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SOME INEQUALITIES RELATED TO THE PRINGSHEIM, STATISTICAL AND σ-CORES OF DOUBLE SEQUENCES

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Abstract

The statistical convergence of double sequences was presented by Mursaleen-Edely and Tripathy in two ways, [10, 16]. The statistical core of double sequences was introduced by Çakan- Altay, [1]. The σ-convergence and σ-core of double sequences were defined by Çakan-Altay-Mursaleen, [5]. In this paper, we will study some new inequalities related to the Pringsheim, statistical and σ-cores of double sequences. To achieve this goal, we will characterize some classes of four-dimensional matrices.

Keywords: Double sequences, four dimensional matrices, core theorems and matrix transformations.

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1. Introduction

Let ℓ_∞ and c be the Banach spaces of real bounded and convergent sequences with the usual supremum norm. Let σ be a one-to-one mapping from $\mathbb{N}$, the set of natural numbers, into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
(ii) $\phi(e) = 1$, where $e = (1,\ 1,\ 1, \ldots)$,
(iii) $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping σ having no finite orbits, that is $\sigma^p(k) \neq k$ for all positive integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ is pth iterate of σ at k.

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Thus, a σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$, [9]. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal.

In the case $\sigma(k) = k+1$, a σ-mean often called a Banach limit and V_σ is the set f of almost convergent sequences, introduced by Lorentz, [6]. It can be shown [14] that

$$V_\sigma = \{x \in \ell_\infty : \lim_p t_{pn}(x) = s \text{ uniformly in } n,\ s = \sigma - \lim x\}$$

where

$$t_{pn}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^p(n)}}{p+1}\ ,\quad t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$. We denote by Z the subset of V_σ consisting of all sequences with σ-limit zero. It is well-known [14] that $x \in \ell_\infty$ if and only if $(\phi(x) - x) \in Z$ and $V_\sigma = Z \oplus \mathbb{R}e$.

A double sequence $x = [x_{j,k}]_{j,k=0}^{\infty}$ is said to be convergent to a number l in the Pringsheim sense or P-convergent if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{j,k} - l| < \varepsilon$ whenever $j,k > N$, [13]. In this case, we write $P - \lim x = l$. In what follows, we will write $[x_{j,k}]$ in place of $[x_{j,k}]_{j,k=0}^{\infty}$. In [11], the concepts P-limit superior and inferior were defined for a double sequence x and the Pringsheim core (P-core) of such a sequence given as $[P - \liminf x, P - \limsup x]$.

A double sequence x is said to be bounded if there exists a positive number M such that $|x_{j,k}| < M$ for all j,k, i.e., if

$$\|x\| = \sup_{j,k} |x_{j,k}| < \infty.$$

Let ℓ_∞^2 be the space of all real bounded double sequences. We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. By c_2^∞, we mean the space of all P-convergent and bounded double sequences.

Let $A = [a_{m,n,j,k}]_{j,k=0}^{\infty}$ be a four dimensional infinite matrix of real numbers for all $m,n = 0,1,\ldots$. The sums

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,n,j,k} x_{j,k}$$

are called the A-transforms of the double sequence x and denoted by Ax. We say that a sequence x is A-summable to the limit s if the A-transform of x exists for all $m,n = 0,1,\ldots$ and convergent in the Pringsheim sense, i.e.,

$$\lim_{p,q\to\infty} \sum_{j=0}^{p}\sum_{k=0}^{q} a_{m,n,j,k} x_{j,k} = y_{m,n}$$

and

$$\lim_{m,n\to\infty} y_{m,n} = s.$$

We say that the matrix $A = [a_{m,n,j,k}]$ is conservative if $x \in c_2^\infty$ implies that $Ax \in c_2^\infty$. In this case, we write $A \in (c_2^\infty, c_2^\infty)$. If A is conservative and $P - \lim Ax = P - \lim x$ for all $x \in c_2^\infty$, we call the matrix A RH-regular and write $A \in (c_2^\infty, c_2^\infty)_{reg}$.

