Hacettepe Journal of Mathematics and Statistics
Volume 42 (2) (2013), 159 – 163

SOME GENERALIZED CONTINUITIES FUNCTIONS ON GENERALIZED TOPOLOGICAL SPACES

Chunfang Cao * † ‖, Jing Yan ‡ ‖, Weiqin Wang * § ‖, Baoping Wang * ¶ ‖

Received 12 : 23 : 2011 : Accepted 12 : 03 : 2012

Abstract

In this paper, we introduce $(g, \alpha g')$-continuous functions, $(g, \sigma g')$-continuous functions, $(g, \pi g')$-continuous functions, and $(g, \beta g')$-continuous functions on generalized topological spaces. These generalized continuous functions are defined by generalized open sets. We discuss some characterizations and some applications of them.

Keywords: (g, g')-continuous functions, $(g, \alpha g')$-continuous functions, $(g, \sigma g')$- continuous functions, $(g, \pi g')$-continuous functions, $(g, \beta g')$-continuous functions

2000 AMS Classification: 54A05; 54D15

1. Introduction

In [1], Császár introduced the notions of generalized topological spaces and two kinds of generalized continuous functions. By using these concepts, Min [8] introduced the notions of $(\alpha g, g')$-continuity, $(\sigma g, g')$-continuity, $(\pi g, g')$-continuity, $(\beta g, g')$-continuity on GTS. In this paper, we introduce the notions of $(g, \alpha g')$-continuous functions, $(g, \sigma g')$-continuous functions, $(g, \pi g')$-continuous functions, and $(g, \beta g')$-continuous functions. We investigate properties of such functions and the relationships among these continuities. Some applications of these functions are given too. For example, we discuss the

*Department of Mathematics Physics and Information, Taizhou Teachers College, Taizhou 225300, Jiangsu, P.R.China
†Email: (C. Cao) ccf85@tom.com
‡Department of Mathematics and Physics , Jiangsu Teachers University of Technology, Changzhou 213001, Jiangsu, P.R.China
Email: (J. Yan) yanjing@jstu.edu.cn
§Email: (W. Wang) jswwq@tom.com
¶Email: (B. Wang) hellowangbp@163.com
‖Supported by the foundation of Taizhou Teachers College(No. 2009-ASL-05).

properties of the product of generalized topological spaces, connectedness of generalized topological spaces and compactness of generalized topological spaces.

Let we recall some notions of generalized topological space in [1]. Let X be a nonempty set and g be a collection of subsets of X. Then g is called a *generalized topology* (briefly GT) on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $\bigcup_{i\in I} G_i \in g$. A set with a GT is said to be a *generalized topological space* (briefly GTS). The elements of g are called *g-open* sets and their complements are called *g-closed* sets. The generalized interior of a subset A of X denoted by $i_g(A)$ is the union of generalized open sets included in A, and the generalized closure of A denoted by $c_g(A)$ is the intersection of generalized closed sets including A. It is easy to verify that $c_g(A) = X - i_g(X - A)$ and $i_g(A) = X - c_g(X - A)$. Let M_g denote the union of all elements of g, we say g is *strong* [3] if $M_g = X$.

Throughout this paper X and X' mean GTS's (X, g) and (X', g'). And the function $f : X \to X'$ denotes a single valued function of a space (X, g) into a space (X', g').

1.1. Definition. [4] Let (X, g) be a generalized topological space and $A \subset X$. Then A is said to be

(1) *g-α-open* if $A \subset i_g(c_g(i_g(A)))$;
(2) *g-σ-open* (*g-semiopen*) if $A \subset c_g(i_g(A))$;
(3) *g-π-open* (*g-preopen*) if $A \subset i_g(c_g(A))$;
(4) *g-β-open* if $A \subset c_g(i_g(c_g(A)))$.

Let we denote by g_X (resp., $\alpha(g_X)$, $\sigma(g_X)$, $\beta(g_X)$, $\pi(g_X)$) the class of all g-open (resp., g-α-open, g-σ-open, g-β-open, g-π-open) sets on X. Obviously $g_X \subset \alpha(g_X) \subset \sigma(g_X) \subset \beta(g_X)$ and $\alpha(g_X) \subset \pi(g_X) \subset \beta(g_X)$.

