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# NEW INTEGRAL INEQUALITIES VIA $(\alpha, m)$-CONVEXITY AND QUASI-CONVEXITY

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**Abstract**

In this paper, we establish some new integral inequalities involving Beta function via $(\alpha, m)$-convexity and quasi-convexity, respectively. Our results in special cases recapture known results.

**Keywords:** Hermite's inequality, Euler Beta function, Hölder's inequality, $(\alpha, m)$-convexity, quasi-convexity

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## 1. Introduction

Let $I$ be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex (see [17, P.1]) if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [27], Toader defined $m$-convexity as follows:

**1.1. Definition.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $m$-convex, where $m \in [0, 1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$.We say that $f$ is $m-$concave if $-f$ is $m-$convex.

In [18], Miheşan defined $(\alpha, m)-$ convexity as follows:

**1.2. Definition.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $(\alpha, m)-$ convex, where $(\alpha, m) \in [0, 1]^2$, if

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$.

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Denote by $K_m^\alpha(b)$ the class of all $(\alpha,m)-$ convex functions on $[0,b]$ for which $f(0)\le 0$. It can be easily seen that for $(\alpha,m)=(1,m)$, $(\alpha,m)-$ convexity reduces to $m-$ convexity and for $(\alpha,m)=(1,1)$, $(\alpha,m)-$ convexity reduces to the concept of usual convexity defined on $[0,b]$, $b>0$. For recent results and generalizations concerning $m-$convex and $(\alpha,m)-$convex functions see [4, 6, 10, 19, 21, 26].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f:[a,b]\to\mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(\lambda x+(1-\lambda)y)\le\max\{f(x),f(y)\}$$

holds for any $x,y\in[a,b]$ and $\lambda\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [14]).

One of the most famous inequalities for convex functions is Hadamard's inequality. This double inequality is stated as follows: Let $f$ be a convex function on some nonempty interval $[a,b]$ of real line $\mathbb{R}$, where $a\neq b$. Then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[19], [22]-[26], [28]). In [4], Bakula et al. establish several Hadamard type inequalities for differentiable $m-$convex and $(\alpha,m)-$convex functions.

Recently, Ion [14] established two estimates on the Hermite-Hadamard inequality for functions whose first derivatives in absolute value are quasi-convex. Namely, he obtained the following results:

**1.3. Theorem.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I$, $a,b\in I$ with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\le\frac{b-a}{4}\left\{\max\left|f'(a)\right|,\left|f'(b)\right|\right\}.$$

**1.4. Theorem.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I$, $a,b\in I$ with $a<b$ and let $p>1$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\le\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\max\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}.$$

In [2], Alomari et al. obtained the following result.

**1.5. Theorem.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I$, $a,b\in I$ with $a<b$ and let $q\ge 1$. If $|f'|^q$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|\le\frac{b-a}{4}\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

In [20], Özdemir et al. used the following lemma in order to establish several integral inequalities via some kinds of convexity.

**1.6. Lemma.** *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $a<b$. Then the equality*

$$\int_a^b(x-a)^p(b-x)^q f(x)dx=(b-a)^{p+q+1}\int_0^1(1-t)^p t^q f(ta+(1-t)b)dt \tag{1.2}$$

*holds for some fixed $p,q>0$.*

Especially, Özdemir et al. [20] discussed the following new results connecting with $m-$convex function and quasi-convex function, respectively:

**1.7. Theorem.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $0 \le a < b < \infty$. If $f$ is $m-$convex on $[a,b]$, for some fixed $m \in (0,1]$ and $p,q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\le (b-a)^{p+q+1} \min\left\{\beta(q+2,p+1)f(a) + m\beta(q+1,p+2) f\left(\frac{b}{m}\right),\right.$$
$$\left.\beta(q+1,p+2)f(b) + m\beta(q+2,p+1) f\left(\frac{a}{m}\right)\right\}, \tag{1.3}$$

*where $\beta(x,y)$ is the Euler Beta function.*

**1.8. Theorem.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $0 \le a < b < \infty$. If $f$ is quasi-convex on $[a,b]$, then for some fixed $p,q > 0$, we have*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \le (b-a)^{p+q+1} \max\{f(a), f(b)\}\beta(p+1,q+1). \tag{1.4}$$

The aim of this paper is to establish some new integral inequalities like those given in Theorems 1.7 and 1.8 for $(\alpha, m)-$convex functions (Section 2) and quasi-convex functions (Section 3), respectively. Our results in special cases recapture Theorems 1.7 and 1.8, respectively. That is, this study is a continuation and generalization of [20].

