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UNIQUENESS RESULTS AND CONVERGENCE OF SUCCESSIVE APPROXIMATIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Jun Wu * and Yicheng Liu † ‡

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Abstract

In this paper, we establish some existence and uniqueness results for an initial value problem with a Caputo fractional derivative. Also, the convergence of successive approximations is exhibited. Our methods are based on the equivalent norm techniques and fixed point theorem.

Keywords: Fractional differential equations, Krasnoselskii-Krein type condition, Equivalent norm.

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1. Introduction

In this paper, we establish the existence and uniqueness results for the initial value problems with a Caputo fractional derivative

$$D^q x(t) = f(t, x(t)), x(t_0) = x_0, t_0 \geq 0,\ t \in [t_0, t_0 + a] := J, \tag{1.1}$$

where $q \in (0, 1)$, $f \in C(J \times R, R)$.

Recently, some Krasnoselskii-Krein-type uniqueness results for the fractional differential equations were presented by Bhaskar, Lakshmikantham and Leela et al, see for

*College of Mathematics and Computer Science, Changsha University of Science Technology, Changsha, 410114, P.R. China Email: junwmath@hotmail.com

†Corresponding Author.

‡Department of Mathematics and System Sciences, College of Science, National University of Defense Technology, Changsha, 410073, P.R. China Email: liuyc2001@hotmail.com

examples [5, 6, 7]. These results are established involving the following Krasnoselskii-Krein-type condition:

$$(A) \qquad |f(t,x)-f(t,y)| \le Gr\Gamma(q)\frac{|x-y|}{(t-t_0)^q},\ t \ne t_0,\ \text{where } Gr \le q, G > 1;$$

$$(B) \qquad |f(t,x)-f(t,y)| \le C|x-y|^{\alpha},$$

where C is constant , $\alpha \in (0,1)$ and $G(1-\alpha) < 1$.

Also, some special fractional differential equations are investigated by many authors, see for examples [1, 2, 3, 4, 8]. Inspired by the above excellent works, in this paper, we follows from this direction to establish some new uniqueness results for fractional differential equations. Our methods are based on the equivalent norm techniques and Banach fixed point theorem.

Let I be a bounded interval and $C(I)$ denote the Banach space consisting of all bounded continuous mappings from I into R with norm $\|u\| = \max\{|u(t)| : t \in I\}$ for $u \in C(I)$. Similarly, $C^q(I)$ is a Banach space equipped with norm

$$\|u\|_q = \|u\| + \sup\{|\frac{u(t_1)-u(t_2)}{|t_2-t_1|^q}| : t_1, t_2 \in I, t_1 \ne t_2\}.$$

2. Main results

At this section, we should state main results in this paper as follows.

2.1. Theorem. *Assume the function f in IVP (1.1) satisfies the following conditions:*

$$(F_1) \qquad |f(t,x)-f(t,y)| \le K\frac{|x-y|}{(t-t_0)^q},\ t \ne t_0,\ \textit{where } K > 0;$$

$$(F_2) \qquad |f(t,x)-f(t,y)| \le l|x-y|^{\alpha},\ \textit{where } l \textit{ is constant and } \alpha \in (0,1);$$

$$(F_3) \qquad K\alpha\frac{\Gamma(\frac{\alpha q}{1-\alpha})}{\Gamma(\frac{q}{1-\alpha})} < 1.$$

Then there exists a unique solution $\varphi(t)$ of IVP (1.1) on J and the successive approximations $\{\varphi_n(t)\}$ defined by

$$\varphi_0(t) = x_0, \tag{2.1}$$

$$\varphi_{n+1}(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,\varphi_n(s))ds, \quad n = 0, 1, \cdots$$

converge uniformly to the unique solution $\varphi(t)$ on J , that is,

$$\lim_{n\to\infty}\|\varphi_n - \varphi\|_q = 0. \tag{2.2}$$

2.2. Remark. It follows the Krasnoselskii-Krein-type condition (A) that $Gr\Gamma(q) \le q\Gamma(q) = \Gamma(1+q) < 1$ for $q \in (0,1)$. The assumption (F_1) in Theorem 1 removes this restriction.

2.3. Theorem. *If there exist two positive constants $\alpha \ge 1$ and l satisfying condition:*

$$|f(t,x)-f(t,y)| \le l|x-y|^{\alpha}, \quad (t,x),(t,y) \in [t_0, t_0+a] \times R.$$

Then there exists a unique solution $\varphi(t)$ of IVP (1.1) on J and (2.2) also holds.

