

Z₃-GRADED DIFFERENTIAL CALCULUS ON THE QUANTUM SPACE \mathcal{R}_q^3

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Abstract

In this work, the Z_3 -graded differential calculus of the extended quantum 3d space is constructed. By using this differential calculus, we obtain the algebra of Cartan-Maurer forms and the corresponding quantum Lie algebra. To give a Z_3 -graded Cartan calculus on the extended quantum 3d space, the noncommutative differential calculus on this space is extended by introducing inner derivations and Lie derivatives.

Keywords: Z_3 -graded quantum 3d space, Lie algebra, inner derivation, Lie derivation

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1. Introduction

The noncommutative differential geometry of quantum groups was introduced by Woronowicz [20],[21]. In this approach the differential calculus on the group is deduced from the properties of the group and it involves functions on the group, differentials, differential forms and derivatives. The other approach, initiated by Wess and Zumino [19], followed Manin's emphasis [10] on the quantum spaces as the primary objects. Differential forms are defined in terms of noncommuting coordinates, and the differential and algebraic properties of quantum groups acting on these spaces are obtained from the properties of the spaces.

The differential calculus on the quantum 3d space similarly involves functions on the 3d space, differentials, differential forms and derivatives. The most important property of this calculus is that the operator d satisfies $d^3 = 0$ $d^l \neq 0, 1 \leq l \leq 2$ and it contains as a consequence, not only first differentials $dx^i, i = 1...3$, but involves also higher order

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differentials $\mathbf{d}^k x^i, k = 1..2$. The exterior differential \mathbf{d} is an operator which gives the mapping from the generators of the 3d space to the differentials:

$$(1.1) \quad \mathbf{d} : a \mapsto \mathbf{d}a, \quad a \in \{x, y, z\}.$$

We demand that the exterior differential \mathbf{d} has to satisfy two properties:

$$(1.2) \quad \mathbf{d} \wedge \mathbf{d} \wedge \mathbf{d} =: \mathbf{d}^3 = 0. \quad (\mathbf{d}^2 \neq 0)$$

and the Z_3 -graded Leibniz rule

$$(1.3) \quad \mathbf{d}(fg) = (\mathbf{d}f)g + j^{grad(f)} f(\mathbf{d}g)$$

where $j = e^{\frac{2\pi i}{3}}$ ($i^2 = -1$) and $grad(f)$ denotes the grade of f and also the exterior differential \mathbf{d} acts on the Cartan-Maurer one forms, i.e, for any forms ω_1 and ω_2

$$\mathbf{d}(\omega_1 \wedge \omega_2) = (\mathbf{d}\omega_1) \wedge \omega_2 + j^{grad(\omega_1)} \omega_1 \wedge (\mathbf{d}\omega_2).$$

There is a relationship of the exterior derivative with the Lie derivative and to describe this relation, we introduce a new operator: the inner derivation. Hence the differential calculus on the quantum 3d space can be extended into a large calculus. We call this new calculus the Cartan calculus. The connection of the inner derivation denoted by i_a and the Lie derivative denoted by \mathcal{L}_a is given by the Cartan formula:

$$\mathcal{L}_a = i_a \circ \mathbf{d} + \mathbf{d} \circ i_a.$$

This and other formulae are explained in [15]- [17]. In section 5, we shall give a brief overview without much discussion.

The extended calculus on the quantum plane was introduced in [7] using the approach of [15]. The Z_3 -graded differential calculus was studied in [11] - [13]. The Z_3 -graded differential geometry of the quantum plane is introduced in [4] and later [9]. In this work we explicitly set up Z_3 -graded differential calculus on the quantum 3d space using approach of [4] and [5]. Also, the scope of the differential calculus was mainly enriched in the series of papers [1], [2], [8], [14], [18].

Let us shortly give a general Z_3 -graded algebraic structure. Let z be a Z_3 -graded variable. Then we say that the variable z satisfies the relation

$$z^3 = 0.$$

If f is an arbitrary function of the variable z , then the $f(z)$ becomes a polynomial of degree two in z , that is,

$$f(z) = a_0 + a_1 z + a_2 z^2,$$

where a_0, a_2, a_1 denote three fixed numbers whose grades are $grad(a_0) = 0, grad(a_2) = 1$ and $grad(a_1) = 2$, respectively.

The cyclic group Z_3 can be represented in the complex plane by means of the cubic roots of 1: let $j = e^{\frac{2\pi i}{3}}$ ($i^2 = -1$). Then one has

$$j^3 = 1 \quad \text{and} \quad j^2 + j + 1 = 0.$$

One can define the Z_3 -graded commutator $[A, B]$ as

$$[A, B]_{Z_3} = AB - j^{ab} BA,$$

where $grad(A) = a$ and $grad(B) = b$. If A and B are j -commutative, then we have

$$AB = j^{ab} BA.$$

Also,

$$grad(A.B) = grad(A) + grad(B).$$

2. Z₃-graded differential algebra

The quantum 3d space is defined as an associative algebra generated by three noncommuting coordinates x, y and z with three quadratic relations [10]

$$(2.1) \quad xy - qyx = 0, \quad yz - qzy = 0, \quad xz - qzx = 0,$$

where q is a non-zero complex number. Here, the coordinates x, y and z with respect to the Z₃-grading are of grade 0. This associative algebra over the complex numbers is known as the algebra of polynomials over the quantum 3d space and we shall denote it by $\text{Fun}(\mathcal{R}_q^3)$. In the limit $q \rightarrow 1$, this algebra is commutative and can be considered as the algebra of polynomials $C[x, y, z]$ over the usual three dimensional space, where x, y and z are the three coordinate functions. We define the extended quantum 3d space to be the algebra that contains \mathcal{R}_q^3 , the unit and x^{-1} , the inverse of x , which obeys $xx^{-1} = 1 = x^{-1}x$. We denote the unital extension of \mathcal{R}_q^3 by \mathcal{A} .

