

OPERATOR VALUED DIRICHLET PROBLEM IN THE PLANE

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Received 26:04:2012 : Accepted 21:06:2012

Abstract

We consider an operator valued Dirichlet problem for harmonic mappings and prove the existence of a Perron-like solution. To formulate the Perron's construction we make use of Olson's notion of spectral order. We introduce a class of operator valued subharmonic mappings and establish some of their elementary properties.

Keywords: Operator theory, Harmonic mappings, Perron method, Spectral order.

2000 AMS Classification: 47A56, 47A63, 47B15

1. Introduction

With this paper we would like to initiate a research on potential theory of harmonic and subharmonic functions in the plane with values in the class of bounded linear operators on a Hilbert space. The main purpose is to describe the solution of the Dirichlet problem using a Perron's method. Making use of the Olson's notion of spectral order we show that there is a Perron-like solution whenever the boundary values are commuting or the Poisson integral of the boundary mapping is a projection (Theorem 4.10).

We recall that inequality in spectral order implies the inequality in the usual order on the class of self-adjoint operators. By Olson \mathcal{S} is a conditionally complete lattice under the partial order \preceq . Making use of functional calculus and spectral order we obtain some generalizations of well known properties of subharmonic functions to the operator valued case. One of the main results of the paper is the maximum principle for subharmonic mappings in terms of the spectral order.

Recently quite a few papers are written about harmonic mappings which take their values in infinite dimensional spaces. We consider in this paper a Dirichlet problem for operator valued harmonic mappings of the complex plane. To formulate the Perron solution we introduce a class of operator valued subharmonic mappings and establish some of their elementary properties which will be used for our purpose. In section 4 we provide an example of a mapping of the form $F(z) = T + zS$, where T and S are

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Hermitian 2×2 matrices such that $F(z)F^*(z)$ fails to be subharmonic. On the contrary an example of subharmonic mappings is given by the following result.

Theorem. The mapping $\text{Log}(F(z)F^*(z))$ is harmonic and $F(z)F^*(z)$ is subharmonic whenever the mapping $F : G \rightarrow \mathcal{A}$ is holomorphic, $F(z)$ is normal, and the spectrum of $F(z)$ does not contain any element from the set

$$\{x + iy \in \mathbb{C} : y = 0, x \leq 0\}.$$

The type of the Dirichlet problem is determined according the WOT, SOT or norm convergence of the solution on the boundary of the domain. It turns out that for a general domain G in the complex plane there is always a SOT continuous up to the boundary of harmonic mapping on G for a given SOT continuous function on the boundary provided that the domain G is regular for the Dirichlet problem for real valued harmonic functions. The theory of operator-valued holomorphic mappings has found many applications in functional analysis. These studies provide a better understanding and a way of formulation of different phenomena about vector-valued function spaces. The ideas and methods of this theory is used in our time not only in mathematical physics, but also in function theory, functional analysis, probability theory, approximation theory and harmonic analysis.

We start with investigating the class of harmonic mappings from an open subset of the complex plane which take values in a von Neumann algebra. Most of the classical results are carried in a natural way to the setting of operator-valued harmonic mappings. We gather several well known information on harmonic mappings in Section 3: A mapping is harmonic if and only if it is weakly harmonic for example. This result has several useful applications. We show that for self-adjoint harmonic mappings the norm is always a subharmonic function and for complex combinations of harmonic mappings the square of its numerical radius is also subharmonic. We prove the main results in section 4.

2. Preliminaries

Basic notation. \mathcal{A} will denote a von Neumann algebra. There exists a Hilbert space \mathcal{H} so that \mathcal{A} is a sub-algebra of $B(\mathcal{H})$ and we will make use of this Hilbert space related to \mathcal{A} throughout. The class of self-adjoint elements in \mathcal{A} is denoted by \mathcal{S} . More generally, we will denote by \mathcal{A} a C^* -algebra. We will denote by \mathcal{S} or $\mathcal{S}_{\mathcal{A}}$ the class of self-adjoint elements in \mathcal{A} . For arbitrary elements $a, b \in B(\mathcal{H})$ we set $[a, b] := ab - ba$. We will denote by $\mathbb{D}(w, r)$ the open disk in the complex plane with center w and radius r and by $\mathbb{T}(w, r)$ its boundary. We denote the open unit disk and unit circle in \mathbb{C} by \mathbb{D} and \mathbb{T} respectively. Let \mathbb{C}_{∞} denote the Riemann sphere.

Suppose \mathcal{A} is a sub-algebra of $B(\mathcal{H})$ for some Hilbert space \mathcal{H} . If $\psi : X \rightarrow \mathcal{A}$ is a mapping from a set X , and $h \in \mathcal{H}$, $k \in \mathcal{H}$, we denote by ψ_h , respectively by $\psi_{h,k}$ the complex-valued function on X defined by

$$\psi_h(x) = \langle \psi(x)h, h \rangle$$

and respectively,

$$\psi_{h,k}(x) = \langle \psi(x)h, k \rangle$$

for any $x \in X$.

