Hacettepe Journal of Mathematics and Statistics
Volume 42 (3) (2013), 259 – 268

APPROXIMATION BY FEJÉR SUMS OF FOURIER TRIGONOMETRIC SERIES IN WEIGHTED ORLICZ SPACES

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Received 09 : 03 : 2012 : Accepted 15 : 06 : 2012

Abstract

In this work we investigate the approximation problems of the functions by Fejér sums of Fourier series in the reflexive weighted Orlicz spaces with Muckenhoupt weights and of the functions by Fejér sums of Faber series in weighted Smirnov-Orlicz classes defined on simply connected domains with a Dini-smooth boundary of the complex plane.

Keywords: Orlicz space, weighted Orlicz space, Boyd indices, Muckenhoupt weight, Fejér sums, weighted Smirnov-Orlicz class, Dini-smooth curve, Faber series.

2000 AMS Classification: 41A10, 42A10, 41A25, 46A25

1. Introduction, main results and some auxiliary results

A convex and continuous function $M : [0,\infty) \to [0,\infty)$ for which $M(0) = 0$, $M(x) > 0$ for $x > 0$ and

$$\lim_{x\to 0} \frac{M(x)}{x} = 0, \quad \lim_{x\to\infty} \frac{M(x)}{x} = \infty$$

is called a Young function.

Let $T := [-\pi, \pi]$, and let M be a Young function. We denote by $L_M(T)$ the linear space Lebesgue measurable functions $f : T \to R$ satisfying the condition

$$\int_T M\left(\alpha |f(t)|\right) dt < \infty$$

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for some $\alpha > 0$. Equipped with the norm

$$\|f\|_{L_M(T)} := \inf\left\{\alpha > 0 : \int\limits_T M\left(\frac{|f(t)|}{\alpha}\right) dt < 1\right\},$$

the space $L_M(T)$ become a Banach space [43, pp.52-68].

The norm $\|\cdot\|_{L_M(T)}$ is called Orlicz norm and the Banach space $L_M(T)$ is called *Orlicz space*. It is known [43, p.50] that every function in $L_M(T)$ is integrable on T, i.e. $L_M(T) \subset L_1(T)$.

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse of the Young function M. The *lower* and *upper indices* α_M, β_M

$$\alpha_M := \lim_{x\to 0} \frac{\log h(x)}{\log x},\ \beta_M := \lim_{x\to\infty} \frac{\log h(x)}{\log x}$$

of the function

$$h : [0,\infty) \to [0,\infty),\ \ h(x) := \lim_{t\to\infty} \sup \frac{M^{-1}(t)}{M^{-1}\left(\frac{t}{x}\right)},\ \ x > 0$$

first considered by Matuszewska and Orlicz [38] are called the *Boyd indices* of the Orlicz space $L_M(T)$. It is known that $0 \le \alpha_M \le 1$. The Boyd indices α_M, β_M said to be nontrivial if $0 < \alpha_M$ and $\beta_M < 1$. The Orlicz space $L_M(T)$ is *reflexive* if and only if $0 < \alpha_M \le \beta_M < 1$, i.e. if the Boyd indices are nontrivial. The detailed information about Orlicz spaces and the Boyd indices can be found in [29] and [6], respectively.

A function ω is called a weight on T if $\omega : T \to [0,\infty]$ is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure). With any given weight ω we associate the *ω-weighted Orlicz space* $L_M(T,\omega)$ consisting of all measurable functions f on T such that

$$\|f\|_{L_M(T,\omega)} := \|f\omega\|_{L_M(T)}.$$

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let ω be a weight function on T. ω is said to satisfy Muckenhoupt's A_p-condition on T if

$$\sup_J \left(\frac{1}{|J|}\int\limits_J \omega^p(t)\,dt\right)^{1/p} \left(\frac{1}{|J|}\int\limits_J \omega^{-p'}(t)\,dt\right)^{1/p'} < \infty,$$

where J is any subinterval of T and $|J|$ denotes its length.

Let us denote by $A_p(T)$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on T.

Note that by [34, Lemma 3.3], [35, Theorem 4.5] and [33, Section 2.3] if $L_M(T)$ is reflexive and ω weight function satisfying the condition $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, then the space $L_M(T,\omega)$ is also reflexive.