2.4. Theorem. *If there exist constants $K > 0$ and $p \in (0,q)$ such that the function f in (1.1) satisfies following condition:*

$$|f(t,x)-f(t,y)| \le K\frac{|x-y|}{(t-t_0)^p}, \quad (t,x),(t,y) \in (t_0, t_0+a] \times R.$$

Then there exists a unique solution $\varphi(t)$ of IVP (1.1) on J and (2.2) also holds.

3. The proofs of main results

We transform the existence of the IVP (1.1) into a fixed point problem. Consider the map $F : C(J) \to C(J)$, defined by,

$$Fu(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s))ds.$$

It is known that the fixed points of F are solutions to the problem (1.1) .

Proof. (**The proof of Theorem 2.1**) First, we choose a $k_0 \in N$ such that $\frac{l}{{k_0}^q} < \frac{2}{(\|x_0\| + \frac{M_0}{\Gamma(q+1)} + 2)^\alpha}$ where $M_0 = \max\{|f(t,0)| + 1 : t \in J\}$. Define the norm $\|\cdot\|_0$ in $C(J)$ by

$$\|u\|_0 = \max\{e^{-k_0(t-t_0)}|u(t)| : t \in J\} \quad \text{for} \quad u \in C(J).$$

Then the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent with $\|u\|_0 \le \|u\|$ for all $u \in C(J)$.

Let $Q = \|x_0\| + \frac{M_0}{\Gamma(q+1)} + 2$ and $B_Q = \{u \in C(J) : \|u\|_0 \le Q\}$. Then, for $u \in B_Q$, by the assumption (F_2), we have

$$\begin{aligned}
|Fu(t)| &\le \|x_0\| + \frac{l}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} |f(s,u(s)) - f(t,0) + f(t,0)| ds \\
&\le \|x_0\| + \frac{M_0}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} ds + \frac{l}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} |u(s)|^\alpha ds \\
&\le \|x_0\| + \frac{M_0}{\Gamma(q+1)} + \frac{l}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} e^{k_0(s-t_0)} ds \|u\|_0^\alpha \\
&\le \|x_0\| + \frac{M_0}{\Gamma(q+1)} + \frac{l}{{k_0}^q} e^{k_0(t-t_0)} \|u\|_0^\alpha .
\end{aligned}$$

Thus

$$\|Fu\|_0 \le \|x_0\| + \frac{M_0}{\Gamma(q+1)} + \frac{l}{{k_0}^q} \|u\|_0^\alpha < Q.$$

This implies $F(B_Q) \subset B_Q$.

On the other hand, for $u \in B_Q$ and $t_1, t_2 \in J (t_1 < t_2)$, we deduce that

$$\begin{aligned}
& |Fu(t_2) - Fu(t_1)| \\
= & \frac{1}{\Gamma(q)} \left| \int_{t_0}^{t_2} (t_2 - s)^{q-1} f(s,u(s)) ds - \int_{t_0}^{t_1} (t_1 - s)^{q-1} f(s,u(s)) ds \right| \\
= & \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s,u(s)) ds + \int_{t_0}^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] f(s,u(s)) ds \right| \\
\le & \frac{M}{\Gamma(q+1)} [2(t_2 - t_1)^q + (t_1 - t_0)^q - (t_2 - t_0)^q] \\
\le & \frac{2M}{\Gamma(q+1)} (t_2 - t_1)^q,
\end{aligned}$$

where $M = \max\{|f(t,x)| : t \in J, x \in B_Q\}$. This means $F(B_Q)$ is an equicontinuous set. By Ascoli-Arzela theorem, we easily deduce that $F(B_Q)$ is relatively compact set. It follows from the continuousness of f that F is complete continuous. By the Leray-Schauder theorem, F has a fixed point $\varphi \in B_Q$.