We now set up a differential calculus on the quantum space \mathcal{R}_q^3 . In order to obtain the commutation relations of the coordinates and their differentials, we shall use the approach of [19] and [4]. In this manner we assume that

$$(2.2) \quad \begin{aligned} xdx &= A_1 dxx, & xdy &= K_{11} dyx + K_{12} dxy, \\ ydx &= K_{21} dxy + K_{22} dyx, & ydy &= A_2 dyy, \\ xdz &= L_{11} dzx + L_{12} dxz, & zdx &= L_{21} dxz + L_{22} dzx, \\ zdz &= A_3 dzz, & ydz &= M_{11} dzy + M_{12} dyz, \\ zdy &= M_{21} dyz + M_{22} dz y, \end{aligned}$$

where the coefficients $A_1, A_2, A_3, K_{ij}, L_{ij}, M_{ij}$ are related with deformation parameter(s). Here, the first order differentials dx, dy, dz with respect to the Z₃-grading are of grade 1. To obtain these coefficients, we also demand that

$$(2.3) \quad dx \wedge dy = F_1 dy \wedge dx, \quad dx \wedge dz = F_2 dz \wedge dx, \quad dy \wedge dz = F_3 dz \wedge dy,$$

where the coefficients F_1, F_2, F_3 are related with deformation parameter(s). Since $d^3 = 0$ (and $d^2 \neq 0$) in the Z₃-graded space, in order to construct a self-consistent theory of differential forms it is necessary to add to the first order differentials of coordinates dx, dy, dz a set of second order differentials d^2x, d^2y, d^2z , which are grade 2 with respect to the Z₃-grading. Appearance of higher order differentials is a peculiar property of a proposed generalization of differential forms.

To obtain the coefficients appearing in (2.2) we shall apply to the exterior differential d the relations (2.2). If we use first four relations in (2.2) and we differentiate them with respect to the Z₃-graded Leibniz rule (1.3) we get

$$(2.4) \quad \begin{aligned} xd^2x &= A_1 d^2xx + (jA_1 - 1) dx dx, \\ xd^2y &= K_{11} d^2yx + K_{12} d^2xy + (jK_{11} + jK_{12}F_1 - F_1) dy dx, \\ yd^2x &= K_{21} d^2xy + K_{22} d^2yx + (jK_{21}F_1 + jK_{22} - 1) dy dx, \\ yd^2y &= A_2 d^2yy + (jA_2 - 1) dy dy. \end{aligned}$$

These relations are not homogeneous in the sense that the commutation relations between the coordinates and second order differentials include first order differentials as well. In order to make homogenous the commutation relations between the coordinates and their second order differentials, we must choose

$$(2.5) \quad \begin{aligned} A_1 &= j^2, & A_2 &= j^2, \\ jK_{11} + jK_{12}F_1 - F_1 &= 0, & jK_{21}F_1 + jK_{22} - 1 &= 0. \end{aligned}$$

Now, using the consistency of calculus and that $d^3 = 0$ (but $d^2 \neq 0$) we have the following relations

$$(2.6) \quad \begin{aligned} K_{12} - qK_{21} &= -1, & K_{11} - qK_{22} &= q, \\ K_{22}(K_{11} - qA_1) &= 0, & K_{22}K_{12} &= 0, & K_{12}(A_1 - qK_{21}) &= 0. \end{aligned}$$

If we assume that $K_{12} = 0$, from the relations (2.5) and (2.6), we easily find the coefficients K_{ij} and F_1 . With similar operations, it can be found other coefficients. Consequently, we see that the commutation relations satisfied by the coordinates and their first order differentials in the form

$$(2.7) \quad \begin{aligned} xdx &= j^2 dx^2, & xdy &= qj^2 dyx, \\ ydx &= q^{-1}dxy + (j^2 - 1)dyx, & ydy &= j^2 dyy, \\ xdz &= qj^2 dzx, & zdx &= q^{-1}dxz + (j^2 - 1)dzx, \\ ydz &= qj^2 dzy, & zdy &= q^{-1}dyz + (j^2 - 1)dz y, \\ zdz &= j^2 dz^2. \end{aligned}$$

The commutation relations between the first order differentials as follows:

$$(2.8) \quad \begin{aligned} dx \wedge dy &= qdy \wedge dx, & dx \wedge dx \wedge dx &= 0, \\ dx \wedge dz &= qdz \wedge dx, & dz \wedge dz \wedge dz &= 0, \\ dy \wedge dz &= qdz \wedge dy, & dy \wedge dy \wedge dy &= 0. \end{aligned}$$

Note that the above relations of differentials among themselves reduce to commutative relations as $q \rightarrow 1$. In fact, this situation is the natural result of the differential operator with the rules (1.2) and (1.3). The commutation relations of the coordinates and their second order differentials now have the form

$$(2.9) \quad \begin{aligned} xd^2x &= j^2 d^2xx, & xd^2y &= qj^2 d^2yx, \\ yd^2x &= q^{-1}d^2xy + (j^2 - 1)d^2yx, & yd^2y &= j^2 d^2yy, \\ xd^2z &= qj^2 d^2zx, & zd^2x &= q^{-1}d^2xz + (j^2 - 1)d^2zx, \\ yd^2z &= qj^2 d^2zy, & zd^2y &= q^{-1}d^2yz + (j^2 - 1)d^2zy, \\ zd^2z &= j^2 d^2zz. \end{aligned}$$