## 2. New integral inequalities for $(\alpha, m)-$ convex functions

**2.1. Theorem.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $0 \le a < b < \infty$. If $f$ is $(\alpha, m)-$convex on $[a,b]$, for some fixed $(\alpha, m) \in (0,1]^2$ and $p,q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\le (b-a)^{p+q+1} \min\left\{\beta(q+\alpha+1,p+1)f(a) + m[\beta(q+1,p+1) - \beta(q+\alpha+1,p+1)] f\left(\frac{b}{m}\right),\right.$$
$$\left.\beta(q+1,p+\alpha+1)f(b) + m[\beta(p+1,q+1) - \beta(q+1,p+\alpha+1)] f\left(\frac{a}{m}\right)\right\}, \tag{2.1}$$

*where $\beta(x,y)$ is the Euler Beta function.*

*Proof.* Since $f$ is $(\alpha, m)-$convex on $[a,b]$, we know that for every $t \in [0,1]$

$$f(ta + (1-t)b) = f\left(ta + m(1-t)\frac{b}{m}\right) \le t^\alpha f(a) + m\left(1-t^\alpha\right) f\left(\frac{b}{m}\right). \tag{2.2}$$

Using Lemma 1.6, with $x = ta + (1-t)b$, then we have

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\le (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q \left(t^\alpha f(a) + m\left(1-t^\alpha\right) f\left(\frac{b}{m}\right)\right) dt$$
$$= (b-a)^{p+q+1} \left[f(a) \int_0^1 (1-t)^p t^{q+\alpha} dt + m f\left(\frac{b}{m}\right) \int_0^1 (1-t)^p t^q \left(1-t^\alpha\right) dt\right].$$

Now, we will make use of the Beta function which is defined for $x,y > 0$ as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

It is known that

$$\int_0^1 t^{q+\alpha}(1-t)^p dt = \beta(q+\alpha+1,p+1),$$

$$\int_0^1 (1-t)^p t^q \left(1-t^\alpha\right) dt = \int_0^1 t^q (1-t)^p dt - \int_0^1 t^{q+\alpha}(1-t)^p dt$$
$$=\beta(q+1,p+1) - \beta(q+\alpha+1,p+1)].$$

Combining all obtained equalities we get

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\leq (b-a)^{p+q+1} \left\{\beta(q+\alpha+1,p+1)f(a) + m[\beta(q+1,p+1) - \beta(q+\alpha+1,p+1)]f\left(\frac{b}{m}\right)\right\}. \tag{2.3}$$

If we choose $x = tb + (1-t)a$, analogously we obtain

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\leq (b-a)^{p+q+1} \left\{\beta(q+1,p+\alpha+1)f(b) + m[\beta(q+1,p+1) - \beta(q+1,p+\alpha+1)]f\left(\frac{a}{m}\right)\right\}. \tag{2.4}$$

Thus, by (2.3) and (2.4) we obtain (2.1), which completes the proof. □

**2.2. Remark.** As a special case of Theorem 2.1 for $\alpha = 1$, that is for $f$ be $m-$convex on $[a,b]$, we recapture Theorem 1.7 due to the fact that

$$\beta(q+1,p+1) - \beta(q+2,p+1) = \beta(q+1,p+1) - \frac{q+1}{p+q+2}\beta(q+1,p+1)$$
$$= \frac{p+1}{p+q+2}\beta(q+1,p+1) = \beta(q+1,p+2)$$

and

$$\beta(q+1,p+1) - \beta(q+1,p+\alpha+1) = \beta(q+2,p+1).$$

**2.3. Corollary.** *In Theorem 2.1, if $p = q$, then* (2.1) *reduces to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$
$$\leq (b-a)^{2p+1} \min\left\{\beta(p+\alpha+1,p+1)f(a) + m[\beta(p+1,p+1) - \beta(p+\alpha+1,p+1)]f\left(\frac{b}{m}\right),\right.$$
$$\left.\beta(p+1,p+\alpha+1)f(b) + m[\beta(p+1,p+1) - \beta(p+1,p+\alpha+1)]f\left(\frac{a}{m}\right)\right\}.$$