Next, we prove the uniqueness of the solution of IVP (1.1). Let $\varphi(t)$ and $\psi(t)$ be two solutions of IVP (1.1), then, by the assumption (F_1), we obtain

$$\begin{aligned} |\varphi(t)-\psi(t)| &= |F\varphi(t)-F\psi(t)| \\ &\leq \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|f(s,\varphi(s))-f(s,\psi(s))|ds \\ &\leq \frac{K}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(s-t_0)^{-q}|\varphi(s)-\psi(s)|ds. \end{aligned} \tag{3.1}$$

Also, by assumption (F_2), we have

$$|\varphi(t)-\psi(t)| \leq \frac{l}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|\varphi(s)-\psi(s)|^{\alpha}ds. \tag{3.2}$$

Let $M_1 = \max_{t\in J}\{|f(t,\varphi(t))-f(t,\psi(t))|\}$, then, for $t \in J$, we obtain

$$|\varphi(t)-\psi(t)| \leq \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|f(s,\varphi(s))-f(s,\psi(s))|ds \leq \frac{M_1}{\Gamma(q+1)}(t-t_0)^q.$$

Substituting it into (3.2), we have

$$|\varphi(t)-\psi(t)| \leq \frac{lM_1^{\alpha}}{\Gamma(q+1)^{1+\alpha}}(t-t_0)^{q+q\alpha}.$$

Using (3.2) and charging by induction, for any $n \in N$, we have

$$|\varphi(t)-\psi(t)| \leq \frac{l^{\sum_{i=0}^{n-1}\alpha^i}M_1^{\alpha^n}}{\Gamma(q+1)^{\sum_{i=0}^{n}\alpha^i}}(t-t_0)^{q\sum_{i=0}^{n}\alpha^i}.$$

Thus

$$|\varphi(t)-\psi(t)| \leq \left(\frac{l}{\Gamma(q+1)}\right)^{\frac{1}{1-\alpha}}(t-t_0)^{\frac{q}{1-\alpha}} \text{ for } t \in J. \tag{3.3}$$

On the other hand, by (3.1), for any $n \in N$, we have

$$\begin{aligned} &|\varphi(t)-\psi(t)| \\ \leq\ & \left(\frac{K}{\Gamma(q)}\right)^n\int_{t_0}^{t}(t-s)^{q-1}(s-t_0)^{-q}\cdots\int_{t_0}^{s}(s-r)^{q-1}(r-t_0)^{-q}|\varphi(r)-\psi(r)|dr\cdots ds. \end{aligned}$$

Substituting (3.3) into the above inequality and using the formulation

$$\int_{t_0}^{t}(t-s)^{q-1}(s-t_0)^{-q}(s-t_0)^{\frac{q}{1-\alpha}}ds = B(q,1+\frac{\alpha q}{1-\alpha})(t-t_0)^{\frac{q}{1-\alpha}},$$

($B(\cdot,\cdot)$ is Beta function $B(x,y)=\int_0^1(1-s)^{x-1}s^{y-1}ds$) we have

$$\begin{aligned} |\varphi(t)-\psi(t)| &\leq \left[\frac{K}{\Gamma(q)}B(q,1+\frac{\alpha q}{1-\alpha})\right]^n\left[\frac{l}{\Gamma(q+1)}\right]^{\frac{1}{1-\alpha}}(t-t_0)^{\frac{q}{1-\alpha}} \\ &\leq \left[\frac{K}{\Gamma(q)}B(q,1+\frac{\alpha q}{1-\alpha})\right]^n\left[\frac{l}{\Gamma(q+1)}\right]^{\frac{1}{1-\alpha}}a^{\frac{q}{1-\alpha}}. \end{aligned} \tag{3.4}$$

By assumption (F_3), we see that

$$\frac{K}{\Gamma(q)}B(q,1+\frac{\alpha q}{1-\alpha}) = \frac{K}{\Gamma(q)}\frac{\Gamma(q)\Gamma(1+\frac{\alpha q}{1-\alpha})}{\Gamma(1+\frac{q}{1-\alpha})} = K\alpha\frac{\Gamma(\frac{\alpha q}{1-\alpha})}{\Gamma(\frac{q}{1-\alpha})} < 1.$$

Letting n go to infinity in (3.4), we conclude that $\varphi(t) \equiv \psi(t)$ for $t \in J$.