It is known [2] that A is conservative if and only if

(1.1) $$P-\lim_{m,n} a_{m,n,j,k} = v_{j,k} \text{ for each } j,k;$$

(1.2) $$P-\lim_{m,n} \sum_j \sum_k a_{m,n,j,k} = v;$$

(1.3) $$P-\lim_{m,n} \sum_j |a_{m,n,j,k}| = v_k \text{ for each } k;$$

(1.4) $$P-\lim_{m,n} \sum_k |a_{m,n,j,k}| = v_j \text{ for each } j;$$

(1.5) $$P-\lim_{m,n} \sum_j \sum_k |a_{m,n,j,k}| \text{ exists};$$

(1.6) $$\|A\| = \sup_{m,n} \sum_j \sum_k |a_{m,n,j,k}| < \infty.$$

It can be also seen [8, 15] that A is RH-regular if and only if (1.1) with $v_{jk}=0$, (1.2) with $v=1$, (1.3) and (1.4) with $v_k=v_j=0$, and the conditions (1.5), (1.6), hold.

For a conservative matrix A, we can define the functional

$$\Gamma(A) = v - \sum_j \sum_k v_{j,k}.$$

In the case A is a RH-regular, $\Gamma(A)=1$.

Let $E \subseteq \mathbb{N}\times\mathbb{N}$ and $E(m,n)=\{(j,k): j\le m, k\le n\}$. Then, the double natural density of E is defined by

$$\delta_2(E) = P-\lim_{m,n} \frac{|E(m,n)|}{mn}$$

if the limit on the right hand side exists; where the vertical bars denotes the cardinality of the set $E(m,n)$.

A real double sequence $x=[x_{j,k}]$ is said to be statistical (or briefly st-) convergent [10, 16] to the number L if for every $\varepsilon>0$, the set $\{(j,k): |x_{j,k}-L|>\varepsilon\}$ has double natural density zero. In this case, we write $\mathrm{st}_2\text{-}\lim x = L$. Let st_2 be the space of all st-convergent double sequences. Clearly, a convergent double sequence is also st-convergent but the converse it is not true, in general. Also, note that a st-convergent double sequence need not be bounded. For example, consider the sequence $x=[x_{j,k}]$ defined by

(1.7) $$x_{j,k} = \begin{cases} jk & , \quad \text{if } j \text{ and } k \text{ are squares,} \\ 1 & , \quad \text{otherwise.} \end{cases}$$

Then, clearly $st_2-\lim x = 1$. Nevertheless x neither convergent nor bounded. The $st_2-\limsup$ and $st_2-\liminf$ of a double sequence were introduced in [1] and also the statistical core of a double sequence was defined by the closed interval $[st_2-\limsup, st_2-\liminf]$.

The σ-convergence of double sequences was introduced in [5] as follows:

A double sequence $x=[x_{j,k}]$ of real numbers is said to be σ-convergent to a limit l if

$$\lim_{p,q} \frac{1}{(p+1)(q+1)} \sum_{s=0}^{p} \sum_{t=0}^{q} x_{\sigma^s(j),\sigma^t(k)} = l \text{ uniformly in } j,k.$$

Also, the σ-core of double sequences was defined in the same paper as the closed interval $[-C_\sigma(-x), C_\sigma(x)]$, where

$$C_\sigma(x) = \limsup_{p,q} \sup_{j,k} \frac{1}{(p+1)(q+1)} \sum_{s=0}^{p} \sum_{t=0}^{q} x_{\sigma^s(j),\sigma^t(k)}.$$

let V_σ^2 be the space of all bounded and σ-convergent double sequences and $Z_\sigma^2 \subset V_\sigma^2$ be the space of all sequences with σ-limit zero.

Some concepts related to the single sequences were extended to the double sequences. For example, we can refer [1, 2, 3, 5, 7, 12, 17, 18] and some others.

In this paper, we will study some new inequalities related to the Pringsheim, statistical and σ-cores of double sequences. To achieve this goal, we also characterize some classes of four-dimensional matrices in the following lemmas.

2. Lemmas

First of all, we shall quote some known results.