**2.4. Theorem.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ such that $f \in L([a,b])$, $0 \leq a < b < \infty$ and let $k > 1$. If $|f|^{\frac{k}{k-1}}$ is $(\alpha,m)-$convex on $[a,b]$, for some fixed $(\alpha,m) \in (0,1]^2$ and $p,q > 0$, then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\leq \frac{(b-a)^{p+q+1}}{(\alpha+1)^{\frac{k-1}{k}}} \left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} \min\left\{\left[|f(a)|^{\frac{k}{k-1}} + \alpha m\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},\right.$$
$$\left.\left[|f(b)|^{\frac{k}{k-1}} + \alpha m\left|f\left(\frac{a}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}\right\}. \tag{2.5}$$

*Proof.* Since $|f|^{\frac{k}{k-1}}$ is $(\alpha, m)-$convex on $[a,b]$ we know that for every $t \in [0,1]$

$$|f(ta+(1-t)b)|^{\frac{k}{k-1}} = \left|f\left(ta+m(1-t)\frac{b}{m}\right)\right|^{\frac{k}{k-1}}$$

$$\leq t^{\alpha}|f(a)|^{\frac{k}{k-1}} + m\left(1-t^{\alpha}\right)\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}.$$

Using Lemma 1.6, with $x = ta+(1-t)b$, then we have

$$\int_a^b (x-a)^p(b-x)^q f(x)dx$$

$$\leq (b-a)^{p+q+1}\left[\int_0^1 (1-t)^{kp}t^{kq}dt\right]^{\frac{1}{k}}\left[\int_0^1 |f(ta+(1-t)b)|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$

$$\leq (b-a)^{p+q+1}\left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}}\left[\int_0^1 t^{\alpha}|f(a)|^{\frac{k}{k-1}}dt + m\int_0^1 (1-t^{\alpha})\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$

$$= (b-a)^{p+q+1}\left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}}\left[\frac{1}{\alpha+1}|f(a)|^{\frac{k}{k-1}} + m\frac{\alpha}{\alpha+1}\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}.$$

If we choose $x = tb+(1-t)a$, analogously we obtain

$$\int_a^b (x-a)^p(b-x)^q f(x)dx$$

$$\leq (b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left[\frac{1}{\alpha+1}|f(b)|^{\frac{k}{k-1}} + m\frac{\alpha}{\alpha+1}\left|f\left(\frac{a}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},$$

which completes the proof. □

**2.5. Corollary.** *In Theorem 2.4, if $p=q$, then* (2.5) *reduces to*

$$\int_a^b (x-a)^p(b-x)^p f(x)dx$$

$$\leq \frac{(b-a)^{2p+1}}{(\alpha+1)^{\frac{k-1}{k}}}\left[\beta(kp+1,kp+1)\right]^{\frac{1}{k}} \min\left\{\left[|f(a)|^{\frac{k}{k-1}} + \alpha m\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},\right.$$

$$\left.\left[|f(b)|^{\frac{k}{k-1}} + \alpha m\left|f\left(\frac{a}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}\right\}.$$

**2.6. Corollary.** *In Theorem 2.4, if $\alpha = 1$, i.e., if $|f|^{\frac{k}{k-1}}$ is $m-$convex on $[a,b]$, then* (2.5) *reduces to*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx$$

$$\leq \frac{(b-a)^{p+q+1}}{2^{\frac{k-1}{k}}}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} \min\left\{\left[|f(a)|^{\frac{k}{k-1}} + m\left|f\left(\frac{b}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},\right.$$

$$\left.\left[|f(b)|^{\frac{k}{k-1}} + m\left|f\left(\frac{a}{m}\right)\right|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}\right\}.$$

**2.7. Remark.** As a special case of Corollary 2.6 for $m = 1$, that is for $|f|^{\frac{k}{k-1}}$ be convex on $[a, b]$, we get

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \le \frac{(b-a)^{p+q+1}}{2^{\frac{k-1}{k}}} \left[\beta(kp+1, kq+1)\right]^{\frac{1}{k}} \left[|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}.$$

