ and $|f'(x)|^q=\frac{1}{x^q}$. Further, we have $\left(\ln|f'(x)|^q\right)''=\frac{q}{x^2}$. This shows that the function $|f'(x)|^q=\frac{1}{x^q}$ is logarithmically convex. On the other hand, we have

$$\begin{aligned}\mu&=\left(\frac{b}{a}\right)^{q/2},\\
g_1(\mu)&=\frac{L(a,b)}{q(b-a)}\left\{\frac{[L_{q/4-1}(a,b)]^{q/4-1}L(a,b)}{a^{q/4}}-1\right\},\\
g_2(\mu)&=\frac{L(a,b)}{q(b-a)}\left\{1-\frac{[L_{q/4-1}(a,b)]^{q/4-1}L(a,b)}{b^{q/4}}\right\}.\end{aligned}$$

Substituting these equations into Theorem 3.10 leads to our required results. □

4.4. Corollary. *For $b>a>0$ and $q\ge 1$, we have*

$$\begin{aligned}&\left|\frac{\ln G(a,b)+\ln A(a,b)}{2}-\ln I(a,b)\right|\\
&\le \frac{b-a}{8}\frac{L(a,b)}{H(a^{1/2},b^{1/2})G(a,b)}\left\{\frac{2}{[L_{-1/2}(a,b)]^{1/2}}-\frac{L(a,b)}{[L_{-3/4}(a,b)]^{3/2}}\right\}\\
&\le \frac{b-a}{8}\frac{1}{H(a,b)}.\end{aligned}\tag{4.4}$$

4.5. Theorem. *Let $b>a>0$ and $\omega\ge 0$.*

(1) *If $s\ge 4$ or $s<1$ with $s\ne -1,-2$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left\{\frac{[H_{\omega,s}(a,b)]^s}{H(a^s,b^s)}+\frac{\left[H_{\omega,s}\left(ab,a^2+b^2\right)\right]^s}{[G(a,b)]^{2s}}\right\}\right.\\
&\quad\left.-\frac{1}{\omega+2}\left\{\frac{[L_{2s+1}(a,b)]^{2s+1}}{A(a,b)[G(a,b)]^{2s}}+\omega\frac{[L_{s+1}(a,b)]^{s+1}}{A(a,b)[G(a,b)]^s}+1\right\}\right|\\
&\le \frac{b-a}{8}\frac{|s|}{\omega+2}A(a,b)\left\{2\frac{A\left(a^{2(s-1)},b^{2(s-1)}\right)}{[G(a,b)]^{2s}}+\omega\frac{A\left(a^{s-2},b^{s-2}\right)}{[G(a,b)]^s}\right\}.\end{aligned}\tag{4.5}$$

(2) *If $s=-1$, then*

$$\begin{aligned}
&\bigg|\frac{1}{2}\bigg\{\frac{H_{\omega,1}(a,b)}{H(a,b)}+\frac{H_{\omega,1}\big(ab,a^2+b^2\big)}{2[A(a,b)]^2}\\
&\quad-\frac{1}{\omega+2}\bigg[\frac{1}{H(a,b)L(a,b)}+\omega\frac{G(a,b)}{A(a,b)}+1\bigg]\bigg\}\bigg| \qquad (4.6)\\
&\le\frac{b-a}{8}\frac{1}{\omega+2}A(a,b)\bigg\{2\frac{A\big(a^4,b^4\big)}{[G(a,b)]^6}+\omega\frac{A\big(a^3,b^3\big)}{[G(a,b)]^5}\bigg\}.
\end{aligned}$$

(3) *If $s=-2$, we have*

$$\begin{aligned}
&\bigg|\frac{1}{2}\bigg\{\frac{[H_{\omega,2}(a,b)]^2}{H\big(a^2,b^2\big)}+\frac{1}{4}\bigg[\frac{H_{\omega,2}\big(ab,a^2+b^2\big)}{A\big(a^2,b^2\big)}\bigg]^2\bigg\}\\
&\quad-\frac{1}{\omega+2}\bigg[\frac{\omega}{H(a,b)L(a,b)}+2\bigg]\bigg| \qquad (4.7)\\
&\le\frac{b-a}{4}\frac{1}{\omega+2}A(a,b)\bigg\{2\frac{A\big(a^6,b^6\big)}{[G(a,b)]^8}+\omega\frac{A\big(a^4,b^4\big)}{[G(a,b)]^4}\bigg\}.
\end{aligned}$$

Proof. Let

$$f(x)=\frac{x^s+\omega x^{s/2}+1}{\omega+2}$$

for $x>0$ and $s\ne0$. Then

$$\begin{aligned}
f'(x)&=\frac{s}{\omega+2}\bigg(x^{s-1}+\frac{\omega}{2}x^{s/2-1}\bigg),\\
(|f'(x)|)''&=\frac{|s|}{\omega+2}x^{s/2-3}\bigg[(s-1)(s-2)x^{s/2}+\frac{\omega}{8}(s-2)(s-4)\bigg].
\end{aligned}$$