2.1. Lemma. [4, Th. 2.1] *$A \in (\ell_\infty^2, Z_\sigma^2)$ if and only if (1.6) holds and*

$$\lim_{p,q\to\infty} \alpha(p,q,j,k,s,t) = 0 \text{ for each } j,k, \tag{2.1}$$

$$\lim_{p,q\to\infty} \sum_j |\alpha(p,q,j,k,s,t)| = 0 \text{ for each } k, \tag{2.2}$$

$$\lim_{p,q\to\infty} \sum_k |\alpha(p,q,j,k,s,t)| = 0 \text{ for each } j, \tag{2.3}$$

$$\lim_{p,q\to\infty} \sum_j \sum_k |\alpha(p,q,j,k,s,t)| = 0, \tag{2.4}$$

where the limits are uniformly in s,t and

$$\alpha(p,q,j,k,s,t) = \frac{1}{(p+1)(q+1)} \sum_{m=0}^{p} \sum_{n=0}^{q} a_{\sigma^m(s),\sigma^n(t),j,k}.$$

2.2. Lemma. [3, Th. 2.1] *$A \in (c_2^\infty, V_\sigma^2)_{reg}$ if and only if (1.6), (2.1), (2.2), (2.3) hold and*

$$\lim_{p,q\to\infty} \sum_j \sum_k \alpha(p,q,j,k,s,t) = 1, \tag{2.5}$$

$$\lim_{p,q\to\infty} \sum_j \sum_k |\alpha(p,q,j,k,s,t)| \text{ exists}, \tag{2.6}$$

where the limits are uniformly in s,t.

One can prove that $A \in (c_2^\infty, V_\sigma^2)$ if and only if the conditions (1.6), (2.2), (2.3, (2.6) hold and

$$\lim_{p,q\to\infty} \alpha(p,q,j,k,s,t) = u_{jk}$$
$$\lim_{p,q\to\infty} \sum_j \sum_k \alpha(p,q,j,k,s,t) = u$$

uniformly in s,t. If $A \in (c_2^\infty, V_\sigma^2)$, we can define the functional

$$\Gamma_\sigma(A) = u - \sum_j \sum_k u_{j,k}.$$

Note that in the case $A \in (c_2^\infty, V_\sigma^2)_{reg}$, $\Gamma_\sigma(A) = 1$.

The class of matrices $(V_\sigma^2, c_2^\infty)_{reg}$ was characterized in Theorem 3.3 of [5]. A little generalization of that result is:

2.3. Lemma. *$A \in (V_\sigma^2, c_2^\infty)$ if and only if A is RH-conservative and*

$$P - \lim_{m,n} \sum_j \sum_k |\Delta_{10}| = 0, \tag{2.7}$$

$$P - \lim_{m,n} \sum_j \sum_k |\Delta_{01}| = 0, \tag{2.8}$$

where $\Delta_{10} = a_{m,n,j,k} - a_{m,n,\sigma(j),k} - v_{j,k} + v_{\sigma(j),k}$ and $\Delta_{01} = a_{m,n,j,k} - a_{m,n,j,\sigma(k)} - v_{j,k} + v_{j,\sigma(k)}$.

2.4. Lemma. [2, Lemma 2.1] *Let A be matrix such that the conditions (1.6), (1.1) with $v_{j,k} = 0$, (1.3) with $v_k = 0$ and (1.4) with $v_j = 0$ hold. Then, for any $y \in \ell_\infty^2$ such that $\|y\| \leq 1$ we have*

$$P - \limsup_{m,n} \sum_j \sum_k a_{m,n,j,k} y_{j,k} = P - \limsup_{m,n} \sum_j \sum_k |a_{m,n,j,k}|. \tag{2.9}$$

2.5. Lemma. [2, Lemma 2.2] *Let $A = [a_{m,n,j,k}]$ be RH-conservative and $\lambda \in \mathbb{R}^+$. Then,*

$$P - \limsup_{m,n} \sum_j \sum_k |a_{m,n,j,k} - v_{j,k}| \leq \lambda \tag{2.10}$$

if and only if

$$P - \limsup_{m,n} \sum_j \sum_k (a_{m,n,j,k} - v_{j,k})^+ \leq \frac{\lambda + \Gamma(A)}{2}$$

and

$$P - \limsup_{m,n} \sum_j \sum_k (a_{m,n,j,k} - v_{j,k})^- \leq \frac{\lambda - \Gamma(A)}{2};$$

where for any $\gamma \in \mathbb{R}$, $\gamma^+ = \max\{0, \gamma\}$ and $\gamma^- = \max\{-\gamma, 0\}$.