**2.8. Theorem.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $0 \le a < b < \infty$ and let $l \ge 1$. If $|f|^l$ is $(\alpha, m)-$convex on $[a, b]$, for some fixed $(\alpha, m) \in (0, 1]^2$ and $p, q > 0$, then*

$$\begin{aligned}
&\int_a^b (x-a)^p (b-x)^q f(x)dx \\
\le &(b-a)^{p+q+1} \left[\beta(p+1, q+1)\right]^{\frac{l-1}{l}} \\
&\times \min\left\{ \left[\beta(q+\alpha+1, p+1)|f(a)|^l + m[\beta(q+1, p+1) - \beta(q+\alpha+1, p+1)] \left|f\left(\frac{b}{m}\right)\right|^l\right]^{\frac{1}{l}}, \right. \\
&\left. \left[\beta(q+1, p+\alpha+1)|f(b)|^l + m[\beta(q+1, p+1) - \beta(q+1, p+\alpha+1)] \left|f\left(\frac{a}{m}\right)\right|^l\right]^{\frac{1}{l}} \right\}. \qquad (2.6)
\end{aligned}$$

*Proof.* Since $|f|^l$ is $(\alpha, m)-$convex on $[a, b]$, we know that for every $t \in [0, 1]$

$$|f(ta + (1-t)b)|^l = \left|f\left(ta + m(1-t)\frac{b}{m}\right)\right|^l \le t^\alpha |f(a)|^l + m\left(1 - t^\alpha\right) \left|f\left(\frac{b}{m}\right)\right|^l.$$

Using Lemma 1.6, with $x = ta + (1-t)b$, then we have

$$\begin{aligned}
&\int_a^b (x-a)^p (b-x)^q f(x)dx \\
=&(b-a)^{p+q+1} \int_0^1 \left[(1-t)^p t^q\right]^{\frac{l-1}{l}} \left[(1-t)^p t^q\right]^{\frac{1}{l}} f(ta + (1-t)b)dt \\
\le&(b-a)^{p+q+1} \left[\int_0^1 (1-t)^p t^q dt\right]^{\frac{l-1}{l}} \left[\int_0^1 (1-t)^p t^q |f(ta + (1-t)b)|^l dt\right]^{\frac{1}{l}} \\
\le&(b-a)^{p+q+1} \left[\beta(q+1, p+1)\right]^{\frac{l-1}{l}} \\
&\times \left[\beta(q+\alpha+1, p+1)|f(a)|^l + m[\beta(q+1, p+1) - \beta(q+\alpha+1, p+1)] \left|f\left(\frac{b}{m}\right)\right|^l\right]^{\frac{1}{l}}.
\end{aligned}$$

If we choose $x = tb + (1-t)a$, analogously we obtain

$$\begin{aligned}
&\int_a^b (x-a)^p (b-x)^q f(x)dx \\
\le&(b-a)^{p+q+1} \left[\beta(p+1, q+1)\right]^{\frac{l-1}{l}} \\
&\times \left[\beta(q+1, p+\alpha+1)|f(b)|^l + m[\beta(q+1, p+1) - \beta(q+1, p+\alpha+1)] \left|f\left(\frac{a}{m}\right)\right|^l\right]^{\frac{1}{l}},
\end{aligned}$$

which completes the proof. □

**2.9. Corollary.** *In Theorem 2.8, if* $p = q$, *then* (2.6) *reduces to*

$$\int_a^b (x-a)^p (b-x)^p f(x)dx$$
$$\leq (b-a)^{2p+1} \left[\beta(p+1,p+1)\right]^{\frac{l-1}{l}}$$
$$\times \min\left\{ \left[\beta(p+\alpha+1,p+1)|f(a)|^l + m[\beta(p+1,p+1) - \beta(p+\alpha+1,p+1)] \left|f\left(\frac{b}{m}\right)\right|^l\right]^{\frac{1}{l}},\right.$$
$$\left.\left[\beta(p+1,p+\alpha+1)|f(b)|^l + m[\beta(p+1,p+1) - \beta(p+1,p+\alpha+1)] \left|f\left(\frac{a}{m}\right)\right|^l\right]^{\frac{1}{l}}\right\}.$$

**2.10. Corollary.** *In Theorem 2.8, if* $\alpha = 1$, *i.e., if* $|f|^l$ *is* $m-$*convex on* $[a,b]$, *then* (2.6) *reduces to*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\leq (b-a)^{p+q+1} \left[\beta(p+1,q+1)\right]^{\frac{l-1}{l}} \min\left\{ \left[\beta(q+2,p+1)|f(a)|^l + m\beta(q+1,p+2) \left|f\left(\frac{b}{m}\right)\right|^l\right]^{\frac{1}{l}},\right.$$
$$\left.\left[\beta(q+1,p+2)|f(b)|^l + m\beta(q+2,p+1) \left|f\left(\frac{a}{m}\right)\right|^l\right]^{\frac{1}{l}}\right\}.$$