This means that when $s\ge4$ or $s<1$ with $s\ne0$ the function $|f'(x)|$ is convex on $[a,b]$. It is easy to see that

$$\frac{1}{2}\bigg[\frac{f(b/a)+f(a/b)}{2}+f\bigg(\frac{b/a+a/b}{2}\bigg)\bigg]=\frac{1}{2}\bigg\{\frac{[H_{\omega,s}(a,b)]^s}{H\big(a^s,b^s\big)}+\frac{\big[H_{\omega,s}\big(ab,a^2+b^2\big)\big]^s}{[G(a,b)]^2}\bigg\}.$$

When $s\ge4$ or $s<1$ with $s\ne0,-1,-2$, we have

$$\begin{aligned}
\frac{1}{b/a+a/b}\int_{a/b}^{b/a}f(x)\,\mathrm{d}x&=\frac{1}{b/a+a/b}\int_{a/b}^{b/a}\frac{x^s+\omega x^{s/2}+1}{\omega+2}\,\mathrm{d}x\\
&=-\frac{1}{\omega+2}\bigg\{\frac{[L_{2s+1}(a,b)]^{2s+1}}{A(a,b)[G(a,b)]^{2s}}+\omega\frac{[L_{s+1}(a,b)]^{s+1}}{A(a,b)[G(a,b)]^{s}}+1\bigg\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{b/a+a/b}{16}\bigg\{\bigg|f'\bigg(\frac{a}{b}\bigg)\bigg|+\bigg|f'\bigg(\frac{b}{a}\bigg)\bigg|\bigg\}&=\frac{b-a}{8}\frac{|s|}{\omega+2}A(a,b)\\
&\quad\times\bigg\{2\frac{A\big(a^{2(s-1)},b^{2(s-1)}\big)}{[G(a,b)]^{2s}}+\omega\frac{A\big(a^{s-2},b^{s-2}\big)}{[G(a,b)]^{s}}\bigg\}.
\end{aligned}$$

Substituting these equations into Corollary 3.4 leads to the inequality (4.5).

When $s=-1$, we have

$$\frac{1}{b/a+a/b}\int_{a/b}^{b/a} f(x)\,\mathrm{d}x = \frac{1}{b/a+a/b}\int_{a/b}^{b/a} \frac{1/x+\omega/x^{1/2}+1}{\omega+2}\,\mathrm{d}x$$

$$= \frac{1}{\omega+2}\left[\frac{1}{H(a,b)L(a,b)}+\omega\frac{G(a,b)}{A(a,b)}+1\right]\Bigg\},$$

$$\frac{1}{2}\left[\frac{f(b/a)+f(a/b)}{2}+f\left(\frac{b/a+a/b}{2}\right)\right] = \frac{1}{2}\left\{\frac{H_{\omega,1}(a,b)}{H(a,b)}+\frac{1}{2}\frac{H_{\omega,1}\left(ab,a^2+b^2\right)}{[A(a,b)]^2}\right\},$$

and

$$\frac{b/a+a/b}{16}\left\{\left|f'\left(\frac{a}{b}\right)\right|+\left|f'\left(\frac{b}{a}\right)\right|\right\} = \frac{b-a}{8}\frac{A(a,b)}{\omega+2}\left\{2\frac{A\left(a^4,b^4\right)}{[G(a,b)]^6}+\omega\frac{A\left(a^3,b^3\right)}{[G(a,b)]^5}\right\}.$$

Substituting these equations into Corollary 3.4 leads to the inequality (4.6).

Finally, when $s=-2$, it is not difficult to obtain that

$$\frac{1}{b/a+a/b}\int_{a/b}^{b/a} f(x)\,\mathrm{d}x = \frac{1}{b/a+a/b}\int_{a/b}^{b/a} \frac{1/x^2+\omega/x+1}{\omega+2}\,\mathrm{d}x$$

$$= \frac{1}{\omega+2}\left[\frac{\omega}{H(a,b)L(a,b)}+2\right],$$

$$\frac{b/a+a/b}{16}\left\{\left|f'\left(\frac{a}{b}\right)\right|+\left|f'\left(\frac{b}{a}\right)\right|\right\} = \frac{b-a}{4}\frac{A(a,b)}{\omega+2}\left\{2\frac{A\left(a^6,b^6\right)}{[G(a,b)]^8}+\omega\frac{A\left(a^4,b^4\right)}{[G(a,b)]^4}\right\},$$

and

$$\frac{1}{2}\left[\frac{f(b/a)+f(a/b)}{2}+f\left(\frac{b/a+a/b}{2}\right)\right]$$

$$= \frac{1}{2}\left\{\frac{[H_{\omega,2}(a,b)]^2}{H\left(a^2,b^2\right)}+\frac{1}{4}\left[\frac{H_{\omega,2}\left(ab,a^2+b^2\right)}{A\left(a^2,b^2\right)}\right]^2\right\}.$$

Substituting these equations into Corollary 3.4 leads to the inequality (4.7). The proof of Theorem 4.5 is complete. □

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