We now apply the exterior differential d to the relations (2.9) then we see that the commutation relations between the first order differentials and the second order differentials as follows:

$$(2.10) \quad \begin{aligned} dx \wedge d^2x &= jd^2x \wedge dx, & dx \wedge d^2y &= qjd^2y \wedge dx, \\ dy \wedge d^2x &= q^{-1}j^2d^2x \wedge dy + (j - j^2)d^2y \wedge dx, \\ dy \wedge d^2y &= jd^2y \wedge dy, & dz \wedge d^2z &= jd^2z \wedge dz, \\ dx \wedge d^2z &= qjd^2z \wedge dx, & dy \wedge d^2z &= qjd^2z \wedge dy, \\ dz \wedge d^2x &= q^{-1}j^2d^2x \wedge dz + (j - j^2)d^2z \wedge dx, \\ dz \wedge d^2y &= q^{-1}j^2d^2y \wedge dz + (j - j^2)d^2z \wedge dy. \end{aligned}$$

Applying the exterior differential d to the relations (2.10), we get the commutation relations between the second order differentials as

$$(2.11) \quad d^2x \wedge d^2y = qd^2y \wedge d^2x, \quad d^2x \wedge d^2z = qd^2z \wedge d^2x, \quad d^2y \wedge d^2z = qd^2z \wedge d^2y.$$

Consequently, we set up an exterior calculus of the higher order differential forms on the quantum 3d space. Next step is to be constructed a structure with Cartan-Maurer one-forms.

3. Cartan-Maurer one forms on \mathcal{A}

In this section we shall define three forms using the generators of \mathcal{A} and investigate their relations with the coordinates, differentials and themselves.

If we call them w_x , w_y and w_z then one can define these forms as follows:

$$(3.1) \quad w_x = dx x^{-1}, \quad w_y = dy x^{-1} - dx x^{-1} y x^{-1}, \quad w_z = dz.$$

We denote the algebra of forms generated by three elements w_x , w_y and w_z by Ω . We can find the commutation relations of these forms with the coordinate functions using the relations (2.7) and (2.1) as follows

$$(3.2) \quad \begin{aligned} x w_x &= j^2 w_x x, & x w_y &= q j^2 w_y x, \\ y w_x &= j^2 w_x y + (j^2 - 1) w_y x, & y w_y &= q w_y y, \\ x w_z &= q j^2 w_z x, & z w_x &= w_x z + (j^2 - 1) w_z, \\ y w_z &= q j^2 w_z y, & z w_y &= w_y z, & z w_z &= j^2 w_z z. \end{aligned}$$

Using together with (2.7) and (2.8) will give the following rules which satisfied by the generators of the algebra Ω with the first order differentials

$$(3.3) \quad \begin{aligned} w_x \wedge dx &= j dx \wedge w_x, & w_x \wedge dy &= j dy \wedge w_x, \\ w_y \wedge dx &= q^{-1} j^2 dx \wedge w_y, & w_y \wedge dy &= q^{-1} j dy \wedge w_y, \\ w_x \wedge dz &= j dz \wedge w_x, & w_z \wedge dx &= q^{-1} dx \wedge w_z, \\ w_y \wedge dz &= j dz \wedge w_y, & w_z \wedge dy &= q^{-1} dy \wedge w_z, \\ w_z \wedge dz &= dz \wedge w_z. \end{aligned}$$

We finally need the relations between the one forms and second order differentials and they are

$$(3.4) \quad \begin{aligned} w_x \wedge d^2 x &= j^2 d^2 x \wedge w_x, & w_x \wedge d^2 y &= j^2 d^2 y \wedge w_x, \\ w_y \wedge d^2 x &= q^{-1} d^2 x \wedge w_y, & w_y \wedge d^2 y &= q^{-1} j^2 d^2 y \wedge w_y, \\ w_x \wedge d^2 z &= j^2 d^2 z \wedge w_x, & w_y \wedge d^2 z &= j^2 d^2 z \wedge w_y, \\ w_x \wedge d^2 x &= q^{-1} j^2 d^2 x \wedge w_z + (j - j^2) d^2 z \wedge dx, \\ w_z \wedge d^2 y &= q^{-1} j^2 d^2 y \wedge w_z + (j - j^2) d^2 z \wedge dy, \\ w_z \wedge d^2 z &= j d^2 z \wedge w_z. \end{aligned}$$

Using (3.2)-(3.4) we now find the commutation rules of the generators of Ω as follows

$$(3.5) \quad \begin{aligned} w_x \wedge w_y &= j^2 w_y \wedge w_x, & w_x \wedge w_x \wedge w_x &= 0, \\ w_y \wedge w_z &= j w_z \wedge w_y, & w_y \wedge w_y \wedge w_y &= 0, \\ w_x \wedge w_z &= j w_z \wedge w_x, & w_z \wedge w_z \wedge w_z &= 0. \end{aligned}$$

We know that the algebra Ω is a Z₃-graded Hopf algebra [6].

Now, we introduce here the commutation relations between the coordinates of the Z₃-graded quantum 3d space and their partial derivatives. We know that the exterior differential d can be expressed in the form

$$(3.6) \quad df = (dx \partial_x + dy \partial_y + dz \partial_z) f.$$

Then, for example,

$$\begin{aligned} d(xf) &= dx f + xdf \\ &= dx(1 + j^2 x \partial_x) f + dy(qj^2 x \partial_y) f + dz(qj^2 x \partial_z) f \\ &= (dx \partial_x x + dy \partial_y x + dz \partial_z x) f \end{aligned}$$

so that

$$(3.7) \quad \begin{aligned} \partial_x x &= 1 + j^2 x \partial_x, & \partial_x y &= q^{-1} y \partial_x, & \partial_y x &= qj^2 x \partial_y, \\ \partial_y y &= 1 + j^2 y \partial_y + (j^2 - 1)x \partial_x, & \partial_x z &= q^{-1} z \partial_x, \\ \partial_z x &= qj^2 x \partial_z, & \partial_y z &= q^{-1} z \partial_y, & \partial_z y &= qj^2 y \partial_z, \\ \partial_z z &= 1 + j^2 z \partial_z + (j^2 - 1)x \partial_x + (j^2 - 1)y \partial_y. \end{aligned}$$