Let $L_M(T,\omega)$ be a weighted Orlicz space, let α_M and β_M be nontrivial, and let $\omega \in A_{\frac{1}{\alpha_M}}(T) \cap A_{\frac{1}{\beta_M}}(T)$. For $f \in L_M(T,\omega)$ we set

$$(\nu_h f)(x) := \frac{1}{2h}\int\limits_{-h}^{h} f(x+t)\,dt,\ 0 < h < \pi, x \in T.$$

By reference [20, Lemma 1] the shift operator ν_h is a bounded linear operator on $L_M(T,\omega)$:

$$\|\nu_h(f)\|_{L_M(T,\omega)} \le C\,\|f\|_{L_M(T,\omega)}.$$

The function

$$\Omega_{M,\omega}^{k}(\delta,f):=\sup_{\substack{0<h_i\leq\delta\\1\leq i\leq k}}\left\|\prod_{i=1}^{k}(I-\nu_{h_i})f\right\|_{L_M(T,\omega)},\delta>0,k=1,2,...$$

is called *k-th modulus of smoothness* of $f\in L_M(T,\omega)$, where I is the identity operator.

It can easily be shown that $\Omega_{M,\omega}^{k}(\cdot,f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega_{M,\omega}^{k}(\delta,f)=0,\ \Omega_{M,\omega}^{k}(\delta,f+g)\leq\Omega_{M,\omega}^{k}(\delta,f)+\Omega_{M,\omega}^{k}(\delta,g)$$

for $f,g\in L_M(T,\omega)$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k(f)\cos kx+b_k(f)\sin kx) \tag{1.1}$$

be the Fourier series of the function $f\in L_1(T)$, where $\alpha_k(f)$ are $\beta_k(f)$ the Fourier coefficients of the function f.

The *n-th partial sums and Fejér sums* of series (1.1) are defined, respectively as

$$S_n(x,f)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k(f)\cos kx+b_k(f)\sin kx),$$

$$\sigma_n(x,f)=\frac{1}{n+1}\sum_{k=0}^{n}S_k(x,f).$$

Note that Fejér sums were introduced by Fejér [9].

The best approximation to $f\in L_M(T,\omega)$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{M,\omega}:=\inf\left\{\|f-T_n\|_{L_M(T,\omega)}:T_n\in\prod\nolimits_n\right\}.$$

Note that the existence of $T_n^*\in\Pi_n$ such that

$$E_n(f)_{M,\omega}=\|f-T_n^*\|_{L_M(T,\omega)}$$

follows, for example, from Theorem 1.1 in [17, p.59].

We put

$$\rho_{n,M}(f)=\|f-\sigma_{n-1}(.,f)\|_{L_M(T,\omega)}.$$

We use $c,c_1,c_2,...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

The problems of approximation theory in weighted, non-weighted Lebesgue spaces and weighted, non-weighted Orlicz spaces have been investigated by several authors (see, for example, [1-5, 8, 11-14,18-28, 30, 31, 36, 37, 39, 40, 42, 44, 45]). Note that the approximation problems by trigonometric polynomials in weighted Lebesgue spaces with weights belonging to the Muckenhoupt class $A_p(T)$ were investigated in [11], [36] and [37].

Detailed information on weighted polynomial approximation can be found in the books [15] and [40].

In this work we obtain the general estimate for the deviation $\rho_{n,M}(f)$ of the function f from its Fejér sums $\sigma_n(f)$ in weighted Orlicz spaces $L_M(T,\omega)$. Note that the estimate obtained in this study depends on sequence of the best approximation $E_n(f)_{M,\omega}$. This result was applied to estimate of approximation of Fejér sums of Faber series in weighted Smirnov-Orlicz classes defined on simply connected domains of the complex plane in terms of the modulus of smoothness.

The following results hold.

1.1. Theorem. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then for $f \in L_M(T,\omega)$, the inequality*

$$\rho_{n,M}(f) = \|f - \sigma_{n-1}(.,f)\|_{L_M(T,\omega)} \leq \frac{c_1}{n} \sum_{m=1}^{n} E_m(f)_{M,\omega}, (n = 1,2,...)$$

holds with a positive constant c_1, not depend on n.

1.2. Corollary. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then for every $f \in L_M(T,\omega)$, the estimate*

$$\|f - \sigma_{n-1}(.,f)\|_{L_M(T,\omega)} \leq \frac{c_2}{n} \sum_{m=1}^{n} \Omega^k_{M,\omega}\left(\frac{1}{m+1}, f\right), \tag{1.2}$$

holds with a $c_2 > 0$ independent of n.

Now, we obtain the analogs of the above results in the weighted Smirnov-Orlicz classes, defined on the finite simple connected domains of the complex plane.

