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A REMARK ON MEROMORPHICALLY MULTIVALENT FUNCTIONS WITH MISSING COEFFICIENTS

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Abstract

The purpose of the present paper is to derive certain properties of meromorphically multivalent functions with missing coefficients.

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1. Introduction

Let $\Sigma_p(n)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^{-p+k} \quad (p, n \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the punctured open unit disk $U^* = \{z : 0 < |z| < 1\} = U \setminus \{0\}$. For functions $f(z)$ and $g(z)$ analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, and we write $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function $w(z)$ in U such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function $g(z)$ is univalent in U, then

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Many important properties and characteristics of various interesting subclasses of the class $\Sigma_p(n)$ of meromorphically multivalent functions were investigated extensively by several authors (see, e.g., [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13]). In this note we shall derive certain properties of meromorphically multivalent functions with missing coefficients.

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2. Main results

Our main result is the following.

2.1. Theorem. *Let $f(z)$ belong to the class $\Sigma_p(n)$ and satisfy*

$$\frac{zf'(z)}{z^{-p}} \prec -p\left(\frac{1+z}{1-z}\right)^{\gamma} \quad (0<\gamma\le 1;\ z\in U). \tag{2.1}$$

Then

$$Re\left\{(1-\delta)\left(\frac{zf'(z)}{-pz^{-p}}\right)^{\frac{1}{\gamma}} + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \rho \quad (|z|<r_n(p,\gamma,\delta,\rho)), \tag{2.2}$$

where $0\le\rho<1$, $0<\delta<\frac{1-\rho}{1+p}$ and $r_n(p,\gamma,\delta,\rho)$ is the smallest root in $(0,1)$ of the equation

$$[1+\rho+(p-1)\delta]r^{2n} - 2(1-\delta+n\delta\gamma)r^n + 1-\rho-(p+1)\delta = 0.$$

The result is sharp.

Proof. From (2.1) we can write

$$\left(\frac{zf'(z)}{-pz^{-p}}\right)^{\frac{1}{\gamma}} = \frac{1+z^n\varphi(z)}{1-z^n\varphi(z)}, \tag{2.3}$$

where $\varphi(z)$ is analytic and $|\varphi(z)|\le 1$ in U. Differentiating both sides of (2.3) logarithmically, we arrive at

$$1+\frac{zf''(z)}{f'(z)} = -p + \frac{2n\gamma z^n\varphi(z)}{1-(z^n\varphi(z))^2} + \frac{2\gamma z^{n+1}\varphi'(z)}{1-(z^n\varphi(z))^2} \quad (z\in U). \tag{2.4}$$

Put $|z|=r<1$ and $\left(\frac{zf'(z)}{-pz^{-p}}\right)^{\frac{1}{\gamma}} = u+iv$ $(u,v\in R)$. Then (2.3) implies that

$$z^n\varphi(z) = \frac{u-1+iv}{u+1+iv} \tag{2.5}$$

and

$$\frac{1-r^n}{1+r^n} \le u \le \frac{1+r^n}{1-r^n}. \tag{2.6}$$

With the help of the Carathéodory inequality(see also [8]):

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-r^2},$$

it follows from (2.5) that

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\delta)\left(\frac{zf'(z)}{-pz^{-p}}\right)^{\frac{1}{\gamma}} + \delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \\
&\ge (1-\delta)u - p\delta + 2n\delta\gamma\mathrm{Re}\left\{\frac{z^n\varphi(z)}{1-(z^n\varphi(z))^2}\right\} - 2\delta\gamma\left|\frac{z^{n+1}\varphi'(z)}{1-(z^n\varphi(z))^2}\right| \\
&\ge (1-\delta)u - p\delta + \frac{n\delta\gamma}{2}\left(u-\frac{u}{u^2+v^2}\right) + \frac{\delta\gamma}{2}\frac{(u-1)^2+v^2-r^{2n}((u+1)^2+v^2)}{r^{n-1}(1-r^2)(u^2+v^2)^{1/2}} \\
&= F_n(u,v) \quad (say)
\end{aligned} \tag{2.7}$$

and

$$\frac{\partial}{\partial v}F_n(u,v) = \delta\gamma v G_n(u,v), \tag{2.8}$$

where $0 < r < 1$, $0 < \delta \leq 1$ and

$$G_n(u,v) = \frac{nu}{(u^2+v^2)^2} + \frac{1-r^{2n}}{r^{n-1}(1-r^2)(u^2+v^2)^{\frac{1}{2}}} + \frac{r^{2n}((u+1)^2+v^2)-((u-1)^2+v^2)}{2r^{n-1}(1-r^2)(u^2+v^2)^{\frac{3}{2}}} > 0 \tag{2.9}$$

because of (2.5) and (2.6). Since $F_n(u,v)$ is a even function of v, from (2.7), (2.8) and (2.9), we see that

$$F_n(u,v) \geq F_n(u,0) = (1-\delta)u + p\delta + \frac{n\delta\gamma}{2}\left(u - \frac{1}{u}\right) + \frac{\delta\gamma}{2r^{n-1}(1-r^2)}\left[(1-r^{2n})\left(u+\frac{1}{u}\right) - 2(1+r^{2n})\right]. \tag{2.10}$$