2.6. Lemma. [2, Th. 2.3] *Let $A = [a_{m,n,j,k}]$ be RH-conservative. Then, for some constant $\lambda \geq |\Gamma(A)|$ and for all $x \in \ell_\infty^2$, one has*

$$P - \limsup_{m,n} \sum_j \sum_k (a_{m,n,j,k} - v_{j,k}) x_{j,k} \leq \frac{\lambda + \Gamma(A)}{2} L(x) - \frac{\lambda - \Gamma(A)}{2} l(x) \tag{2.11}$$

if and only if (2.10) holds.

2.7. Lemma. *$A \in (st_2 \cap \ell_\infty^2, V_\sigma^2)$ if and only if $A \in (c_2^\infty, V_\sigma^2)$ and*

$$\lim_{p,q\to\infty} \sum_{j\in E} \sum_{k\in E} \left|\alpha(p,q,j,k,s,t) - u_{j,k}\right| = 0 \tag{2.12}$$

for every $E \subseteq \mathbb{N} \times \mathbb{N}$ with $\delta_2(E) = 0$.

Proof. Let $A \in (st_2 \cap \ell_\infty^2, V_\sigma^2)$. Then, since $c_2^\infty \subset st_2 \cap \ell_\infty^2$, $A \in (c_2^\infty, V_\sigma^2)$. Now, let us define a sequence $z = [z_{j,k}]$ by using a sequence $x = [x_{j,k}] \in \ell_\infty^2$ as follows

$$z_{j,k} = \begin{cases} x_{j,k} & , \quad \text{if } j,k \in E, \\ 0 & , \quad \text{otherwise} \end{cases}$$

where $E \subseteq \mathbb{N} \times \mathbb{N}$ with $\delta_2(E) = 0$. Then, it is clear that $z \in st_2$ and $st_2 - \lim z = 0$. So, by the assumption, $Az \in Z_\sigma^2$. On the other hand, since

$$Az = \sum_j \sum_k a_{m,n,j,k} x_{j,k},$$

the matrix $B = [b_{m,n,j,k}]$ defined by, for all $m, n \in \mathbb{N}$,

$$b_{m,n,j,k} = \begin{cases} a_{m,n,j,k} - u_{j,k} & , \quad \text{if } j,k \in E, \\ 0 & , \quad \text{otherwise} \end{cases}$$

is in the class $(\ell^2_\infty, Z^2_\sigma)$. Thus, the condition (2.12) follows from Lemma 2.1.

To the sufficiency, suppose that $A \in (c^\infty_2, V^2_\sigma)$ and the condition (2.12) holds. Choose a sequence $x \in st_2 \cap \ell^2_\infty$ with $st_2 - \lim x = L$, say. Then, for any $\varepsilon > 0$, $\delta_2(E) = \delta_2(\{(j,k) : |x_{j,k} - L| > \varepsilon\}) = 0$ and $|x_{j,k} - L| \le \varepsilon$ whenever $j,k \ne E$. Now, we can write

$$\sum_j \sum_k a_{m,n,j,k} x_{j,k} = \sum_j \sum_k a_{m,n,j,k}(x_{j,k} - L) + L \sum_j \sum_k a_{m,n,j,k}.$$