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := ext\Gamma$. Further let

$$T := \{w \in \mathbb{C} : |w| = 1\}, D := int\,T \texttt{ and } D^- := ext\,T.$$

Let $w = \varphi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0,$$

and let ψ denote the inverse of φ.

Let $w = \varphi_1(z)$ denote a function that maps the domain G conformally onto the disk $|w| < 1$.

The inverse mapping of φ_1 will be denoted by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\varphi_1(z)| = r$ under the mapping $z = \psi_1(w)$.

Let us denote by E_p, where $p > 0$, the class of all functions $f(z) \neq 0$ that are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p\,|dz|$$

is bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class.* If the function $f(z)$ belongs to E_p, then $f(x)$ has definite limiting values $f(z')$ almost every where on Γ, over all nontangential paths; $|f(z')|$ is summable on Γ; and

$$\lim_{r\to 1} \int_{\Gamma_r} |f(z)|^p\,|dz| = \int_{\Gamma} |f(z')|^p\,|dz|.$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [10, pp. 438-453] and [16, pp. 168-185].

Let $L_M(T,\omega)$ is a weighted Orlicz space defined on Γ. We define also the *ω-weighted Smirnov-Orlicz class* $E_M(G,\omega)$ as

$$E_M(G,\omega) := \{f \in E_1(G) : f \in L_M(\Gamma,\omega)\}.$$

With every weight function ω on Γ, we associate another weight ω_0 on T defined by

$$\omega_0(t) := \omega(\psi(t)), t \in T.$$

For $f \in L_M(\Gamma, \omega)$ we define the function

$$f_0(t) := f(\psi(t)), t \in T.$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \leq t\}, t \geq 0.$$

The curve Γ is called *Dini-smooth* if it has a parameterization

$$\Gamma : \varphi_0(s), \ 0 \leq s \leq 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(t, \varphi_0')}{t} dt < \infty$$

and $\varphi_0'(s) \neq 0$ [41, p. 48].

If Γ Dini-smooth curve, then there exist [46] the constants c_3 and c_4 such that

$$0 \leq c_3 \leq |\psi'(t)| \leq c_4 < \infty, \ |t| > 1. \tag{1.3}$$

Note that if Γ is a Dini-smooth curve, then by (1.3) we have $f_0 \in L_M(\Gamma, \omega_0)$ and $f \in L_M(\Gamma, \omega)$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'}$ and let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p -condition on Γ if

$$\sup_{z\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int_{\Gamma\cap D(z,r)} |\omega(\tau)|^p |d\tau|\right)^{1/p}\left(\frac{1}{r}\int_{\Gamma\cap D(z,r)} [\omega(\tau)]^{-p'} |d\tau|\right)^{1/p'} < \infty,$$

where $D(z, r)$ is an open disk with radius r and centered z.

Let us denote by $A_p(\Gamma)$ the set of all weight functions satisfying Muckenhoupt's A_p -condition on Γ. For a detailed discussion of Muckenhoupt weights on curves, see, e.g. [7].

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(s)ds}{s - z}, \ z \in G$$

is analytic in G. Note that if $0 < \alpha_M \leq \beta_M < 1$, $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$ and $f \in L_M(\Gamma, \omega)$, then by Lemma1 in [25] $f^+ \in E_M(G, \omega)$.

Let $\varphi_k(z)$, $k = 0, 1, 2, ...$ be the Faber polynomials for G. The Faber polynomials $\varphi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(t)}{\psi(t) - z} = \sum_{k=0}^{\infty} \frac{\varphi_k(z)}{t^{k+1}}, \ z \in G, \ t \in D^- \tag{1.4}$$

and the equalities

$$\varphi_k(z) = \frac{1}{2\pi i}\int_T \frac{t^k \psi'(t)}{\psi(t) - z} dt \ \ z \in G, \tag{1.5}$$

$$\varphi_k(z) = \varphi^k(z) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi^k(s)}{s - z} ds, \ z \in G^- \tag{1.6}$$

hold [45, p.33-48].

Let $f \in E_M(G,\omega)$. Since $f \in E_1(G)$, we have

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\int\limits_{T} \frac{f(\psi(t))\,\psi'(t)}{\psi(t)-z}dt,$$

for every $z \in G$. Considering this formula and expansion (1.4), we can associate with f the formal series

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\,\varphi_k(z), \tag{1.7}$$

where

$$a_k(f) := \frac{1}{2\pi i}\int\limits_{T} \frac{f(\psi(t))}{t^{k+1}}dt.$$

This series is called the *Faber series* expansion of f, and the coefficients $a_k(f)$ are said to be the *Faber coefficients* of f.