Furthermore, let $\{\varphi_n(t)\}$ be a sequence defined by (2.1) and $\varphi(t)$ be a solution of IVP (1.1). Setting $\psi_n(t)=\varphi_{n+1}(t)-\varphi(t)$, $k_n(t)=f(t,\varphi_n(t))-f(t,\varphi(t))$, for any $t_1,t_2\in(t_0,t_0+a]$ and $t_1<t_2$, we have

$$\begin{aligned}
|\psi_n(t_2)-\psi_n(t_1)| &= \frac{1}{\Gamma(q)}|\int_{t_0}^{t_2}(t_2-s)^{q-1}k_n(s)ds-\int_{t_0}^{t_1}(t_1-s)^{q-1}k_n(s)ds| \\
&= \frac{1}{\Gamma(q)}|\int_{t_1}^{t_2}(t_2-s)^{q-1}k_n(s)ds+\int_{t_0}^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]k_n(s)ds| \\
&\le \frac{l}{\Gamma(q+1)}[2(t_2-t_1)^q+(t_1-t_0)^q-(t_2-t_0)^q]\|\varphi_n-\varphi\|^\alpha \\
&\le \frac{2l}{\Gamma(q+1)}(t_2-t_1)^q\|\varphi_n-\varphi\|^\alpha.
\end{aligned}$$

This means

$$\sup\{|\frac{\psi_n(t_1)-\psi_n(t_2)}{(t_2-t_1)^q}|:t_1,t_2\in J,t_1\neq t_2\}\le\frac{2l}{\Gamma(q+1)}\|\varphi_n-\varphi\|^\alpha.$$

Thus

$$\|\varphi_n-\varphi\|_q\le\|\varphi_n-\varphi\|+\frac{2l}{\Gamma(q+1)}\|\varphi_{n-1}-\varphi\|^\alpha.$$

This implies that

$$\lim_{n\to\infty}\|\varphi_n-\varphi\|_q=0.$$

This completes the proof of Theorem 2.1.

□

For $u,v\in C(J)$, we have, for $t\in J$,

$$\begin{aligned}
|Fu(t)-Fv(t)| &\le \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|f(s,u(s))-f(s,v(s))|ds \\
&\le \frac{l}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|u(s))-v(s)|^\alpha ds.
\end{aligned}$$

Define an operator $T:C(J,R^+)\to C(J,R^+)$ by

$$Tx(t)=\frac{l}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x^\alpha(s)ds.$$

In order to prove the Theorem 2.2, we establish two key lemmas.

3.1. Lemma. *If the assumptions in Theorem 2.2 hold, then there exists a function $y\in C(J,R^+)$ such that*

$$Ty(t)\le\gamma y(t)\ \textit{and}\ \gamma\in(0,\min\{1,(\frac{\Gamma(1+q)}{M_0a^q})^{\frac{\alpha-1}{\alpha}}\}),$$

where $M_0=\min\{|f(t,x_0)|+1:t\in J\}$.

Proof. Let $\eta\in(t_0,t_0+a]$ be a constant satisfying

$$\frac{l(\eta-t_0)^q}{\Gamma(1+q)}+\frac{la^{\frac{q}{2\alpha}}(\eta-t_0)^{q-\frac{q}{2\alpha}}}{\Gamma(1+q)}+\frac{lB(q,1-\frac{q}{2})a^{\frac{q}{2}+\frac{q}{2\alpha}}(\eta-t_0)^{\frac{q}{2}-\frac{q}{2\alpha}}}{\Gamma(q)}<\min\{1,(\frac{\Gamma(1+q)}{M_0a^q})^{\frac{\alpha-1}{\alpha}}\}).$$

Take $\gamma = \frac{l(\eta-t_0)^q}{\Gamma(1+q)} + \frac{la^{\frac{q}{2\alpha}}(\eta-t_0)^{q-\frac{q}{2\alpha}}}{\Gamma(1+q)} + \frac{lB(q,1-\frac{q}{2})a^{\frac{q}{2}+\frac{q}{2\alpha}}(\eta-t_0)^{\frac{q}{2}-\frac{q}{2\alpha}}}{\Gamma(q)}$ and define the function y from J into R by

$$y(t) = \begin{cases} 1, & \text{if } \ t \in [t_0, \eta], \\ (\frac{t-t_0}{\eta-t_0})^{-\frac{q}{2\alpha}}, & \text{if } \ t \in (\eta, t_0 + a]. \end{cases}$$

We prove that $Ty(t) \le \gamma y(t)$ for $t \in [t_0, t_0 + a]$.