Order. Let \mathcal{A} be a von Neumann algebra of operators on a Hilbert space \mathcal{H} . Let \mathcal{S} be the real vector space of self-adjoint elements of \mathcal{A} . For two operators a and b in \mathcal{S} , $a \leq b$ means $\langle ah, h \rangle \leq \langle bh, h \rangle$ for all $h \in \mathcal{H}$. With this order, \mathcal{S} is a partially ordered vector space but not a vector lattice.

Another order, so called the spectral order on \mathcal{S} is defined by Olson in [8]. Let a and b be self-adjoint elements of the von Neumann algebra \mathcal{A} . Let E_a and E_b be the resolutions of the identity of a and b , respectively. We write $a \preceq b$ if

$$E_a(t) \geq E_b(t), \quad t \in \mathbb{R}.$$

The order " \preceq " is called spectral order. It was proved in [8] that \mathcal{S} is a conditionally complete lattice under the partial order \preceq . Some of the basic properties of the spectral order which will be used frequently in the text are collected in the next proposition.

2.1. Proposition. *Let $a, b \in \mathcal{S}$.*

- i. $a \succeq 0$ if and only if $a \geq 0$. The two orders, therefore, have the same positive cones.
- ii. If $a \preceq b$, then $a \leq b$.
- iii. If a and b commute, then $a \preceq b$ if and only if $a \leq b$.
- iv. Suppose a and b are also bounded positive. Then $a \preceq b$ if and only if $a^k \leq b^k$ for $k = 1, 2, \dots$
- v. $a \preceq b$ if and only if $-a \preceq -b$.
- vi. Let $a_j, b_j \in \mathcal{S}$ so that $kI \leq a_j \preceq b_j, j \geq 1$, for some $k \in \mathbb{R}$, and

$$\text{strong } \lim_j a_j = a, \quad \text{strong } \lim_j b_j = b.$$

Then $a \preceq b$.

- vii. Let $a_1, \dots, a_k \in \mathcal{S}$ be nonnegative elements. Then

$$\text{strong } \lim_{r \rightarrow \infty} (a_1^r + \dots + a_k^r)^{1/r} = \sup_{1 \leq j \leq k} a_j$$

and the limit is increasing in usual order. Likewise it follows that

$$\text{strong } \lim_{r \rightarrow \infty} (a_1^{-r} + \dots + a_k^{-r})^{-1/r} = \inf_{1 \leq j \leq k} a_j$$

and the limit is decreasing in usual order.

- viii. Let $S \geq 0$ and T be a projection. Then $S \preceq T$ if and only if $S \leq T$.

Proof. The first four properties are in [8]. Property v. although appears in several papers we could find a proof in [9]. To prove part vi. we may assume without loss of generality that $a_j \geq 0$ and $b_j \geq 0$ since for any two operators a and b it is true that $a \preceq b$ if and only if $a - kI \preceq b - kI$. The condition $a_j \preceq b_j$ implies $a_j^k \leq b_j^k$ for every $k \geq 1$ by part iv. Hence $a^k \leq b^k$ for every $k \geq 1$ and the result follows again using part iv. Property vii. above is proved for positive bounded operators in [8]. It is proved in full generality in [6]. If $0 \leq S \leq I$, then $S^k \leq S$ for every $k \geq 1$. Property viii. then follows from iv. \square

The following version of *Jensen's inequality* is proved in [2].

2.2. Theorem. *Let \mathcal{A} be a C^* -algebra and \mathcal{B} be a von Neumann algebra. Let $a = a^* \in \mathcal{A}$, f be a monotone convex real-valued function on an open interval containing the spectrum of a , and φ be a unital positive map from \mathcal{A} to \mathcal{B} . Then $f(\varphi(a)) \preceq \varphi(f(a))$.*

A real-valued function defined on an interval O of \mathbb{R} is said to be *spectral order preserving* on O if $a \preceq b$ implies that $f(a) \preceq f(b)$ for every $a, b \in \mathcal{S}$ whenever the spectra of a and b are contained in O . The next result helps us to generate spectral order preserving functions.

2.3. Proposition. *a. Let $f : O \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$ be spectral order preserving functions on intervals O and J , respectively, so that $g(J) \subset O$. Then $f \circ g$ is spectral order preserving on J .*

b. Let $g(t)$ be a non-decreasing continuous function on an open interval J so that $g(J) \subset (0, \infty)$, and let $p \geq 0$ be a number. Then $g^p(t)$ is spectral order preserving on J .

Proof. Part a. is rather apparent. Let $g(t)$ be a function as in b., and let $p \geq 0$ be a number. Then $g^{kp} > 0$ and non-decreasing on J for every integer $k \geq 1$. To prove the second part take operators $a \preceq b$ with their spectra lying in J . Then $0 \leq g^{kp}(a) \leq g^{kp}(b)$ for every k by Proposition 2.1. Again by Proposition 2.1, this implies that $g^p(a) \preceq g^p(b)$. \square

We will borrow a result from [1] which will be used later in the next sections. Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras, $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ a faithful conditional expectation (that is, a projection of norm 1 so that $\Phi(a) > 0$ whenever $a > 0$), and $a \in \mathcal{M}$ a positive element. We let

$$a_+ := \inf\{b \in \mathcal{S}_{\mathcal{N}} : a \preceq b\},$$

the spectral order majorant of a in \mathcal{N} .