The complement of g-α-open set (resp., g-σ-open, g-π-open, g-β-open set) is said to be *g-α-closed* (resp., *g-σ-closed*, *g-π-closed*, *g-β-closed*). $i_\alpha(A)$ (resp., $i_\beta(A), i_\sigma(A), i_\pi(A)$) is denoted by the union of g-α-open (resp.,g-β-open, g-σ-open, g-π-open) sets included in A, and $c_\alpha(A)$ (resp., $c_\beta(A), c_\sigma(A), c_\pi(A)$) is denoted by the intersection of g-α-closed (resp., g-β-closed, g-σ-closed, g-π-closed) sets including A.

2. On generalized continuity

2.1. Definition. Let (X, g) and (X', g') be GTS's . Then a function $f : X \to X'$ is said to be

(1)[1] (g, g')*-continuous* if $f^{-1}(V)$ is g-open set in X for every g-open set V of X'.
(2) $(g, \alpha g')$*-continuous* if $f^{-1}(V)$ is g-open set in X for every g-α-open set V of X'.
(3) $(g, \sigma g')$*-continuous* if $f^{-1}(V)$ is g-open set in X for every g-σ-open set V of X'.
(4) $(g, \pi g')$*-continuous* if $f^{-1}(V)$ is g-open set in X for every g-π-open set V of X'.
(5) $(g, \beta g')$*-continuous* if $f^{-1}(V)$ is g-open set in X for every g-β-open set V of X'.

2.2. Remark. From the definitions stated above, we obtain the following relationship.

$(g, \beta g')$*-continuous* $\to$ $(g, \sigma g')$*-continuous* $\to$ $(g, \alpha g')$*-continuous* $\to$ (g, g')*-continuous*
$(g, \beta g')$*-continuous* $\to$ $(g, \pi g')$*-continuous* $\to$ $(g, \alpha g')$*-continuous* $\to$ (g, g')*-continuous*

2.3. Example. Let $X = X' = \{a, b, c, d\}$ and $g = g' = \{\emptyset, \{a\}, \{a, b, c\}\}$. Then

$$\alpha g' = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}.$$

We consider a function $f : (X, g) \to (X', g')$ defined by $f(a) = a, f(b) = b, f(c) = c, f(d) = d$. Then f is (g, g')-continuous. However $f^{-1}(\{a, c\}) = \{a, c\}$ is not in g. So f is not $(g, \alpha g')$-continuous.

2.4. Example. Let $X = X' = \{a, b, c\}$ and $g' = g = \{\emptyset, \{a, b\}\}$. Then

$$\alpha g' = \{\emptyset, \{a, b\}\}, \quad \sigma g' = \{\emptyset, \{c\}, \{a, b\}, X\}, \quad \pi g' = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

We consider a function $f:(X,g)\to(X',g')$ defined by $f(a)=a, f(b)=b, f(c)=c$. Then f is $(g,\alpha g')$-continuous . However $f^{-1}(\{c\})=\{c\}$ is not in g and $f^{-1}(\{a\})=\{a\}$ is not in g. So f is neither $(g,\sigma g')$-continuous nor $(g,\pi g')$-continuous.

2.5. Example. Let $X=X'=\{a,b,c\}$ and $g=\{\emptyset,\{a\},\{b\},\{a,b\}\}$, $g'=\{\emptyset,\{a,b\}\}$. Then

$$\pi g'=\{\emptyset,\{a\},\{b\},\{a,b\}\},\quad \beta g'=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}.$$

We consider a function $f:(X,g)\to(X',g')$ defined by $f(a)=a, f(b)=b, f(c)=c$. Then f is $(g,\pi g')$-continuous . However $f^{-1}(\{c\})=\{c\}$ is not in g. So f is not $(g,\beta g')$-continuous .