For $t \in [t_0, \eta]$, we have

$$\begin{aligned} Ty(t) &= \frac{l}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} ds = \frac{l}{q\Gamma(q)} (t-t_0)^q \\ &\le \frac{l(\eta-t_0)^q}{\Gamma(1+q)} < \gamma y(t). \end{aligned}$$

For $t \in (\eta, t_0 + a]$, recalling that $B(x, y) = \int_0^1 (1-s)^{x-1} s^{y-1} ds$, we have

$$\begin{aligned} Ty(t) &= \frac{l}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (y(s))^{\alpha} ds \\ &= \frac{l}{\Gamma(q)} \int_{t_0}^{\eta} (t-s)^{q-1} ds + \frac{l}{\Gamma(q)} \int_{\eta}^{t} (t-s)^{q-1} (\frac{s-t_0}{\eta-t_0})^{-\frac{q}{2}} ds \\ &\le (\frac{t-t_0}{\eta-t_0})^{\frac{q}{2\alpha}} \frac{l}{\Gamma(q)} \int_{\eta}^{t} (t-s)^{q-1} (\frac{s-t_0}{\eta-t_0})^{-\frac{q}{2}} ds (\frac{t-t_0}{\eta-t_0})^{-\frac{q}{2\alpha}} \\ &\quad + \frac{l}{\Gamma(q)} \int_{t_0}^{\eta} (\eta-s)^{q-1} ds \\ &= (t-t_0)^{\frac{q}{2}+\frac{q}{2\alpha}} (\eta-t_0)^{\frac{q}{2}-\frac{q}{2\alpha}} \frac{l}{\Gamma(q)} \int_{\frac{\eta-t_0}{t-t_0}}^{1} (1-z)^{q-1} z^{-\frac{q}{2}} dz (\frac{t-t_0}{\eta-t_0})^{-\frac{q}{2\alpha}} \\ &\quad + \frac{l}{\Gamma(q)} \int_{t_0}^{\eta} (\eta-s)^{q-1} ds \\ &\le \frac{l(\eta-t_0)^q}{\Gamma(1+q)} + a^{\frac{q}{2}+\frac{q}{2\alpha}} (\eta-t_0)^{\frac{q}{2}-\frac{q}{2\alpha}} \frac{lB(q, 1-\frac{q}{2})}{\Gamma(q)} (\frac{t-t_0}{\eta-t_0})^{-\frac{q}{2\alpha}} \\ &\le [\frac{la^{\frac{q}{2\alpha}} (\eta-t_0)^{q-\frac{q}{2\alpha}}}{\Gamma(1+q)} + a^{\frac{q}{2}+\frac{q}{2\alpha}} \frac{lB(q, 1-\frac{q}{2})}{\Gamma(q)} (\eta-t_0)^{\frac{q}{2}-\frac{q}{2\alpha}}] (\frac{t-t_0}{\eta-t_0})^{-\frac{q}{2\alpha}} \\ &< \gamma y(t). \end{aligned}$$

This completes the proof of Lemma 3.1. □

Define the norm $\|\cdot\|_y$ in $C(J)$ by

$$\|u\|_y = \max\{\frac{1}{y(t)} |u(t)| : t \in J\} \quad \text{for} \quad u \in C(J),$$

where $y(t)$ is given in Lemma 1. Then the norms $\|\cdot\|$ and $\|\cdot\|_y$ are equivalent.

3.2. Lemma. *Assume the assumptions of Theorem* 2.2 *hold, then* $\|Fu - Fv\|_y \le \gamma \|u - v\|_y^{\alpha}$.

Proof. Noting $|u(t)| \leq y(t)\|u\|_y$, we have

$$\begin{aligned} |Fu(t) - Fv(t)| &\leq \frac{l}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}(y(s))^{\alpha} ds \|u - v\|_y^{\alpha} \\ &= Ty(t)\|u - v\|_y^{\alpha} \leq \gamma y(t)\|u - v\|_y^{\alpha}, \end{aligned}$$

for $t \in [t_0, t_0 + a]$. By the definition of the norm $\|\cdot\|_y$, we see that

$$\|Fu - Fv\|_y \leq \gamma \|u - v\|_y^{\alpha}.$$

This completes the proof of Lemma 3.2. □

Proof. (**The proof of Theorem 2.2**)

CASE 1: $\alpha = 1$**.**

By Lemma 3.2, we see that F is a contractive mapping for the norm $\|\cdot\|_y$. It is easy to prove that there exists a unique function $\varphi \in C(J)$ satisfying

$$\varphi(t) = F\varphi(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, \varphi(s)) ds.$$

Thus the IVP (1.1) has a unique solution $\varphi(t)$.