2.4. Theorem. [1, Theorem 9] *The sequence $[\Phi(a^n)]^{1/n}$ converges in SOT to a_+ .*

For last we will state a result from [10] for commuting n -tuples of operators.

2.5. Theorem. [10] *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function which is increasing in each variable. Let $a_k, b_k \in \mathcal{S}$, $k = 1, \dots, n$, be operators such that $a_k \preceq b_k$, and $[a_j, a_k] = [b_j, b_k] = 0$ for every $j, k = 1, \dots, n$. Then*

$$F(a_1, \dots, a_n) \preceq F(b_1, \dots, b_n).$$

Taking $F(t) = t_1 + \dots + t_n$, $t \in \mathbb{R}^n$ in Theorem 2.5 we obtain the following result.

2.6. Corollary. *Let $a_k, b_k \in \mathcal{S}$, $k = 1, \dots, n$, be operators such that $a_k \preceq b_k$, and $[a_j, a_k] = [b_j, b_k] = 0$ for every $j, k = 1, \dots, n$. Then*

$$a_1 + \dots + a_n \preceq b_1 + \dots + b_n.$$

3. Harmonic mappings on the plane

In this section we will introduce the harmonic mappings on the complex plane. We will present the Dirichlet problem for operator valued harmonic mappings on the plane. The main theorem of this section is that if a domain is regular for the classical real valued Dirichlet problem, then it is regular for the operator valued Dirichlet problem.

First we shall gather the well-known properties of harmonic mappings. Let G be an open set in \mathbb{C} . The target space of the harmonic mappings which will be defined in this section is always the class of self-adjoint elements \mathcal{S} of a von Neumann algebra \mathcal{A} .

3.1. Definition. A mapping $u : G \rightarrow \mathcal{S}$ is called harmonic if $\Delta u = 0$ in G .

We denote the class of harmonic mappings by $\text{Har}[G, \mathcal{S}]$. Let us denote by $\text{Har}[G, \mathcal{A}]$ the complex linear combination of all self-adjoint harmonic mappings. There are natural examples of harmonic mappings. Let $F : G \rightarrow \mathcal{A}$ be a holomorphic mapping, then $\text{Re } F$ and $\text{Im } F$ belong to $\text{Har}[G, \mathcal{S}]$. Note that the class of all holomorphic mappings from G to \mathcal{A} belongs to $\text{Har}[G, \mathcal{A}]$.

σ -Dirichlet problem on the disk. We denote by σ the weak operator topology (WOT), the strong operator topology (SOT) or the norm topology on the class \mathcal{S} of selfadjoint elements in \mathcal{A} . Let G be a domain in \mathbb{C} . Given a function $\varphi : \partial G \rightarrow \mathcal{S}$ continuous in σ find a harmonic mapping $u \in \text{Har}[G, \mathcal{S}]$ so that

$$\lim_{z \rightarrow \zeta} u(z) = \varphi(\zeta)$$

in σ for every $\zeta \in \partial G$. This problem is the generalization of the classical Dirichlet problem to the operator-valued potential theory. A domain G is said to be regular for

the σ -Dirichlet problem or simply σ -regular if for every σ -continuous \mathcal{S} -valued function φ from ∂G the σ -Dirichlet problem has a solution. Although the statements and proofs in this section are quite standard, we could not find a good reference for the material presented here. We refer to [11] for the proof in the classical case.

3.2. Theorem. *a. There is at most one solution to the Dirichlet problem when G is a bounded domain in \mathbb{C} .*

b. The Poisson transform $P[\varphi, \mathbb{D}]$ is harmonic on \mathbb{D} for any mapping $\varphi \in L^1[\mathbb{T}, \mathcal{S}]$.

c. If φ is continuous at a point $\zeta_0 \in \mathbb{T}$, then $\lim_{z \rightarrow \zeta_0} P[\varphi, \mathbb{D}](z) = \varphi(\zeta_0)$ in norm.

For general domains the solution of the Dirichlet problem is again quite similar to the real valued case. Let G be a domain in \mathbb{C}_∞ such that ∂G is non-polar. It is well-known that there is a unique harmonic measure ω_G for G (see [11, Theorem 4.3.2]). Let $\varphi : \partial G \rightarrow \mathcal{S}$ be a bounded mapping so that the complex-valued function $\varphi_{h,k}$ is Borel for every $h, k \in \mathcal{H}$. We set

$$P[\varphi, G](z) := \int_{\partial G} \varphi(\zeta) d\omega_G(z, \zeta), \quad z \in G.$$

Note that

$$\langle P[\varphi, G](z)h, h \rangle = P[\varphi_h, G](z)$$

and the real-valued function $P[\varphi_h, G](z)$ is harmonic in G for every $h \in \mathcal{H}$. It turns out that the mapping $P[\varphi, G](z)$ is harmonic. The big part of the next theorem is obtained in [3] as a consequence of Theorem 17 there.