The *n-th partial sums and Fejér sums* of the series (1.7) are defined, respectively, as

$$S_n(z,f) = \sum_{k=0}^{n} a_k(f)\,\varphi_k(z),$$

$$\sigma_n(z,f) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(z,f).$$

Let Γ be a Dini-smooth curve. Using the nontangential boundary values of f_0^+ on T we define the *$r-th$ modulus of smoothness* of $f \in L_M(\Gamma,\omega)$ as

$$\Omega^k_{\Gamma,M,\omega}(\delta,f) := \Omega^k_{M,\omega_0}\left(\delta,f_0^+\right),\ \delta > 0,$$

for $k = 1,2,3,...$

The following theorem holds.

1.3. Theorem. *Let Γ be a Dini-smooth curve, $L_M(\Gamma,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \le \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$. Then for $f \in E_M(G,\omega)$ the inequality*

$$\|f - \sigma_{n-1}(.,f)\|_{L_M(\Gamma,\omega)} \le \frac{c_5}{n}\sum_{m=1}^{n} \Omega^k_{\Gamma,M,\omega}\left(\frac{1}{m+1},f\right)$$

holds with a constant $c_5 > 0$ independent of k.

Let $P :=$ {all polynomials (with no restriction on the degree)}, and let $P(D)$ be the set of traces of members of P on D. We define the operator

$$T : P(D) \to E_M(G,\omega)$$

as

$$T(P)(z) := \frac{1}{2\pi i}\int\limits_{T} \frac{P(w)\,\psi'(w)}{\psi(w)-z}dw,\quad z \in G.$$

Then using (1.5) and (1.6) we get

$$T\left(\sum_{k=0}^{n} a_k w^k\right) = \sum_{k=0}^{n} a_k(f)\,\varphi_k(z),\quad z \in G.$$

The following theorems hold for the linear operator T [25].

1.4. Theorem. *Let Γ be a Dini-smooth curve and $L_M(\Gamma)$ be a reflexive Orlicz space. If $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, then the linear operator $T : P(D) \to E_M(G,\omega)$ is bounded.*

1.5. Theorem. *If Γ is a Dini-smooth curve, $0 < \alpha_M \leq \beta_M < 1$ and $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, then the operator*

$$T : E_M(D, \omega_0) \to E_M(G, \omega)$$

is one-to-one and onto.

The following theorem was proved in [20, Theorem 2].

1.6. Theorem. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then for every $f \in L_M(T,\omega)$ the estimate*

$$E_n(f)_{M,\omega} \leq c_6 \Omega^k_{M,\omega}\left(\frac{1}{n+1}, f\right), \quad k = 1, 2, ...$$

holds with a constant $c_6 > 0$ independent of n.

2. Proofs of the Main Results

Proof of Theorem 1.1 We can write the following equality:

$$\sigma_{n-1}(f) = \frac{1}{n}\sum_{m=0}^{n-1} S_m(f) = \frac{1}{n}\left\{S_0(f) + \sum_{i=1}^{j-1}\sum_{m=2^{i-1}}^{2^i-1} S_m(f) + \sum_{m=2^{j-1}}^{n-1} S_m(f)\right\}. \tag{1.8}$$

Using (1.8) we get

$$f - \sigma_{n-1}(f) = \frac{1}{n}\left\{(f - S_0(f)) + \sum_{i=1}^{j-1}\sum_{m=2^{i-1}}^{2^i-1}(f - S_m(f)) + \sum_{m=2^{j-1}}^{n-1}(f - S_m(f))\right\} \tag{1.9}$$

By using ineguality

$$\|f - S_n(., f)\|_{L_M(T,\omega)} \leq c_7 E_n(f)_{M,\omega}$$

given [20] and (1.9) we obtain

$$\left.\begin{aligned} &\|f - \sigma_{n-1}(f)\|_{L_M(T,\omega)} \leq \\ &\leq \frac{c_8}{n}\left[E_1(f)_{M,\omega} + \sum_{i=1}^{j-1}\left(2^i + 2^{i-1} - 1\right)E_{2^{i-1}}(f)_{M,\omega} + \left(2n - 2^{j-1} - 1\right)E_{n-2^{j-1}}(f)_{M,\omega}\right] \\ &\leq \frac{c_9}{n}\left[E_1(f)_{M,\omega} + E_1(f)_{M,\omega} + \sum_{i=2}^{j-1} 2^{i-1}E_{2^{i-1}}(f)_{M,\omega} + \left(2n - 2^{j-1}\right)E_{n-2^{j-1}}(f)_{M,\omega}\right]. \end{aligned}\right\} \tag{1.10}$$