Since $A \in (c^\infty_2, V^2_\sigma)$, by Lemma 2.2

$$\begin{aligned} \sigma - \lim \sum_j \sum_k a_{m,n,j,k} x_{j,k} &= P - \lim_{p,q} \sum_j \sum_k \alpha(p,q,j,k,s,t) x_{j,k} \\ &= P - \lim_{p,q} \sum_j \sum_k \alpha(p,q,j,k,s,t)(x_{j,k} - L) + L\alpha. \end{aligned}$$

On the other hand, since

$$\begin{aligned} &\Big| \sum_j \sum_k \alpha(p,q,j,k,s,t)(x_{j,k} - L) - \sum_j \sum_k v_{j,k}(x_{j,k} - L) \Big| \\ &= \Big| \sum_j \sum_k [\alpha(p,q,j,k,s,t) - v_{j,k}](x_{j,k} - L) \Big| \\ &\le \|x_{j,k} - L\| \sum_{j,k \in E} |\alpha(p,q,j,k,s,t) - v_{j,k}| + \varepsilon \|A\|, \end{aligned}$$

from (2.12) we get that

$$\lim_{p,q} \sum_j \sum_k \alpha(p,q,j,k,s,t)(x_{j,k} - L) = \sum_j \sum_k v_{j,k}(x_{j,k} - L).$$

This completes the proof. □

3. The Main Results

3.1. Theorem. *Let $A = [a_{m,n,j,k}]$ be RH-conservative. Then, for some constant $\lambda \ge |\Gamma(A)|$ and for all $x \in \ell^2_\infty$, we have*

$$P - \limsup_{m,n} \sum_j \sum_k (a_{m,n,j,k} - v_{j,k}) x_{j,k} \le \frac{\lambda + \Gamma(A)}{2} C_\sigma(x) + \frac{\lambda + \Gamma(A)}{2} C_\sigma(-x) \tag{3.1}$$

if and only if (2.7), (2.8) and (2.10) hold.

Proof. Firstly, let (3.1) holds. Then, since $C_\sigma(x) \le L(x)$ and $C_\sigma(-x) \le -l(x)$ for all $x \in \ell^2_\infty$, the necessity of (2.10) follows from Lemma 2.6. Now, for all $m,n,j,k \in \mathbb{N}$, define a matrix $B = [b_{m,n,j,k}]$ by $b_{m,n,j,k} = a_{m,n,j,k} - v_{j,k}$ and then a matrix $C = [c_{m,n,j,k}]$ with $c^{mn}_{jk} = (b_{m,n,j,k} - b_{m,n,\sigma(j),k})$. Then, C satisfies the hypothesis of Lemma 2.4. Hence, for a $y \in \ell^2_\infty$ such that $\|y\| \le 1$, we have (2.9) with $c_{m,n,j,k}$ in place of $a_{m,n,j,k}$. Also, for the same y, as in Theorem 3.3 in [5] we can write

$$\sum_j \sum_k c_{m,n,j,k} y_{\sigma(j),k} = \sum_j \sum_k b_{m,n,j,k}(y_{j,k} - y_{\sigma(j),k}). \tag{3.2}$$

So, we have from (3.1) that

$$P-\limsup_{m,n}\sum_j\sum_k|c_{m,n,j,k}|=P-\limsup_{m,n}\sum_j\sum_k b_{m,n,j,k}(y_{j,k}-y_{\sigma(j),k})$$
$$\le\frac{\lambda+\Gamma(A)}{2}C_\sigma(y_{j,k}-y_{\sigma(j),k})+\frac{\lambda+\Gamma(A)}{2}C_\sigma(y_{\sigma(j),k}-y_{j,k}).$$

Since $(y_{j,k}-y_{\sigma(j),k})\in Z_\sigma^2$, we get the necessity of (2.7). By the same argument one can prove the necessity of (2.8).