3.3. Theorem. *Let G be an open set in \mathbb{C} . Let $u : G \rightarrow \mathcal{S}$ be a mapping which is locally integrable on G . The following statements are equivalent:*

- (1) *u is harmonic on G .*
- (2) *$u(z) = P[u, \Delta](z)$ for any disk Δ compactly belonging to G and for every $z \in \Delta$.*
- (3) *$u(z) = P[u, \Omega](z)$ for any open set Ω compactly belonging to G and for every $z \in \Omega$.*
- (4) *The complex valued function $u_{h,k}$ is harmonic for every $h, k \in \mathcal{H}$.*
- (5) *The real valued function u_h is harmonic on G for every $h \in \mathcal{H}$.*
- (6) *For every open disk Δ in G , u is the real part of a holomorphic function from Δ to \mathcal{A} on Δ .*
- (7) *$\psi \circ u : G \rightarrow \mathbb{R}$ is harmonic for every continuous functional $\psi : \mathcal{S} \rightarrow \mathbb{R}$.*
- (8) *If Δ is an open disk which is relatively compact in G , then $u = \mathbf{Re} F$ for some holomorphic mapping $F : \Delta \rightarrow \mathcal{S}$.*

Thus all the natural topologies, norm topology, SOT and the WOT in $B(\mathcal{H})$ give rise to the same class of harmonic mappings. We state a corollary of Theorem 3.3 (see also [3] and [5]).

3.4. Corollary. *If $(u_n)_{n \geq 1}$ is a sequence of harmonic mappings on an open set $G \subset \mathbb{C}$ that converge locally uniformly in WOT to a mapping u , then u is harmonic on G .*

Theorem 3.3 has several other consequences. The next one is trivial.

3.5. Corollary. *Suppose G is a bounded domain in \mathbb{C} . Let $u, v \in \text{Har}[G, \mathcal{S}]$. If $u \leq v$ on ∂G , then $u \leq v$ on G .*

Using Theorem 2.2 we prove the integral version of Jensen's inequality.

3.6. Corollary. *Let G be an open set in \mathbb{C} , \mathcal{A} and \mathcal{C} be C^* -algebras so that $\mathcal{A} \subset L^1[\partial G, \mathcal{S}_{\mathcal{C}}]$, and $u \in \mathcal{A}$. Let f be a monotone convex real-valued function on an open interval containing the spectrum of $u(z)$ for every $z \in \partial G$. Then*

$$f(P[u, G])(z) \preceq P[f(u), G](z)$$

for every $z \in G$.

Proof. Take a point $z \in G$. Let $\mathcal{B} = \mathbb{C}$, and $\varphi(v) = P[v, G](z)$ for any $v \in \mathcal{A}$. Clearly φ is a unital positive map from \mathcal{A} to \mathcal{B} . Hence Theorem 2.2 can be applied. \square

The next result of this section shows that the norm of a harmonic mapping is subharmonic.

3.7. Theorem. *Let G be an open subset of \mathbb{C} , and let $u \in \text{Har}[G, \mathcal{S}]$. Then $\|u\|$ belongs to $\text{SH}[G]$.*

Proof. We have

$$\|u(z)\| = \sup\{|\psi \circ u(z)| : \psi \in \mathcal{S}^*, \|\psi\| = 1\}$$

for every $z \in G$. Since $\psi \circ u \in \text{Har}[G]$ by Theorem 3.3 the conclusion follows by noting that norm is continuous. \square

Recall that for any operator $T \in B(\mathcal{H})$ we define the numerical range $W[T]$ of T as the set

$$W[T] := \{\langle Tx, x \rangle : x \in \mathcal{H}\}.$$

The numerical radius $w[T]$ of T is defined as the number

$$w[T] := \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

If T is self-adjoint, then the numerical radius $w(T)$ coincides with the spectral radius $\rho(T)$ which is the same as the norm $\|T\|$ of T . We prove in the next result that the square of the numerical radius of a harmonic mapping is subharmonic.

3.8. Theorem. *Let $u \in \text{Har}[G, B(\mathcal{H})]$. Then the function $v(z)$ defined by*

$$v(z) := (w[u(z)])^2, \quad z \in G$$

is continuous subharmonic on G .

Proof. It is sufficient to note that we have

$$v(z) = \sup_{\|x\|=1} |\langle u(z)x, x \rangle|^2,$$

the functions $|\langle u(z)x, x \rangle|^2$ are subharmonic on G for every $x \in \mathcal{H}$ and that $v(z)$ is continuous. This proves that v is subharmonic on G . \square

4. Subharmonic mappings and Perron method

In the previous section we gathered some necessary information on harmonic mappings. In this section we describe the solution of the operator valued Dirichlet problem using a Perron-like method. We introduce a notion of subharmonic mappings to describe Perron's construction. The following example is important. There by $\text{Log } z$ we denote the principal branch of the complex logarithm.

4.1. Theorem. *The mapping*

$$\text{Log}(F(z)F^*(z))$$

belongs to $\text{Har}[G, \mathcal{S}]$ whenever $F : G \rightarrow \mathcal{A}$ is holomorphic, $F(z)$ is normal, and the spectrum of $F(z)$ does not contain any element from the set

$$\{x + iy \in \mathbb{C} : y = 0, x \leq 0\}$$

for every $z \in G$.