By [20] the following inequality holds:

$$2^{i-1}E_{2^{i-1}}(f)_{M,\omega} \leq 2\sum_{m=2^{i-2}+1}^{2^{i-1}} E_m(f)_{M,\omega}. \tag{1.11}$$

Selecting j such that $2^j \leq n < 2^{j+1}$, from (1.11) we get

$$\left(2n - 2^{j-1}\right)E_{n-2^{j-1}}(f)_{M,\omega} \leq \frac{2n - 2^{j-1}}{n - 2^{j-1} - 2^{j-2}}\sum_{m=2^{i-2}+1}^{n-2^{j-1}} E_m(f)_{M,\omega} \tag{1.12}$$

$$= \left(2 + \frac{2^j}{n - 2^{j-1} - 2^{j-2}}\right)\sum_{m=2^{i-2}+1}^{n-2^{j-1}} E_m(f)_{M,\omega} \leq c_{10}\sum_{m=2^{j-2}+1}^{n} E_m(f)_{M,\omega}$$

By (1.10), (1.11) and (1.12) we obtain

$$\|f-\sigma_{n-1}(f)\|_{L_M(T,\omega)} \le \frac{c_{11}}{n}\left\{E_1(f)_{M,\omega}+\sum_{i=2}^{j-1}\sum_{m=2^{i-2}+1}^{2^i-1}E_m(f)_{M,\omega}+\right. \tag{1.13}$$
$$\left.\sum_{m=2^{j-2}+1}^{n}E_m(f)_{M,\omega}\right\}\le\frac{c_{12}}{n}\sum_{m=1}^{n}E_m(f)_{M,\omega},$$

which completes the proof of Theorem 1.1.

Proof of Corollary 1.2. By Theorem 1.6 the following inequality holds

$$E_n(f)_{M,\omega}\le c_{13}\Omega^k_{M,\omega}\left(\frac{1}{n+1},f\right)\quad k=1,2,... \tag{1.14}$$

Then using (1.13) and (1.14) we obtain inequality (1.2).

Proof of Theorem 1.3. Let $f\in E_M(G,\omega)$. Then by Theorem 1.5 the operator $T:E_M(D,\omega_0)\to E_M(G,\omega)$ is bounded, one-to-one and onto and $T\left(f_0^+\right)=f$. The function f has the following Faber series

$$f(z)\sim\sum_{k=0}^{\infty}a_k(f)\varphi_k(z).$$

Since $\omega_0\in A_{1/\alpha_M}(T)\cap A_{1/\beta_M}(T)$, by Lemma 1 in [25, p. 760] we have $f_0^+\in E_M(D,\omega_0)$. For the function f_0^+ the following Taylor expansion holds:

$$f_0^+(w)=\sum_{k=0}^{\infty}a_k(f)w^k.$$

It is known that $f_0^+\in E_1(D)$ and boundary function $f_0^+\in L_M(T,\omega_0)$. Then using [16, Th. 3.4] for the function f_0^+ we have Fourier expansion

$$f_0^+(w)\sim\sum_{k=0}^{\infty}a_k(f)w^{ikt}.$$

Using the boundedness of the operator T Theorem 1.1 and Corollary 1.2 we get

$$\|f-\sigma_{n-1}(\cdot,f)\|_{L_M(\Gamma,\omega)}=\left\|T\left(f_0^+\right)-T\left(\sigma_{n-1}\left(\cdot,f_0^+\right)\right)\right\|_{L_M(\Gamma,\omega)}\le$$
$$\le c_{14}\left\|f_0^+-\sigma_{n-1}\left(\cdot,f_0^+\right)\right\|_{L_M(T,\omega_0)}\le\frac{c_{15}}{n}\sum_{m=1}^{n}E_m\left(f_0^+\right)_{M,\omega}\le$$
$$\le\frac{c_{16}}{n}\sum_{m=1}^{n}\Omega^k_{M,\omega_0}\left(\frac{1}{m+1},f_0^+\right)=\frac{c_{17}}{n}\sum_{m=1}^{n}\Omega^k_{\Gamma,M,\omega}\left(\frac{1}{m+1},f\right).$$

The proof of Theorem 1.3 is completed.

Acknowledgements

The author wishes to express deep gratitude to the referee for valuable suggestions.

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