Let us now calculate the minimum value of $F_n(u,0)$ on the closed interval $\left[\frac{1-r^n}{1+r^n},\ \frac{1+r^n}{1-r^n}\right]$. Noting that

$$\frac{1-r^{2n}}{r^{n-1}(1-r^2)} \geq n \quad \text{(see [12])}$$

and (2.6), we deduce from (2.10) that

$$\begin{aligned} \frac{d}{du}F_n(u,0) &= 1-\delta+\frac{\delta\gamma}{2}\left[\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}+n\right) - \frac{1}{u^2}\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}-n\right)\right] \\ &\geq 1-\delta+\frac{\delta\gamma}{2}\left[\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}+n\right) - \left(\frac{1+r^n}{1-r^n}\right)^2\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}-n\right)\right] = 1-\delta+\frac{2\delta\gamma I_n(r)}{(1-r^n)^2}, \end{aligned} \tag{2.11}$$

where

$$I_n(r) = \frac{n}{2}(1+r^{2n}) - r(1+r^2+\cdots+r^{2n-2}).$$

Also

$$I_n'(r) = n^2r^{2n-1} - (1+3r^2+\cdots+(2n-1)r^{2n-2}).$$

$I_1'(r) = r-1 < 0$. Suppose that $I_n'(r) < 0$. Then

$$\begin{aligned} I_{n+1}'(r) &= (n+1)^2r^{2n+1} - (2n+1)r^{2n} - (1+3r^2+\cdots+(2n-1)r^{2n-2}) \\ &< n^2r^{2n} - (1+3r^2+\cdots+(2n-1)r^{2n-2}) \\ &< I_n'(r) < 0. \end{aligned}$$

Hence, by virtue of the mathematical induction, we have $I_n'(r) < 0$ for all $n \in N$ and $0 \leq r < 1$. This implies that

$$I_n(r) > I_n(1) = 0 \quad (n \in N;\ 0 \leq r < 1). \tag{2.12}$$

In view of (2.11) and (2.12), we see that

$$\frac{d}{du}F_n(u,0) > 0 \quad \left(\frac{1-r^n}{1+r^n} \leq u \leq \frac{1+r^n}{1-r^n}\right). \tag{2.13}$$

Further it follows from (2.7), (2.10) and (2.13) that

$$\begin{aligned}\operatorname{Re}\left\{(1-\delta)\left(\frac{zf'(z)}{-pz^{-p}}\right)^{\frac{1}{\gamma}}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}-\rho &\geq F_n\left(\frac{1-r^n}{1+r^n},0\right)-\rho\\ &= (1-\delta)\frac{1-r^n}{1+r^n}+\delta\frac{-p-2n\gamma r^n+pr^{2n}}{1-r^{2n}}-\rho\\ &= \frac{J_n(r)}{1-r^{2n}}, \qquad (2.14)\end{aligned}$$

where

$$J_n(r)=[1+\rho+(p-1)\delta]r^{2n}-2(1-\delta+n\delta\gamma)r^n+1-\rho-(p+1)\delta.$$

Note that $J_n(0)=1-\rho-(p+1)\delta>0$ and $J_n(1)=-2n\delta\gamma<0$. If we let $r_n(p,\gamma,\delta,\rho)$ denote the smallest root in $(0,1)$ of the equation $J_n(r)=0$, then (2.14) yields the desired result (2.2).

To see that the bound $r_n(p,\gamma,\delta,\rho)$ is the best possible, we consider the function

$$f(z)=-p\int_0^z t^{-p-1}\left(\frac{1+t^n}{1-t^n}\right)^{\gamma}dt. \qquad (2.15)$$

It is clear that for $z=r\in(r_n(p,\gamma,\delta,\rho),1)$,

$$(1-\delta)\left(\frac{rf'(r)}{-pr^{-p}}\right)^{\frac{1}{\gamma}}+\delta\left(1+\frac{rf''(r)}{f'(r)}\right)-\rho=\frac{J_n(r)}{1-r^{2n}}<0,$$

which shows that the bound $r_n(p,\gamma,\delta,\rho)$ can not be increased.

Taking $\gamma=1$ in Theorem, we get the following.

2.2. Corollary. *Let $f(z)$ belong to the class $\Sigma_p(n)$ and satisfy*

$$\frac{zf'(z)}{z^{-p}}\prec -p\left(\frac{1+z}{1-z}\right) \quad (z\in U). \qquad (2.16)$$

Then

$$Re\left\{(1-\delta)\left(-\frac{zf'(z)}{pz^{-p}}\right)+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\rho \quad (|z|<r_n(p,1,\delta,\rho)), \qquad (2.17)$$

where $0\leq\rho<1$, $0<\delta<\frac{1-\rho}{1+p}$ and $r_n(p,1,\delta,\rho)$ is the smallest root in $(0,1)$ of the equation

$$[1+\rho+(p-1)\delta]r^{2n}-2(1-\delta+n\delta)r^n+1-\rho-(p+1)\delta=0.$$

The result is sharp.

Letting $p=1$ in Theorem, we have

2.3. Corollary. *Let $f(z)$ belong to the class $\Sigma_p(n)$ and satisfy*

$$\frac{f'(z)}{z^{-2}}\prec -\left(\frac{1+z}{1-z}\right)^{\gamma} \quad (0<\gamma\leq 1;\ z\in U). \qquad (2.18)$$

Then

$$Re\left\{(1-\delta)\left(-\frac{f'(z)}{z^{-2}}\right)^{\frac{1}{\gamma}}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\rho \quad (|z|<r_n(1,\gamma,\delta,\rho)), \qquad (2.19)$$

where $0\leq\rho<1$, $0<\delta<\frac{1-\rho}{2}$ and $r_n(1,\gamma,\delta,\rho)$ is the smallest root in $(0,1)$ of the equation

$$(1+\rho)r^{2n}-2(1-\delta+n\delta\gamma)r^n+1-\rho-2\delta=0.$$

The result is sharp. □

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