Proof. By functional calculus for normal operators $\text{Log } F^*(z) = (\text{Log } F(z))^*$ and $\text{Log } F(z)F^*(z) = \text{Log } F(z) + \text{Log } F^*(z)$ for every $z \in G$. Since the mapping $\text{Log } F(z)$ is holomorphic, we see that the mapping $\text{Log } F(z)F^*(z)$ is harmonic in G . \square

We recall that for any $\phi : \partial G \rightarrow \mathbb{R}$ the associated Perron function $H_G\phi : G \rightarrow \mathbb{R}$ is defined by

$$H_G\phi(z) := \sup_{u \in \mathcal{U}[\phi]} u(z),$$

where $\mathcal{U}[\phi]$ is the class of all subharmonic functions u on G so that $\limsup_{z \rightarrow \zeta} u(z) \leq \phi(\zeta)$ for every $\zeta \in \partial G$. It is clear that when there is a harmonic solution of the Dirichlet problem for ϕ in G , then $H_G\phi(z)$ is the solution. An application of the classical potential theory gives the following result.

4.2. Theorem. *Let G be a domain in \mathbb{C}_∞ such that ∂G is non-polar. Let $\varphi : \partial G \rightarrow \mathcal{S}$ be a bounded mapping so that the complex-valued function $\varphi_{h,k}$ is Borel for every $h, k \in \mathcal{H}$. The following statements hold:*

- a. *For every $h \in \mathcal{H}$ the function $H_G\varphi_h$ coincides with the function $P[\varphi_h, G]$ on G .*
- b. *If for every $h, k \in \mathcal{H}$ the function $\varphi_{h,k}$ is continuous at nearly every point in ∂G , then*

$$\text{SOT} - \lim_{z \in G, z \rightarrow \zeta} P[\varphi, G](z) = \varphi(\zeta)$$

for nearly every point $\zeta \in \partial G$.

- c. *Let ζ_0 be a regular boundary point of G . If φ is SOT-continuous at ζ_0 , then*

$$\text{SOT} - \lim_{z \in G, z \rightarrow \zeta_0} P[\varphi, G](z) = \varphi(\zeta_0).$$

- d. *If G is a regular domain and $\varphi : \partial G \rightarrow \mathcal{S}$ is SOT-continuous, then $P[\varphi, G](z)$ is the unique harmonic mapping on G such that*

$$\text{SOT} - \lim_{z \in G, z \rightarrow \zeta} P[\varphi, G](z) = \varphi(\zeta)$$

for all $\zeta \in \partial G$.

Proof. The proofs follow immediately from [11] Theorem 4.3.3 for part a., Corollary 4.2.6 for part b., Theorem 4.1.5 for part c., and from Corollary 4.1.8 for part d. \square

Theorem 4.2 shows that if a domain in the complex plane is regular with respect to the classical Dirichlet problem, then it is regular for the operator-valued Dirichlet problem. We will now define a class of subharmonic mappings. There is no exactly one way of a description of subharmonic mappings in the operator theoretic setup. Various classes of subharmonic mappings and their relations within them will be studied in another project. So let us give the definition of a subharmonic mapping that fits to our purpose.

4.3. Definition. Let $u : G \rightarrow \mathcal{S}$ be a SOT-continuous function. We say that u is subharmonic in G if

(SH) for every open domain Ω compactly belonging to G we have the inequality

$$u(z) \preceq P[u, \Omega](z)$$

for every $z \in \Omega$.

We say that u is superharmonic in G if $-u$ is subharmonic in G .

We will denote the class of subharmonic and superharmonic mappings in G by $\text{SH}[G, \mathcal{S}, \preceq]$ and $\text{SPH}[G, \mathcal{S}, \preceq]$, respectively. In view of Theorem 3.3 the class $\text{Har}[G, \mathcal{S}] \subset \text{SH}[G, \mathcal{S}, \preceq]$. The following observation follows immediately from Proposition 2.1 part v.

4.4. Proposition. *Let $u : G \rightarrow \mathcal{S}$ be continuous. Then $u \in \text{Har}[G, \mathcal{S}]$ if and only if both u and $-u$ belong to $\text{SH}[G, \mathcal{S}, \preceq]$.*

A subharmonic real-valued function composed with a convex function is again subharmonic. We have similar results for operator-valued mappings.

4.5. Theorem. *Let \mathcal{A} be a von Neumann algebra. Let $u \in \text{SH}[G, \mathcal{S}, \preceq]$ and f be a monotone convex spectral order preserving real-valued function on an open interval containing the spectrum of every $u(z)$, $z \in G$. Then $f(u) \in \text{SH}[G, \mathcal{S}, \preceq]$.*

Proof. Let Δ be a disk compactly belonging to G . If $u \in \text{SH}[G, \mathcal{S}, \preceq]$ and f is spectral order preserving, then $f(u(z)) \preceq f(P[u, \Delta](z))$ for every $z \in G$. By Theorem 2.2 $f(P[u, \Delta](z)) \preceq P[f(u), \Delta](z)$. Then $f(u(z)) \preceq P[f(u), \Delta](z)$. Hence $f(u) \in \text{SH}[G, \mathcal{S}, \preceq]$. \square

As an application of these observations we present the following corollary.