Furthermore, noting that $\varphi_{n+1}(t) = F\varphi_n(t)$ for $n = 1, 2, \cdots$, we have

$$\begin{aligned} \|\varphi_{n+1} - \varphi\|_y &= \|F\varphi_n - F\varphi\|_y \leq \gamma\|\varphi_n - \varphi\|_y \\ &\leq \cdots \leq \gamma^{n+1}\|\varphi_0 - \varphi\|_y \leq \frac{M_2 a^q}{\Gamma(1+q)}\gamma^{n+1}, \end{aligned}$$

where $M_2 = \sup_{t \in J} |f(t, \varphi(t))|$. This implies that the successive sequence $\{\varphi_n(t)\}$ converge uniformly to the unique solution $\varphi(t)$ on J with $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$.

Similar as the proof of Theorem 2.1, we have

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_q = 0.$$

Thus the Theorem 2.2 holds for the case $\alpha = 1$.

CASE 2: $\alpha > 1$**.**

In this case, by Lemma 3.1, we see that $\gamma < (\frac{\Gamma(1+q)}{M_0 a^q})^{\frac{\alpha-1}{\alpha}}$. This means that

$$\gamma^{\frac{\alpha}{\alpha-1}} \frac{M_0 a^q}{\Gamma(q+1)} < 1.$$

By Lemma 3.2, we have

$$\begin{aligned} \|\varphi_{n+1} - \varphi_n\|_y &= \|F\varphi_n - F\varphi_{n-1}\|_y \leq \gamma\|\varphi_n - \varphi_{n-1}\|_y^{\alpha} \\ &\leq \cdots \leq \gamma^{\sum_{i=0}^{n} \alpha^i} \|\varphi_1 - \varphi_0\|_y^{\alpha^n} \\ &\leq \gamma^{\sum_{i=0}^{n} \alpha^i} \|\varphi_1 - \varphi_0\|^{\alpha^n} \\ &\leq \gamma^{-\frac{1}{\alpha-1}} (\gamma^{\frac{\alpha}{\alpha-1}} \frac{M_0 a^q}{\Gamma(1+q)})^{\alpha^n}. \end{aligned}$$

It follows from $\gamma_{k_3}^{\frac{\alpha}{\alpha-1}} \frac{M_0 a^q}{\Gamma(q+1)} < 1$ that $\{\varphi_n(t)\}$ is a Cauchy sequence for the norm $\|\cdot\|_y$. Thus $\{\varphi_n(t)\}$ also is a Cauchy sequence for the norm $\|\cdot\|$.

Setting $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$, then, by Lebesgue's dominated convergence theorem, we obtain

$$\varphi(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, \varphi(s)) ds.$$

Thus the IVP (1.1) has a solution $\varphi(t)$.

Following the above arguments and Lemma 3.2, the uniqueness and convergence are obvious. The proofs are omitted.

For all, the results of Theorem 2.2 hold for $\alpha \geq 1$. This completes the proof of Theorem 2.2. □

With the similar arguments in the proof of Theorem 2.2, we should establish the following two lemmas under the assumptions of Theorem 2.3.

3.3. Lemma. *If the assumptions in Theorem* 2.3 *hold, then there exist an increasing function* $b \in C(J, R^+)$ *and a constant* $\delta \in (0,1)$ *such that*

$$Hb(t) := \frac{K}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}(s-t_0)^{-p} b(s) ds \leq \delta b(t).$$

Proof. Similarly, we choose a positive number $\eta \in J$ such that

$$\frac{K(\eta - t_0)^{q-p} B(q, 1-p)}{\Gamma(q)} + K(\eta - t_0)^{q-p} < 1.$$

Also, let $\delta = \frac{K(\eta-t_0)^{q-p}B(q,1-p)}{\Gamma(q)} + K(\eta - t_0)^{q-p}$ and define an increasing function b from J into R by

$$b(t) = \begin{cases} 1, & \text{if } t \in [t_0, \eta], \\ e^{\frac{t-\eta}{\eta - t_0}}, & \text{if } t \in (\eta, t_0 + a]. \end{cases}$$

We prove that $Hb(t) \leq \delta b(t)$ for $t \in [t_0, t_0 + a]$.