2.6. Example. Let $X=X'=\{a,b,c\}$ and $g=\{\emptyset,\{c\},\{a,b\},X\}$, $g'=\{\emptyset,\{a,b\}\}$. Then

$$\sigma g'=\{\emptyset,\{c\},\{a,b\},X\},\quad \beta g'=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}.$$

We consider a function $f:(X,g)\to(X',g')$ defined by $f(a)=a, f(b)=b, f(c)=c$. Then f is $(g,\sigma g')$-continuous . However $f^{-1}(\{a,c\})=\{a,c\}$ is not in g. So f is not $(g,\beta g')$-continuous.

2.7. Theorem. *For a function $f:(X,g)\to(X',g')$, the following are equivalent*

(1) f is $(g,\alpha g')$-continuous (resp., $(g,\sigma g')$-continuous, $(g,\pi g')$-continuous, $(g,\beta g')$-continuous).

(2) $f^{-1}(V)$ is a g-open set in X, for each g-α-open (resp., g-σ-open, g-π-open, g-β-open) set V in X'.

(3) $f^{-1}(F)$ is a g-closed set in X, for each g-α-closed (resp., g-σ-closed, g-π-closed, g-β-closed) set F in X'.

(4) $c_g(f^{-1}(B))\subset f^{-1}(c_\alpha(B))$ (resp., $c_g(f^{-1}(B))\subset f^{-1}(c_\sigma(B))$, $c_g(f^{-1}(B))\subset f^{-1}(c_\pi(B))$, $c_g(f^{-1}(B))\subset f^{-1}(c_\beta(B))$) for each subset B of X'.

(5)$f^{-1}(i_\alpha(B))\subset i_g(f^{-1}(B))$ (resp., $f^{-1}(i_\sigma(B))\subset i_g(f^{-1}(B))$, $f^{-1}(i_\pi(B))\subset i_g(f^{-1}(B))$, $f^{-1}(i_\beta(B))\subset i_g(f^{-1}(B))$) for each subset B of X'.

(6)$f(c_g(A))\subset c_\alpha(f(A))$ (resp., $f(c_g(A))\subset c_\sigma(f(A))$, $f(c_g(A))\subset c_\pi(f(A))$, $f(c_g(A))\subset c_\beta(f(A))$) for each subset A of X.

Proof. We only prove the case of $(g,\alpha g')$-continuity. The others are similar.

(1) $\Leftrightarrow$ (2) It is obviously by definition.

(2) $\Rightarrow$ (3) Let F be any g-α-closed subset of X', set $V=X'-F$, so V is a g-α-open set in X'. By (2) $f^{-1}(V)$ is a g-open set in X. So $f^{-1}(F)=X-f^{-1}(X'-F)=X-f^{-1}(V)$ is a g-closed set in X. (3) $\Rightarrow$ (2) is similar.

(3) $\Rightarrow$ (4) Let B be any subset of X', since $c_\alpha(B)$ is a g-α-closed set in X'. By (3) $f^{-1}(c_\alpha(B))$ is a g-closed set in X. Thus $c_g(f^{-1}(c_\alpha(B)))\subset f^{-1}(c_\alpha(B))$. So $c_g(f^{-1}(B))\subset f^{-1}(c_\alpha(B))$.

(4) $\Leftrightarrow$ (5) It follows from the conditions of $c_g(A)=X-i_g(X-A)$ and $i_g(A)=X-c_g(X-A)$.

(4) $\Rightarrow$ (6) Let A be any subset of X. By (4)

$$c_g(A)\subset c_g(f^{-1}(f(A)))\subset f^{-1}(c_\alpha(f(A)))$$

Then we have $f(c_g(A))\subset f(f^{-1}(c_\alpha(f(A)))\subset c_\alpha(f(A))$.