4.6. Corollary. *Let \mathcal{A} be a von Neumann algebra and let $u \in \text{SH}[G, \mathcal{S}, \preceq]$. Then*

- a. $e^u \in \text{SH}[G, \mathcal{S}, \preceq]$;
- b. if $u(z) \geq 0$ for all $z \in G$, then $u^p \in \text{SH}[G, \mathcal{S}, \preceq]$ for every number $p \geq 1$.

Proof. The functions t^p and e^t are spectral order preserving on the positive real line by Proposition 2.3. Therefore if $u(z) \geq 0$ for all $z \in G$, then Theorem 4.5 can be applied and hence a. and b. hold. For part a. suppose that u is an arbitrary mapping in $\text{SH}[G, \mathcal{S}, \preceq]$. Take a disk Δ compactly belonging to G and let $m = 2 \max\{\|u(z)\| : z \in \overline{\Delta}\}$ so that $u + mI > 0$ on $\overline{\Delta}$. Observe that $e^u \in \text{SH}[\Delta, \mathcal{S}, \preceq]$ if and only if $e^{u+mI} \in \text{SH}[\Delta, \mathcal{S}, \preceq]$. Then use the first argument to conclude the proof. \square

4.7. Corollary. *Let $F : G \rightarrow \mathcal{A}$ be holomorphic, $F(z)$ be normal, and suppose that the spectrum of $F(z)$ does not contain any element from the set*

$$\{x + iy \in \mathbb{C} : y = 0, x \leq 0\}$$

for every $z \in G$. Then the mapping $F(z)F^(z)$ belongs to $\text{SH}[G, \mathcal{S}, \preceq]$.*

The next result is the formulation of the *maximum principle* for subharmonic mappings.

4.8. Theorem. *Let G be a domain in \mathbb{C} and $u \in \text{SH}[G, \mathcal{S}, \preceq]$. For every relatively compact open subset Ω in G and for every $z \in \Omega$ we have*

$$u(z) \preceq \sup_{w \in \partial\Omega} u(w).$$

Proof. Let Ω in G be a relatively compact open subset and $z \in \Omega$. Since u is subharmonic in G we have

$$u(z) \preceq P[u, \Omega](z).$$

Hence by Corollary 3.6

$$u^n(z) \preceq (P[u, \Omega](z))^n \preceq P[u^n, \Omega](z)$$

for every integer $n \geq 1$. Note that the map $v \mapsto P[u, \Omega](z)$ is a faithful conditional expectation from $\mathcal{M} := L^\infty[\partial\Omega, \mathcal{A}]$ onto $\mathcal{N} :=$ set of constant operator valued mappings from $\partial\Omega$ to \mathcal{A} . We identify \mathcal{N} with \mathcal{A} . By Theorem 2.4 the sequence $(P[u^n, \Omega](z))^{1/n}$ converges in strong operator topology to the operator

$$u_+ = \inf\{a \in \mathcal{S} : u(\zeta) \preceq a \text{ for every } \zeta \in \partial\Omega\}.$$

We will show that u_+ equals $\sup_{w \in \partial\Omega} u(w)$. Since $u(\zeta) \preceq \sup_{w \in \partial\Omega} u(w)$ for every $\zeta \in \partial\Omega$ we have $u_+ \preceq \sup_{w \in \partial\Omega} u(w)$. On the other hand since $u(\zeta) \preceq u_+$ for every $\zeta \in \partial\Omega$ we have $\sup_{w \in \partial\Omega} u(w) \preceq u_+$. Hence $u_+ = \sup_{w \in \partial\Omega} u(w)$. To finish the proof we have

$$u(z) \preceq (P[u^n, \Omega](z))^{1/n}$$

for every $n \geq 1$. Taking limit and using vi. of Proposition 2.1 we get

$$u(z) \preceq \sup_{w \in \partial\Omega} u(w).$$

This completes the proof. \square

In the next result we prove that uniform limit of subharmonic mappings is again subharmonic. An analog result holds for superharmonic mappings.

4.9. Theorem. *Let G be a domain in \mathbb{C} and $u_n \in \text{SH}[G, \mathcal{S}, \preceq]$. If u_n converges in strong operator topology to a mapping u locally uniformly in G , then u belongs to $\text{SH}[G, \mathcal{S}, \preceq]$.*

Proof. Let Ω be a relatively open subset of G and $z \in \Omega$. Then

$$u_n(z) \preceq P[u_n, G](z)$$

for every n . By vi. of Proposition 2.1 we get that $u(z) \preceq P[u, G](z)$. Thus $u \in \text{SH}[G, \mathcal{S}, \preceq]$. \square

Perron method. To describe the solution of the Dirichlet problem in classical potential theory one uses the Perron method. It is quite useful in applications. In this section we extend the Perron method to certain operator valued settings. In these results supremum is with respect to the spectral order. The purpose of the next result is to extend this classical result to the case for which the boundary data is commutative.