Using the fact that $d^3 = 0$, we find

$$(3.8) \quad \partial_x \partial_y = qj^2 \partial_y \partial_x, \quad \partial_x \partial_z = qj^2 \partial_z \partial_x, \quad \partial_y \partial_z = qj^2 \partial_z \partial_y.$$

To complete the scheme, we need the relations partial differentials with first and second order differentials. And it is computed as follows

$$(3.9) \quad \begin{aligned} \partial_x dx &= j dx \partial_x + (j - 1) dy \partial_y + (j - 1) dz \partial_z, \\ \partial_x dy &= q^{-1} j dy \partial_x, & \partial_y dx &= q dx \partial_y, \\ \partial_y dy &= j dy \partial_y + (j - 1) dz \partial_z, & \partial_x dz &= q^{-1} j dz \partial_x, \\ \partial_z dx &= q dx \partial_z, & \partial_y dz &= q^{-1} j dz \partial_y, \\ \partial_z dy &= q dy \partial_z, & \partial_z dz &= j dz \partial_z, \end{aligned}$$

and

$$(3.10) \quad \begin{aligned} \partial_x d^2 x &= j d^2 x \partial_x + (j - 1) d^2 y \partial_y + (j - 1) d^2 z \partial_z, \\ \partial_x d^2 y &= q^{-1} j d^2 y \partial_x, & \partial_x d^2 z &= q^{-1} j d^2 z \partial_x, \\ \partial_y d^2 x &= q d^2 x \partial_y, & \partial_y d^2 y &= j d^2 y \partial_y + (j - 1) d^2 z \partial_z, \\ \partial_y d^2 z &= q^{-1} j d^2 z \partial_y, & \partial_z d^2 x &= q d^2 x \partial_z, \\ \partial_z d^2 y &= q d^2 y \partial_z, & \partial_z d^2 z &= j d^2 z \partial_z. \end{aligned}$$

4. Quantum Lie algebra

In this section, we construct an algebra generated by the Maurer-Cartan forms which are subjected to certain commutation relations. In order to obtain the quantum Lie algebra corresponding to the Maurer-Cartan forms we first write the Cartan-Maurer forms as

$$(4.1) \quad dx = w_x x, \quad dy = w_x y + w_y x, \quad dz = w_z.$$

The differential d can then be expressed in the form

$$(4.2) \quad d = w_x T_x + w_y T_y + w_z T_z.$$

Here T_x , T_y and T_z are the (quantum) Lie algebra generators. We now shall obtain the commutation relations of these generators. Considering an arbitrary function f of the coordinates of the Z_3 -graded quantum 3d space and using that $d^3 = 0$ one has

$$d^2 f = (dw_x) T_x f + (dw_y) T_y f + (dw_z) T_z f + j w_x dT_x f + j w_y dT_y f + j w_z dT_z f,$$

and

$$\begin{aligned} d^3 f &= (d^2 w_x)T_x f + (d^2 w_y)T_y f + (d^2 w_z)T_z f + j^2 w_x d^2 T_x f + \\ &\quad j^2 w_y d^2 T_y f + j^2 w_z d^2 T_z f - (dw_x)dT_x f - (dw_y)dT_y f - (dw_z)dT_z f. \end{aligned}$$

So we need the two-forms. Applying the exterior differential d to the expressions in (3.1) one has

$$(4.3) \quad \begin{aligned} dw_x &= d^2 x x^{-1} - j w_x \wedge w_x, & dw_z &= d^2 z, \\ dw_y &= d^2 y x^{-1} - d^2 x x^{-1} y x^{-1} + j^2 w_y \wedge w_x. \end{aligned}$$

These two-forms with the one-forms satisfy the following relations

$$\begin{aligned} w_x \wedge dw_x &= j dw_x \wedge w_x, & w_x \wedge dw_y &= j dw_y \wedge w_x + (j - j^2) dw_x \wedge w_y, \\ w_y \wedge dw_x &= j^2 dw_x \wedge w_y, & w_y \wedge dw_y &= j dw_y \wedge w_y, \\ w_x \wedge dw_z &= j^2 dw_z \wedge w_x, & w_z \wedge dw_x &= j dw_x \wedge w_z + (j - j^2) dw_z \wedge w_x, \\ w_y \wedge dw_z &= j^2 dw_z \wedge w_y, & w_z \wedge dw_y &= j dw_y \wedge w_z + (j - j^2) dw_z \wedge w_y, \\ w_z \wedge dw_z &= j dw_z \wedge w_z. \end{aligned}$$

Using these relations we get

$$(4.4) \quad d^2 w_x = 0, \quad d^2 w_y = 0, \quad d^2 w_z = 0.$$

After making this, it is easy to find the quantum Lie algebra:

$$(4.5) \quad T_x T_y - T_y T_x = 0, \quad T_x T_z - T_z T_x = 0, \quad T_y T_z - T_z T_y = 0.$$

The commutation relation (4.5) of the Lie algebra generators should be consistent with monomials of the coordinates of the Z₃-graded quantum 3d space. To do this, we evaluate the commutation relations between the generators of algebra and the coordinates. The commutation relations of the generators with the coordinates can be extracted from the Leibniz rule:

$$\begin{aligned} d(xf) &= dx f + x df \\ &= w_x(x + j^2 x T_x) f + w_y(q j^2 x T_y) f + w_z(q j^2 x T_z) f \\ &= (w_x T_x + w_y T_y + w_z T_z) x f \end{aligned}$$

This yields

$$(4.6) \quad \begin{aligned} T_x x &= x + j^2 x T_x, & T_x y &= y + j^2 y T_x, & T_y x &= q j^2 x T_y, \\ T_y y &= x + q y T_y + (j^2 - 1) x T_x, & T_x z &= z T_x, & T_z x &= q j^2 x T_z, \\ T_y z &= z T_y, & T_z y &= q j^2 y T_z, & T_z z &= 1 + j^2 z T_z + (j^2 - 1) T_x. \end{aligned}$$

Now, we illustrate the connection between the relations in this section, and the relations obtained at the end of the section 3.