**2.11. Remark.** As a special case of Corollary 2.10 for $m = 1$, that is for $|f|^l$ be convex on $[a,b]$, we get

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\leq (b-a)^{p+q+1} \left[\beta(p+1,q+1)\right]^{\frac{l-1}{l}} \left[\beta(q+2,p+1)|f(a)|^l + \beta(q+1,p+2)\left|f(b)\right|^l\right]^{\frac{1}{l}}.$$

## 3. New integral inequalities for quasi-convex functions

**3.1. Theorem.** *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *such that* $f \in L([a,b])$, $0 \leq a < b < \infty$ *and let* $k > 1$. *If* $|f|^{\frac{k}{k-1}}$ *is quasi-convex on* $[a,b]$, *for some fixed* $p, q > 0$, *then*

$$\int_a^b (x-a)^p (b-x)^q f(x)dx \leq (b-a)^{p+q+1} \left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} \left(\max\left\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right\}\right)^{\frac{k-1}{k}}. \tag{3.1}$$

*Proof.* By Lemma 1.6, Hölder's inequality, the definition of Beta function and the fact that $|f|^{\frac{k}{k-1}}$ is quasi-convex on $[a,b]$, we have

$$\int_a^b (x-a)^p (b-x)^q f(x)dx$$
$$\leq (b-a)^{p+q+1} \left[\int_0^1 (1-t)^{kp} t^{kq} dt\right]^{\frac{1}{k}} \left[\int_0^1 |f(ta+(1-t)b)|^{\frac{k}{k-1}} dt\right]^{\frac{k-1}{k}}$$
$$\leq (b-a)^{p+q+1} \left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}} \left[\int_0^1 \max\left\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right\} dt\right]^{\frac{k-1}{k}}$$
$$= (b-a)^{p+q+1} \left[\beta(kq+1,kp+1)\right]^{\frac{1}{k}} \left[\max\left\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right\}\right]^{\frac{k-1}{k}},$$

which completes the proof. □

**3.2. Corollary.** *Let $f$ be as in Theorem 3.1. Additionally, if*

(1) $f$ *is increasing, then we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} f(b).$$

(2) $f$ *is decreasing, then we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} f(a).$$

**3.3. Theorem.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\leq a<b<\infty$ and let $l\geq 1$. If $|f|^l$ is quasi-convex on $[a,b]$, for some fixed $p,q>0$, then*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1}\beta(p+1,q+1)\left(\max\left\{|f(a)|^l,|f(b)|^l\right\}\right)^{\frac{1}{l}}, \tag{3.2}$$

*where $\beta(x,y)$ is the Euler Beta function.*

*Proof.* By Lemma 1.6, Hölder's inequality, the definition of Beta function and the fact that $|f|^l$ is quasi-convex on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b (x-a)^p(b-x)^q f(x)dx \\
=&(b-a)^{p+q+1}\int_0^1 \left[(1-t)^p t^q\right]^{\frac{l-1}{l}}\left[(1-t)^p t^q\right]^{\frac{1}{l}} f(ta+(1-t)b)dt \\
\leq&(b-a)^{p+q+1}\left[\int_0^1 (1-t)^p t^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 (1-t)^p t^q |f(ta+(1-t)b)|^l dt\right]^{\frac{1}{l}} \\
\leq&(b-a)^{p+q+1}\left[\beta(q+1,p+1)\right]^{\frac{l-1}{l}}\left[\max\left\{|f(a)|^l,|f(b)|^l\right\}\beta(q+1,p+1)\right]^{\frac{1}{l}} \\
=&(b-a)^{p+q+1}\beta(p+1,q+1)\left(\max\left\{|f(a)|^l,|f(b)|^l\right\}\right)^{\frac{1}{l}},
\end{aligned}$$

which completes the proof. □

**3.4. Corollary.** *Let $f$ be as in Theorem 3.3. Additionally, if*

(1) $f$ *is increasing, then we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1}\beta(p+1,q+1)f(b).$$

(2) $f$ *is decreasing, then we have*

$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1}\beta(p+1,q+1)f(a).$$

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