For $t \in [t_0, \eta]$, recalling $B(x, y) = \int_0^1 (1-s)^{x-1} s^{y-1} ds$, we have

$$\begin{aligned} Hb(t) &= \frac{K}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}(s-t_0)^{-p} ds = \frac{K}{\Gamma(q)} (t-t_0)^{q-p} \int_0^1 (1-z)^{q-1} z^{1-p-1} dz \\ &= \frac{KB(q,1-p)}{\Gamma(q)} (t-t_0)^{q-p} \leq \frac{K(\eta-t_0)^{q-p} B(q,1-p)}{\Gamma(q)} < \delta b(t). \end{aligned}$$

For $t \in (\eta, t_0 + a]$, we have

$$\begin{aligned} Hb(t) &= \frac{K}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}(s-t_0)^{-p} b(s) ds \\ &= \frac{K}{\Gamma(q)} \int_{t_0}^{\eta} (t-s)^{q-1}(s-t_0)^{-p} ds + \frac{K}{\Gamma(q)} \int_{\eta}^{t} (t-s)^{q-1}(s-t_0)^{-p} e^{\frac{s-\eta}{\eta-t_0}} ds \\ &\leq \frac{K}{\Gamma(q)} \int_{t_0}^{\eta} (\eta-s)^{q-1}(s-t_0)^{-p} ds + \frac{K}{\Gamma(q)} \int_{\eta}^{t} (t-s)^{q-1}(s-t_0)^{-p} e^{\frac{s-\eta}{\eta-t_0}} ds \\ &\leq \frac{K(\eta-t_0)^{q-p} B(q,1-p)}{\Gamma(q)} + \frac{K}{\Gamma(q)} \int_{\eta}^{t} (t-s)^{q-1}(s-t_0)^{-p} e^{-\frac{t-s}{\eta-t_0}} ds e^{\frac{t-\eta}{\eta-t_0}} \\ &\leq [\frac{K(\eta-t_0)^{q-p} B(q,1-p)}{\Gamma(q)} + \frac{K(\eta-t_0)^{q-p}}{\Gamma(q)} \int_0^{\frac{t}{\eta-t_0}} z^{q-1} e^{-z} dz] e^{\frac{t-\eta}{\eta-t_0}} \\ &\leq [\frac{K(\eta-t_0)^{q-p} B(q,1-p)}{\Gamma(q)} + K(\eta-t_0)^{q-p}] b(t) \\ &= \delta b(t). \end{aligned}$$

Thus there exist a positive function $b(t)$ and a constant $\delta \in (0,1)$ such that

$$Hb(t) \leq \delta b(t).$$

□

3.4. Lemma. *Under the assumptions of Theorem 2.3, the operator F is contraction mapping in the sense of equivalent norm.*

Proof. Define norm $\|\cdot\|_b$ in $C(J)$ by

$$\|u\|_b = \max\{\frac{1}{b(t)}|u(t)| : t \in J\} \quad \text{for} \quad u \in C(J),$$

where $b(t)$ is given in Lemma 3.3. Then the two norms $\|\cdot\|$ and $\|\cdot\|_b$ are equivalent.

For $u, v \in C(J)$, by Lemma 3.3, we have, for $t \in J$,

$$\begin{aligned}
|Fu(t) - Fv(t)| &\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} |f(s, u(s)) - f(s, v(s))| ds \\
&\leq \frac{K}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (s-t_0)^{-p} |u(s)) - v(s)| ds \\
&\leq \frac{K}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (s-t_0)^{-p} b(s) ds \|u - v\|_b \\
&= \delta b(t) \|u - v\|_b.
\end{aligned}$$

Thus

$$\|Fu - Fv\|_b \leq \delta \|u - v\|_b.$$

This implies that the operator F is contractive in the sense of equivalent norm $\|\cdot\|_b$. □

Proof. (**Proof of Theorem 2.3**) By Lemma 3.4 and the similar arguments in the proof Theorem 2.2, we claim that there exists a unique solution $\varphi(t)$ of IVP (1.1) on J. Also, It is easy to prove that $(\psi_n(t) = \varphi_{n+1}(t) - \varphi(t))$

$$\sup\{|\frac{\psi_n(t_1) - \psi_n(t_2)}{(t_2 - t_1)^q}| : t_1, t_2 \in J, t_1 \neq t_2\} \leq \frac{2l}{\Gamma(q+1)} \sup_{t \in J} |f(t, \varphi_n(t)) - f(t, \varphi(t))|.$$

Noting that $\varphi_n(t)$ converge uniformly to $\varphi(t)$ on J, we conclude that

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_q = 0 \text{ uniformly on } J.$$

This completes the proof of Theorem 2.3.

□

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