We know that the exterior differential d can be expressed in the form (4.2), which we repeat here,

$$(4.7) \quad df = (w_x T_x + w_y T_y + w_z T_z) f.$$

Considering (3.6) together (4.7) and using (4.1) one has

$$(4.8) \quad T_x \equiv x \partial_x + y \partial_y \quad T_y \equiv x \partial_y \quad T_z \equiv \partial_z.$$

Using the relations (3.7) and (3.8) one can check that the relation of the generators in (4.8) coincide with (4.5). It can also be verified that, the action of the generators in (4.8) on the coordinates coincide with (4.6).

5. Extended calculus on the quantum 3d space

The Lie derivative is closely related to the exterior derivative. The exterior derivative and the Lie derivative are set to cover the idea of a derivative in different ways. These differences can be hasped together by introducing the idea of an antiderivation which is called an inner derivation.

5.1. Inner derivations. In order to obtain the commutation rules of the coordinates with inner derivations, we shall use the approach of [4]. Similarly other relations can also obtain.

Let us begin with some information about the inner derivations. Generally, for a smooth vector field X on a manifold the inner derivation, denoted by i_X , is a linear operator which maps k -forms to $(k - 1)$ -forms. If we define the inner derivation i_X on the set of all differential forms on a manifold, we know that i_X is an antiderivation of degree -1 :

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta)$$

where α and β are both differential forms. The inner derivation i_X acts on 0- and 1-forms as follows:

$$i_X(f) = 0, \quad i_X(df) = X(f).$$

We now wish to find the commutation relations between the coordinates x, y, z and the inner derivations associated with them. In order to obtain the commutation rules of the coordinates with inner derivations, we shall assume that they are of the following form

$$(5.1) \quad \begin{aligned} i_x x &= A_1 x i_x + A_2 y i_y + A_3 z i_z, & i_x y &= A_4 y i_x + A_5 x i_y, \\ i_x z &= A_6 z i_x + A_7 x i_z, & i_y x &= A_8 x i_y + A_9 y i_x, \\ i_y y &= A_{10} y i_y + A_{11} x i_x + A_{12} z i_z, & i_y z &= A_{13} z i_y + A_{14} y i_z, \\ i_z x &= A_{15} x i_z + A_{16} z i_x, & i_z y &= A_{17} y i_z + A_{18} z i_y, \\ i_z z &= A_{19} z i_z + A_{20} x i_x + A_{21} y i_y. \end{aligned}$$

The coefficients A_k ($1 \leq k \leq 21$) will be determined in terms of the deformation parameters q and j . But the use of the relations (2.1) does not give rise any solution in terms of the parameters q and j . However, we have, at least, the system

$$\begin{aligned} A_5(A_1 - qA_8) &= 0, & A_2 A_{11} - q^2 A_5 A_9 &= 0, & A_2 A_{14} &= 0, \\ A_8(A_{10} - qA_4) &= 0, & A_3 A_{20} - q^2 A_7 A_{16} &= 0, & A_3 A_{18} &= 0, \quad etc. \end{aligned}$$

To find the coefficients, we need the commutation relations of the inner derivations with the differentials of x, y and z . Since

$$i_{X_i}(dX_j) = \delta_{ij}$$

we can assume that the relations between the differentials and the inner derivations are of the following form

$$\begin{aligned}
i_x \wedge dx &= 1 + a_1 dx \wedge i_x + a_2 dy \wedge i_y + a_3 dz \wedge i_z, \\
i_x \wedge dy &= a_4 dy \wedge i_x + a_5 dx \wedge i_y, \\
i_x \wedge dz &= a_6 dz \wedge i_x + a_7 dx \wedge i_z, \\
i_y \wedge dx &= a_8 dx \wedge i_y + a_9 dy \wedge i_x, \\
(5.2) \quad i_y \wedge dy &= 1 + a_{10} dy \wedge i_y + a_{11} dx \wedge i_x + a_{12} dz \wedge i_z, \\
i_y \wedge dz &= a_{13} dz \wedge i_y + a_{14} dy \wedge i_z, \\
i_z \wedge dx &= a_{15} dx \wedge i_z + a_{16} dz \wedge i_x, \\
i_z \wedge dy &= a_{17} dy \wedge i_z + a_{18} dz \wedge i_y, \\
i_z \wedge dz &= 1 + a_{19} dz \wedge i_z + a_{20} dx \wedge i_x + a_{21} dy \wedge i_y.
\end{aligned}$$

Applying i_x , i_y and i_z to the relations (2.7) one gets

$$\begin{aligned}
A_1 &= j^2, & A_2 &= 0, & A_3 &= 0, & A_4 &= q^{-1}, \\
A_5 &= 0, & A_6 &= q^{-1}, & A_7 &= 0, & A_8 &= qj^2, \\
A_9 &= 0, & A_{10} &= j^2, & A_{11} &= j^2 - 1, & A_{12} &= 0, \\
A_{13} &= q^{-1}, & A_{14} &= 0, & A_{15} &= qj^2, & A_{16} &= 0, \\
A_{17} &= qj^2, & A_{18} &= 0, & A_{19} &= j^2, & A_{20} &= j^2 - 1, & A_{21} &= j^2 - 1,
\end{aligned}$$

and

$$\begin{aligned}
a_3(qA_1 - A_{15}) &= 0, & A_2a_9 - a_2A_9 &= 0, & A_2a_{12} &= 0, \\
A_3(a_{15} - qa_1) &= 0, & A_3a_{16} - a_3A_{16} &= 0, & A_3A_{12} &= 0, \quad etc.
\end{aligned}$$