4.10. Theorem. *Let G be a bounded domain in \mathbb{C} which is regular for the classical real-valued Dirichlet problem. Let φ be a norm continuous \mathcal{S} -valued function from ∂G which has a commutative range. Let $\mathcal{U}_c[\varphi]$ be the class of all mappings $u \in \text{SH}[G, \mathcal{S}, \preceq] \cap C[\overline{G}, \mathcal{S}_{\text{SOT}}]$ so that $u(\zeta) \preceq \varphi(\zeta)$ for every $\zeta \in \partial G$ and the range of $u|_{\partial G}$ is commutative. Then the mapping*

$$H_c[\varphi, G](z) := \sup \{u(z) : u \in \mathcal{U}_c[\varphi]\}, \quad z \in \overline{G}$$

is harmonic on G , SOT-continuous on \overline{G} , and $P[\varphi, G](z) = H_c\varphi(z)$ for every $z \in \overline{G}$. Moreover,

$$\lim_{z \rightarrow \zeta} P[\varphi, G](z) = \varphi(\zeta)$$

in SOT for every $\zeta \in \partial G$.

Proof. We only need to prove the equality of $H_c[\varphi, G](z)$ and $P[\varphi, G](z)$, $z \in \overline{G}$, since all the other statements about $P[\varphi, G](z)$ are already proved in Theorem 4.2. Clearly $P[\varphi, G](z) \preceq H_c[\varphi, G](z)$ for $z \in G$ and $P[\varphi, G](z) = H_c[\varphi, G](z) = \varphi(z)$ when $z \in \partial G$. To prove the reverse inequality take any $z \in G$ and $u \in \mathcal{U}_c[\varphi]$. We set $a := u|_{\partial G}$. Let ω_G be the harmonic measure for G . Given $n \geq 1$ we can find $z_1, \dots, z_n \in \partial G$ and relatively open subsets $I_1, \dots, I_n \subset \partial G$ with $z_k \in I_k$, $k = 1, \dots, n$ so that

$$\begin{aligned} \lim_n \left\| P[\varphi, G](z) - \sum_{k=1}^n \varphi(z_k) \omega_G(z, I_k) \right\| &= \\ &= \lim_n \left\| P[a, G](z) - \sum_{k=1}^n a(z_k) \omega_G(z, I_k) \right\| = 0. \end{aligned}$$

By Corollary 2.6 we have

$$\sum_{k=1}^n a(z_k) \omega_G(z, I_k) \preceq \sum_{k=1}^n \varphi(z_k) \omega_G(z, I_k)$$

for every $n \geq 1$. By vi. of Proposition 2.1 we see that

$$u(z) \preceq P[a, G](z) \preceq P[\varphi, G](z).$$

Taking supremum over all mappings $u \in \mathcal{U}_c[\varphi]$ we get that $H_c[\varphi, G](z) \preceq P[\varphi, G](z)$. Thus $H_c[\varphi, G](z) = P[\varphi, G](z)$. \square

In the next result we consider the case for which $P[a, G](z)$ is a projection.

4.11. Theorem. *Let G be a domain in \mathbb{C} which is regular for the classical real-valued Dirichlet problem. Let $z \in G$ and φ be a norm continuous \mathcal{S} -valued function from ∂G so that $P[\varphi, G](z)$ is a projection. Let $U_+[\varphi]$ be the class of all nonnegative mappings $u \in \text{SH}[G, \mathcal{S}, \preceq] \cap C[\overline{G}, \mathcal{S}_{\text{OT}}]$ so that $u(\zeta) \preceq \varphi(\zeta)$ for every $\zeta \in \partial G$. Then the operator*

$$H_+[\varphi, G](z) := \sup \{u(z) : u \in U_+[\varphi]\}$$

coincides with $P[\varphi, G](z)$.

Proof. Let $u \in U_+[\varphi]$. Then $u \leq P[\varphi, G]$ in G , in particular, $u(z) \leq P[\varphi, G](z)$. From viii. of Proposition 2.1 we have $u(z) \preceq P[\varphi, G](z)$. Taking supremum over all such $u \in U_+[\varphi]$ we get $H_+[\varphi, G](z) \preceq P[\varphi, G](z)$. The reverse inequality is already true. Hence $H_+[\varphi, G](z)$ coincides with $P[\varphi, G](z)$. \square

In the example below we construct continuous mappings a, b on $\partial \mathbb{D}$ with the following properties:

- i. $a(\zeta) \leq b(\zeta)$ for every $\zeta \in \partial \mathbb{D}$.
- ii. $P[b, \mathbb{D}](0)$ is a projection.
- iii. $P[a, \mathbb{D}](0) \not\leq P[b, \mathbb{D}](0)$.