Conversely, suppose that (2.7), (2.8) and (2.10) hold. For any $x\in\ell_\infty^2$, let us write again (3.2). Since $(x_{j,k}-x_{\sigma(j),k})\in Z_\sigma^2$,

$$P-\lim_{m,n}\sum_j\sum_k b_{m,n,j,k}(x_{j,k}-x_{\sigma(j),k})=0.$$

Thus, by taking infimum over $z\in Z_\sigma^2$ in (2.11), we get that

$$\inf_{z\in Z_\sigma^2}\left\{P-\limsup_{m,n}\sum_j\sum_k b_{m,n,j,k}(x_{j,k}+z_{j,k})\right\}\le\frac{\lambda+\Gamma(A)}{2}\inf_{z\in Z_\sigma^2}L(x+z)-$$
$$\frac{\lambda-\Gamma(A)}{2}\inf_{z\in Z_\sigma^2}l(x+z)=\frac{\lambda+\Gamma(A)}{2}W_p(x)+\frac{\lambda-\Gamma(A)}{2}W_p(-x).$$

On the other hand, since $P-\lim Bz=0$ for $z\in Z_\sigma^2$,

$$\inf_{z\in Z_\sigma^2}\left\{P-\limsup_{m,n}\sum_j\sum_k b_{m,n,j,k}(x_{j,k}+z_{j,k})\right\}\ge P-\limsup_{m,n}\sum_j\sum_k b_{m,n,j,k}x_{j,k}+$$
$$\inf_{z\in Z_\sigma^2}\left\{P-\limsup_{m,n}\sum_j\sum_k b_{m,n,j,k}z_{j,k}\right\}=P-\limsup_{m,n}\sum_j\sum_k b_{m,n,j,k}x_{j,k}.$$

Where B is as in the part of necessity. Since $W_p(x)=C_\sigma(x)$ for all $x\in\ell_\infty^2$ (see [5]), we conclude that (3.1) holds and the proof is completed. □

In the case $\Gamma(A)>0$ and $\lambda=\Gamma(A)$, we have the following result.

3.2. Theorem. *Let A be RH-conservative and $x\in\ell_\infty^2$. Then,*

$$P-\limsup_{m,n}\sum_j\sum_k(a_{m,n,j,k}-v_{j,k})x_{j,k}\le\Gamma(A)C_\sigma(x)$$

if and only if (2.7), (2.8) hold and

$$P-\limsup_{m,n}\sum_j\sum_k|a_{m,n,j,k}-v_{j,k}|=\Gamma(A). \tag{3.3}$$

We should state that in the case $\sigma(n)=n+1$, Theorems 3.1-3.2 reduces to the Theorems 2.5-2.6 in [2].

3.3. Theorem. *Let $A=[a_{m,n,j,k}]$ be RH-conservative. Then, for some constant $\lambda\ge|\Gamma(A)|$ and for all $x\in\ell_\infty^2$, one has*

$$P-\limsup_{m,n}\sum_j\sum_k(a_{m,n,j,k}-v_{j,k})x_{jk}\le\frac{\lambda+\Gamma(A)}{2}\beta(x)+\frac{\lambda+\Gamma(A)}{2}\alpha(-x) \tag{3.4}$$

if and only if (2.10) holds and

$$P-\limsup_{m,n}\sum_{j\in E}\sum_{k\in E}|a_{m,n,j,k}-v_{j,k}|=0 \tag{3.5}$$

for every $E\in\mathbb{N}\times\mathbb{N}$ with $\delta_2(E)=0$, where $\beta(x)=st_2-\limsup x$ and $\alpha(x)=st_2-\liminf x$.

Proof. Let (3.4) holds. Since $\beta(x) \leq L(x)$ and $\alpha(-x) \leq -l(x)$ (see [1]), we get the necessity of (2.10) from Lemma 2.6. Now, for any $E \in \mathbb{N} \times \mathbb{N}$ with $\delta_2(E) = 0$, let us define a matrix $D = [d_{m,n,j,k}]$ by

$$d_{m,n,j,k} = \begin{cases} a_{m,n,j,k} - v_{j,k} & , \quad \text{if } j,k \in E, \\ 0 & , \quad \text{otherwise;} \end{cases}$$

for all $j,k,m,n \in \mathbb{N}$. Then, the matrix D satisfies the conditions of Lemma 2.4. For the same E, let us choose a sequence $(y_{j,k})$ as follows:

$$y_{j,k} = \begin{cases} 1 & , \quad \text{if } j,k \in \mathbb{N}, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{3.6}$$