To find the coefficients a_k ($1 \leq k \leq 21$), we use the expression

$$i_a \circ d - F_1 d \circ i_a = \partial_a, \quad \text{for } a \in \{x, y, z\}$$

For example, using the first relation in (5.1) with the relations (3.7) we obtain

$$F_1 = 1, \quad a_1 = j^2, \quad a_2 = 0 = a_3.$$

Other coefficients can be similarly obtained. Consequently, we have the following commutation relations:

- the commutation relations of the inner derivations with x , y and z

$$\begin{aligned}
(5.3) \quad i_x x &= j^2 x i_x, & i_x y &= q^{-1} y i_x, & i_x z &= q^{-1} z i_x, \\
i_y x &= qj^2 x i_y, & i_y y &= j^2 y i_y + (j^2 - 1)x i_x, & i_y z &= q^{-1} z i_y, \\
i_z x &= qj^2 x i_z, & i_z y &= qj^2 y i_z, \\
i_z z &= j^2 z i_z + (j^2 - 1)x i_x + (j^2 - 1)y i_y.
\end{aligned}$$

- the commutation relations between the first order differentials and the inner derivations

$$\begin{aligned}
(5.4) \quad & i_x \wedge dx = 1 + j^2 dx \wedge i_x, \quad i_x \wedge dy = q^{-1} dy \wedge i_x, \\
& i_x \wedge dz = q^{-1} dz \wedge i_x, \quad i_y \wedge dx = qj^2 dx \wedge i_y, \\
& i_y \wedge dy = 1 + j^2 dy \wedge i_y + (j^2 - 1) dx \wedge i_x, \\
& i_y \wedge dz = q^{-1} dz \wedge i_y, \quad i_z \wedge dx = qj^2 dx \wedge i_z, \\
& i_z \wedge dy = qj^2 dy \wedge i_z, \\
& i_z \wedge dz = 1 + j^2 dz \wedge i_z + (j^2 - 1) dx \wedge i_x + (j^2 - 1) dy \wedge i_y.
\end{aligned}$$

- the commutation relations between the second order differentials and the inner derivations

$$\begin{aligned}
(5.5) \quad & i_x \wedge d^2x = j^2 d^2x \wedge i_x + (j^2 - j) d^2y \wedge i_y + (j^2 - j) d^2z \wedge i_z, \\
& i_x \wedge d^2y = q^{-1} j^2 d^2y \wedge i_x, \quad i_x \wedge d^2z = q^{-1} j^2 d^2z \wedge i_x, \\
& i_y \wedge d^2x = qj d^2x \wedge i_y, \\
& i_y \wedge d^2y = j^2 d^2y \wedge i_y + (j^2 - j) d^2z \wedge i_z, \\
& i_y \wedge d^2z = q^{-1} j^2 d^2z \wedge i_y, \quad i_z \wedge d^2x = qj d^2x \wedge i_z, \\
& i_z \wedge d^2y = qj d^2y \wedge i_z, \quad i_z \wedge d^2z = j^2 d^2z \wedge i_z.
\end{aligned}$$

- the relations of the inner derivations with the partial derivatives $\partial_x, \partial_y, \partial_z$

$$\begin{aligned}
(5.6) \quad & i_x \partial_x = j \partial_x i_x, \quad i_x \partial_y = q \partial_y i_x, \quad i_x \partial_z = q \partial_z i_x, \\
& i_y \partial_x = q^{-1} j \partial_x i_y + (j - 1) \partial_y i_x, \\
& i_y \partial_y = j \partial_y i_y, \quad i_y \partial_z = q \partial_z i_y, \\
& i_z \partial_x = q^{-1} j \partial_x i_z + (j - 1) \partial_z i_x, \\
& i_z \partial_y = q^{-1} j \partial_y i_z + (j - 1) \partial_z i_y, \\
& i_z \partial_z = j \partial_z i_z.
\end{aligned}$$

5.2. Lie derivatives. In this section we find the commutation rules of the Lie derivatives with functions, *i.e.* the elements of the algebra \mathcal{A} , their differentials, etc., using the approach of [5].

We know, from the classical differential geometry, that the Lie derivative \mathcal{L} can be defined as a linear map from the exterior algebra into itself which takes k -forms to k -forms. For a 0-form, that is, an ordinary function f , the Lie derivative is just the contraction of the exterior derivative with the vector field X :

$$\mathcal{L}_X f = i_X df.$$

For a general differential form, the Lie derivative is likewise a contraction, taking into account the variation in X :

$$\mathcal{L}_X \alpha = i_X d\alpha + d(i_X \alpha).$$

For the \mathbb{Z}_3 -graded differential form, the Lie derivative is obtained as the following formula

$$\mathcal{L}_a \alpha = i_a d\alpha - jd(i_a \alpha).$$

For example, if we apply this formula to the first relation in (5.3), using the relations (5.4) we get

$$\begin{aligned}\mathcal{L}_x x &= (i_x \mathbf{d} - j \mathbf{d} i_x) x \\ &= 1 + j^2 x \{i_x \mathbf{d} - j \mathbf{d} i_x\} + (j^2 - 1) \mathbf{d} x \wedge i_x \\ &= 1 + j^2 x \mathcal{L}_x + (j^2 - 1) \mathbf{d} x \wedge i_x.\end{aligned}$$