Hence, as in this example, it may happen that $P[b, \mathbb{D}](z) \neq H[b, \mathbb{D}](z)$ for some $z \in \mathbb{D}$, where

$$H[b, \mathbb{D}](z) := \sup \{u(z) : u \in \text{SH}[\mathbb{D}, \mathcal{S}] \cap C[\overline{\mathbb{D}}, \mathcal{S}], u \leq b \text{ in } \partial \mathbb{D}\}.$$

4.12. Example. In [8, page 543] 2×2 matrices P and A were constructed so that $0 \leq P - A$, P is a projection and $A \not\leq P$. More precisely,

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix}, \quad P - A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$

Let $0 < \varepsilon < \pi$. For any 2×2 matrix X we set

$$g_X(e^{it}) := \begin{cases} X & \text{if } 0 \leq t \leq \frac{\pi - \varepsilon}{2} \text{ or } \frac{3\pi + \varepsilon}{2} \leq t \leq 2\pi, \\ \frac{2}{\varepsilon} \left[\left(\frac{\pi}{2} - t \right) X + \left(t - \frac{\pi - \varepsilon}{2} \right) I \right] & \text{if } \frac{\pi - \varepsilon}{2} \leq t \leq \frac{\pi}{2}, \\ \frac{2}{\varepsilon} \left[\left(\frac{\pi + \varepsilon}{2} - t \right) I + \left(t - \frac{\pi}{2} \right) A \right] & \text{if } \frac{\pi}{2} \leq t \leq \frac{\pi + \varepsilon}{2}, \\ A & \text{if } \frac{\pi + \varepsilon}{2} \leq t \leq \frac{3\pi - \varepsilon}{2}, \\ \frac{2}{\varepsilon} \left[\left(\frac{3\pi}{2} - t \right) A + \left(t - \frac{3\pi - \varepsilon}{2} \right) I \right] & \text{if } \frac{3\pi - \varepsilon}{2} \leq t \leq \frac{3\pi}{2}, \\ \frac{2}{\varepsilon} \left[\left(\frac{3\pi + \varepsilon}{2} - t \right) I + \left(t - \frac{3\pi}{2} \right) X \right] & \text{if } \frac{3\pi}{2} \leq t \leq \frac{3\pi + \varepsilon}{2}. \end{cases}$$

It can be checked easily that

$$\frac{1}{2\pi} \int_0^{2\pi} g_X(e^{it}) dt = \frac{2\pi - \varepsilon}{4\pi} (X + A) + \frac{\varepsilon}{2\pi} I.$$

Now we let

$$a(e^{it}) := \frac{4\pi}{2\pi - \varepsilon} g_0(e^{it}) - \frac{2\varepsilon}{2\pi - \varepsilon}, \quad \text{and} \quad b(e^{it}) := \frac{4\pi}{2\pi - \varepsilon} g_{P-A}(e^{it}) - \frac{2\varepsilon}{2\pi - \varepsilon}$$

for any $0 \leq t \leq 2\pi$. From the construction it is readily seen that $a(e^{it}) \leq b(e^{it})$ for any $0 \leq t \leq 2\pi$,

$$P[a, \mathbb{D}](0) = \frac{1}{2\pi} \int_0^{2\pi} a(e^{it}) dt = A,$$

and

$$P[b, \mathbb{D}](0) = \frac{1}{2\pi} \int_0^{2\pi} b(e^{it}) dt = P.$$

Since $A \not\leq P$ we have $P[a, \mathbb{D}](0) \not\leq P[b, \mathbb{D}](0)$. In this example $P[b, \mathbb{D}](0) \prec H[b, \mathbb{D}](0)$.

Following Example 4.12 and in contrast to Corollary 4.7 we construct a linear mapping $F(z)$ with positive self-adjoint coefficients so that the mapping $\ln F(z)F^*(z)$ is not harmonic.

4.13. Example. With the same notation as in Example 4.12 let $S := (A + 100I)^{1/2}$, and $T := (P - A)^{1/2}$. We set

$$F(z) := S + zT$$

for every $z \in \mathbb{C}$. Note that the spectrum of $F(z)$ does not contain any element from the set

$$\{z = x + iy \in \mathbb{C} : y = 0, x \leq 0\}$$

whenever $|z| \leq 2$. Let $\Delta := \mathbb{D}(0, 2)$. We claim that the mapping $u(z) := \ln F(z)F^*(z)$ is not harmonic in Δ . Suppose on the contrary that it is harmonic. By Corollary 4.6 then the mapping $F(z)F^*(z)$ belongs to $\text{SH}[\Delta, \mathcal{S}_{2 \times 2}]$, where $\mathcal{S}_{2 \times 2}$ is the class of self-adjoint 2×2 matrices. Let us show that this is not the case. We compute

$$\frac{1}{2\pi} \int_0^{2\pi} F(e^{it})F^*(e^{it}) dt = S^2 + T^2 = P + 100I.$$

Thus

$$F(0)F^*(0) = A + 100I \not\leq P + 100I = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it})F^*(e^{it}) dt.$$

Hence $F(z)F^*(z) \notin \text{SH}[\Delta, \mathcal{S}_{2 \times 2}]$. Therefore $\ln F(z)F^*(z)$ is not harmonic in Δ .

Acknowledgements

We would like to thank Mohan Ravichandran for introducing us with the notion of spectral order and to Artur Planeta who brought the result in Theorem 2.5 to our attention.

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