Then, since $st_2 - \lim y = \beta(y) = \alpha(y) = 0$, from (3.4) we get that

$$P - \limsup_{m,n} \sum_{j \in E} \sum_{k \in E} |a_{m,n,j,k} - v_{j,k}| \leq \frac{\lambda + \Gamma(A)}{2}\beta(x) + \frac{\lambda + \Gamma(A)}{2}\alpha(-x) = 0.$$

Conversely, suppose that (2.10) and (3.5) hold. Let $E_1 = \{(j,k) : x_{j,k} > \beta(x) + \varepsilon\}$ and $E_2 = \{(j,k) : x_{j,k} < \alpha(x) - \varepsilon\}$. Then, since $\delta_2(E_1) = \delta_2(E_2) = 0$ (see [1]), $\delta_2(E) = \delta_2(E_1 \cap E_2) = 0$. Now, we can write

$$\sum_j \sum_k (a_{m,n,j,k} - v_{j,k})x_{j,k} = \sum_{j \in E} \sum_{k \in E} (a_{m,n,j,k} - v_{j,k})x_{j,k} + \sum_{j \notin E} \sum_{k \notin E} (a_{m,n,j,k} - v_{j,k})^+ x_{j,k} - \sum_{j \notin E} \sum_{k \notin E} (a_{m,n,j,k} - v_{j,k})^- x_{j,k}. \tag{3.7}$$

Now, by (3.5), one can see that the first sum on the right hand side of (3.7) goes to zero. So, from Lemma 2.5 we have (3.4) and the proof is completed. □

In the case $\Gamma(A) > 0$ and $\lambda = \Gamma(A)$, we have the following result.

3.4. Theorem. *Let A be RH-conservative and $x \in \ell_\infty^2$. Then,*

$$P - \limsup_{m,n} \sum_j \sum_k (a_{m,n,j,k} - v_{j,k})x_{j,k} \leq \Gamma(A)\beta(x)$$

if and only if (3.3) and (3.5) hold.

Here, we should note that when A is RH-regular, Theorem 3.4 reduced to the Theorem 3.4 in [1].

3.5. Theorem. *Let $A \in (c_2^\infty, V_\sigma^2)$. Then, for some constant $\lambda \geq |\Gamma_\sigma(A)|$ and for all $x \in \ell_\infty^2$, one has*

$$P - \limsup_{p,q} \sup_{s,t} \sum_j \sum_k (\alpha(p,q,j,k,s,t) - u_{j,k})x_{j,k} \leq \frac{\lambda + \Gamma_\sigma(A)}{2}\beta(x) + \frac{\lambda + \Gamma_\sigma(A)}{2}\alpha(-x) \tag{3.8}$$

(2.12) hold and

$$P - \limsup_{p,q} \sup_{s,t} \sum_j \sum_k |\alpha(p,q,j,k,s,t) - u_{j,k}| \leq \lambda. \tag{3.9}$$

Proof. Let us define a matrix $B = [b_{m,n,j,k}]$ by $b_{m,n,j,k} = (\alpha(p,q,j,k,s,t) - u_{j,k})$. Then, since $A \in (c_2^\infty, V_\sigma^2)$, the matrix B satisfies the hypothesis of Lemma 2.4. So, for a $y \in \ell_\infty^2$ such that $\|y\| \leq 1$, we have (2.9) with $b_{m,n,j,k}$ in place of $a_{m,n,j,k}$. Now, by choosing the sequence $y = (y_{j,k})$ in (3.6) for which $st_2 - \lim y = 0$ we get the necessity of (3.9) from (3.8). On the other hand, the necessity of (2.12) can be obtained easily by using the same method for the necessity of (3.5).

Conversely, let us choose again the set E in Theorem 3.3 and write (3.7) with $b_{m,n,j,k}$ in place of $a_{m,n,j,k}$, where $b_{m,n,j,k}$ is as above. Then, the proof can be seen by the same reasons for the sufficiency of Theorem 3.3. □

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