Other relations can be similarly obtained. Consequently, we have the following commutation relations:

- the relations between the Lie derivatives and the elements of \mathcal{A}

$$\begin{aligned}(5.7) \quad \mathcal{L}_x x &= 1 + j^2 x \mathcal{L}_x + (j^2 - 1) \mathbf{d} x \wedge i_x, \\ \mathcal{L}_x y &= q^{-1} y \mathcal{L}_x + q^{-1} (1 - j) \mathbf{d} y \wedge i_x, \\ \mathcal{L}_x z &= q^{-1} z \mathcal{L}_x + q^{-1} (1 - j) \mathbf{d} z \wedge i_x, \\ \mathcal{L}_y x &= q j^2 x \mathcal{L}_y + q (j^2 - 1) \mathbf{d} x \wedge i_y, \\ \mathcal{L}_y y &= 1 + j^2 y \mathcal{L}_y + (j^2 - 1) x \mathcal{L}_x \\ &\quad + (j^2 - 1) \mathbf{d} y \wedge i_y + (1 - j^2) (j - 1) \mathbf{d} x \wedge i_x, \\ \mathcal{L}_y z &= q^{-1} z \mathcal{L}_y + q^{-1} (1 - j) \mathbf{d} z \wedge i_y, \\ \mathcal{L}_z x &= q j^2 x \mathcal{L}_z + q (j^2 - 1) \mathbf{d} x \wedge i_z, \\ \mathcal{L}_z y &= q j^2 y \mathcal{L}_z + q (j^2 - 1) \mathbf{d} y \wedge i_z, \\ \mathcal{L}_z z &= 1 + j^2 z \mathcal{L}_z + (j^2 - 1) x \mathcal{L}_x + (j^2 - 1) y \mathcal{L}_y + (j^2 - 1) \mathbf{d} z \wedge i_z \\ &\quad + (1 - j^2) (j - 1) \mathbf{d} x \wedge i_x + (1 - j^2) (j - 1) \mathbf{d} y \wedge i_y.\end{aligned}$$

- The relations of the Lie derivatives with the first order differentials

$$\begin{aligned}(5.8) \quad \mathcal{L}_x \mathbf{d} x &= \mathbf{d} x \mathcal{L}_x, \quad \mathcal{L}_x \mathbf{d} y = q^{-1} j \mathbf{d} y \mathcal{L}_x, \quad \mathcal{L}_x \mathbf{d} z = q^{-1} j \mathbf{d} z \mathcal{L}_x, \\ \mathcal{L}_y \mathbf{d} x &= q \mathbf{d} x \mathcal{L}_y, \quad \mathcal{L}_y \mathbf{d} y = \mathbf{d} y \mathcal{L}_y + (1 - j) \mathbf{d} x \mathcal{L}_x, \\ \mathcal{L}_y \mathbf{d} z &= q^{-1} j \mathbf{d} z \mathcal{L}_y, \quad \mathcal{L}_z \mathbf{d} x = q \mathbf{d} x \mathcal{L}_z, \quad \mathcal{L}_z \mathbf{d} y = q \mathbf{d} y \mathcal{L}_z, \\ \mathcal{L}_z \mathbf{d} z &= \mathbf{d} z \mathcal{L}_z + (1 - j) \mathbf{d} x \mathcal{L}_x + (1 - j) \mathbf{d} y \mathcal{L}_y.\end{aligned}$$

- The relations of the Lie derivatives with the second order differentials

$$\begin{aligned}(5.9) \quad \mathcal{L}_x \mathbf{d}^2 x &= j \mathbf{d}^2 x \mathcal{L}_x + (j - 1) \mathbf{d}^2 y \mathcal{L}_y + (j - 1) \mathbf{d}^2 z \mathcal{L}_z, \\ \mathcal{L}_x \mathbf{d}^2 y &= q^{-1} j \mathbf{d}^2 y \mathcal{L}_x, \quad \mathcal{L}_x \mathbf{d}^2 z = q^{-1} j \mathbf{d}^2 z \mathcal{L}_x, \\ \mathcal{L}_y \mathbf{d}^2 x &= q \mathbf{d}^2 x \mathcal{L}_y, \quad \mathcal{L}_y \mathbf{d}^2 y = j \mathbf{d}^2 y \mathcal{L}_y + (j - 1) \mathbf{d}^2 z \mathcal{L}_z, \\ \mathcal{L}_y \mathbf{d}^2 z &= q^{-1} j \mathbf{d}^2 z \mathcal{L}_y, \quad \mathcal{L}_z \mathbf{d}^2 x = q \mathbf{d}^2 x \mathcal{L}_z, \\ \mathcal{L}_z \mathbf{d}^2 y &= q \mathbf{d}^2 y \mathcal{L}_z, \quad \mathcal{L}_z \mathbf{d}^2 z = j \mathbf{d}^2 z \mathcal{L}_z.\end{aligned}$$

Other commutation relations can be similarly obtained. To complete the description of the above scheme, we get below the remaining commutation relations as follows:

- the Lie derivatives and partial derivatives

$$\begin{aligned}
(5.10) \quad & \mathcal{L}_x \partial_x = \partial_x \mathcal{L}_x, \quad \mathcal{L}_x \partial_y = qj^2 \partial_y \mathcal{L}_x, \quad \mathcal{L}_x \partial_z = qj^2 \partial_z \mathcal{L}_x, \\
& \mathcal{L}_y \partial_x = q^{-1} \partial_x \mathcal{L}_y + (1 - j^2) \partial_y \mathcal{L}_x, \\
& \mathcal{L}_y \partial_y = \partial_y \mathcal{L}_y, \quad \mathcal{L}_y \partial_z = qj^2 \partial_z \mathcal{L}_y, \\
& \mathcal{L}_z \partial_x = q^{-1} \partial_x \mathcal{L}_z + (1 - j^2) \partial_z \mathcal{L}_x, \\
& \mathcal{L}_z \partial_y = q^{-1} \partial_y \mathcal{L}_z + (1 - j^2) \partial_z \mathcal{L}_y, \\
& \mathcal{L}_z \partial_z = \partial_z \mathcal{L}_z.
\end{aligned}$$

- the inner derivations

$$\begin{aligned}
(5.11) \quad & i_x \wedge i_y = qj^2 i_y \wedge i_x, \quad i_x \wedge i_x \wedge i_x = 0, \\
& i_x \wedge i_z = qj^2 i_z \wedge i_x, \quad i_y \wedge i_y \wedge i_y = 0, \\
& i_y \wedge i_z = qj^2 i_z \wedge i_y, \quad i_z \wedge i_z \wedge i_z = 0.
\end{aligned}$$

- the Lie derivatives and the inner derivations

$$\begin{aligned}
(5.12) \quad & i_x \mathcal{L}_x = \mathcal{L}_x i_x, \quad i_x \mathcal{L}_y = q \mathcal{L}_y i_x, \quad i_x \mathcal{L}_z = q \mathcal{L}_z i_x, \\
& i_y \mathcal{L}_x = q^{-1} j^2 \mathcal{L}_x i_y, \quad i_y \mathcal{L}_y = \mathcal{L}_y i_y, \quad i_y \mathcal{L}_z = q \mathcal{L}_z i_y, \\
& i_z \mathcal{L}_x = q^{-1} j^2 \mathcal{L}_x i_z, \quad i_z \mathcal{L}_y = q^{-1} j^2 \mathcal{L}_y i_z, \quad i_z \mathcal{L}_z = \mathcal{L}_z i_z.
\end{aligned}$$

- the Lie derivatives

$$(5.13) \quad \mathcal{L}_x \mathcal{L}_y = qj \mathcal{L}_y \mathcal{L}_x, \quad \mathcal{L}_x \mathcal{L}_z = qj \mathcal{L}_z \mathcal{L}_x, \quad \mathcal{L}_y \mathcal{L}_z = qj \mathcal{L}_z \mathcal{L}_y.$$

Appendix: Quantum matrices in \mathbb{Z}_3 -graded space

In this appendix we shall investigate the quantum matrices in \mathbb{Z}_3 -graded quantum space. We know, from section 2, that the \mathbb{Z}_3 -graded quantum space is generated by coordinates x, y and z , and the commutation rules (2.1), which we repeat here,

$$(5.14) \quad xy - qyx = 0, \quad yz - qzy = 0, \quad xz - qzx = 0.$$

These relations define a deformation of the algebra of functions on the space generated by x, y, z and we have denoted it by \mathcal{A} . The \mathbb{Z}_3 -graded dual quantum space \mathcal{A}^* is generated by dx, dy and dz with the relations

$$\begin{aligned}
(5.15) \quad & dx \wedge dy = qdy \wedge dx, \quad dx \wedge dx \wedge dx = 0, \\
& dx \wedge dz = qdz \wedge dx, \quad dz \wedge dz \wedge dz = 0, \\
& dy \wedge dz = qdz \wedge dy, \quad dy \wedge dy \wedge dy = 0.
\end{aligned}$$

Let T be a 3×3 matrix in \mathbb{Z}_3 -graded space,

$$(5.16) \quad T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where a_{ij} $i, j = 1, 2, 3$ with respect to the \mathbb{Z}_3 -grading are of grade 0.

We now consider linear transformations with the following properties:

$$(5.17) \quad T : \mathcal{A} \longrightarrow \mathcal{A}, \quad T : \mathcal{A}^* \longrightarrow \mathcal{A}^*.$$

The action on the elements of \mathcal{A} of T is $\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We assume that the entries of T are commutative with the elements of \mathcal{A} . As a consequence

of the linear transformations in (5.17) the elements

$$(5.18) \quad \begin{aligned} \hat{x} &= a_{11}x + a_{12}y + a_{13}z, \\ \hat{y} &= a_{21}x + a_{22}y + a_{23}z, \\ \hat{z} &= a_{31}x + a_{32}y + a_{33}z \end{aligned}$$

should satisfy the relations (5.14). Applying the exterior differential d to the relations (5.18) one has

$$(5.19) \quad \begin{aligned} d\hat{x} &= a_{11}dx + a_{12}dy + a_{13}dz, \\ d\hat{y} &= a_{21}dx + a_{22}dy + a_{23}dz, \\ d\hat{z} &= a_{31}dx + a_{32}dy + a_{33}dz. \end{aligned}$$

These elements must satisfy the relations (5.15). Consequently, we have the following commutation relations between the matrix elements of T :

$$(5.20) \quad \begin{aligned} a_{ij}a_{ik} &= qa_{ik}a_{ij}, \quad j < k, \\ a_{ji}a_{ki} &= qa_{ki}a_{ji}, \quad j < k, \\ a_{ik}a_{lj} &= a_{lj}a_{ik}, \quad i < l \quad \text{and} \quad j > k, \\ a_{ij}a_{lk} - a_{lk}a_{ij} &= (q - q^{-1})a_{ik}a_{lj}, \quad i < l \quad \text{and} \quad j < k. \end{aligned}$$

We know that $GL_q(3)$ is a quantum group with the above relations. We obtained the same relations in our work in Z_3 -graded space. But there is a difference from $GL_q(3)$ because of the property of the q -parameter. In our work q should satisfy the following relations:

$$q^2 + q + 1 = 0, \quad q^3 = 1.$$

An interesting problem is the construction of a differential calculus on the Z_3 -graded quantum group $GL_{q,j}(3|0)$ using the methods of this paper and [3]. Work on this issue is in progress.

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