(6) $\Rightarrow$ (3) For any g-α-closed set F in X', by (6) $f(c_g(f^{-1}(F)))\subset c_\alpha(f(f^{-1}(F)))\subset c_\alpha(F)$. This implies $c_g(f^{-1}(F))\subset f^{-1}(c_\alpha(F))=f^{-1}(F)$. So $f^{-1}(F)$ is a g-closed set in X. □

3. Some applications

Let $K \neq \emptyset$ be an index set and $(X_k, g_k)(k \in K)$ a class of GTS's. $X = \prod_{k\in K} X_k$ is the Cartesian product of the sets X_k. Let us consider all sets of the form $\prod_{k\in K} B_k$ where $B_k \in g_k$ and, with the exception of a finite number of indices k, $B_k = M_{g_k}$. We denote $\mathfrak{B}$ the collection of all these sets. We call $g = g(\mathfrak{B})$ having $\mathfrak{B}$ as a base the product of the GT's g_k and denote it by $P_{k\in K} g_k$. The GTS (X, g) is called the product of the GTS's (X_k, g_k). We denote by p_k the projection $X \to X_k$ and $x_k = p_k(x)$ for each $x \in X$.

3.1. Lemma. [6] *Let $A = \prod_{k\in K} A_k \subset \prod_{k\in K} X_k$ and K_0 be a finite subset of K. If $A_k \in \{M_{g_k}, X_k\}$ for each $k \in K - K_0$, then $iA = \prod_{k\in K} i_k A_k$.*

3.2. Lemma. [6] *If every g_k is strong, then each p_k is (g, g_k)-continuous (resp,. $(\alpha g, \alpha g_k)$-continuous, $(\beta g, \beta g_k)$-continuous, $(\sigma g, \sigma g_k)$-continuous, $(\pi g, \pi g_k)$-continuous).*

3.3. Theorem. *Let X be a strong GTS. Let $f : X \to X'$ be a function and $h : X \to X \times X'$ be the graph function of f defined by $h(x)=(x, f(x))$ for each $x \in X$. If h is $(g, \alpha g')$-continuous (resp,. $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous), then f is $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous).*

Proof. We only prove the case of $(g, \alpha g')$-continuity. The others are similar.

Let V be any g-α-open set of X'. Then $X \times V$ is a g-α-open set of $X \times X'$ by Thorem 4.3[5]. Since h is $(g, \alpha g')$-continuous, $h^{-1}(X \times V) = f^{-1}(V)$ is a g-open set in X. Thus f is $(g, \alpha g')$-continuous. □

3.4. Remark. When we assert that h is $(g, \beta g')$-continuous ($(g, \sigma g')$-continuous), the condition that X is strong can be omitted.

Question 1: When we assert that h is $(g, \alpha g')$-continuous ($(g, \pi g')$-continuous), can the condition that X is strong be omitted?

Question 2: Whether is the conclusion valid that if f is $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous) then h is $(g, \alpha g')$-continuous (resp,. $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous) ?

3.5. Theorem. *If a function $f : X \to \prod_{k\in K} X'_k$ is $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous), and every X'_k is strong, then $p_k \circ f : X \to X'_k$ is $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous) for each $k \in K$, where p_k is the projection of $\prod_{k\in K} X'_k$ onto X'_k.*

Proof. We only prove the case of $(g, \alpha g')$-continuity. The others are similar.

Let V_k be any g-α-open set of X'. By lemma 3.2 p_k is $(\alpha g, \alpha g')$-continuous, so $p_k^{-1}(V_k)$ is a g-α-open set in $\prod_{k\in K} X'_k$. Since f is $(g, \alpha g')$-continuous, then $f^{-1}(p_k^{-1}(V_k)) = (p_k \circ f)^{-1}(V_k)$ is a g-open set in X. Therefore $p_k \circ f$ is $(g, \alpha g')$-continuous. □

3.6. Theorem. *Let X_k, X'_k be strong GTS's and $f_k : X_k \to X'_k$. If the product function $f : \prod_{k\in K} X_k \to \prod_{k\in K} X'_k$ is $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous), then $f_k : X_k \to X'_k$ is $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous,$(g, \sigma g')$-continuous,$(g, \pi g')$-continuous) for each $k \in K$.*

Proof. We only prove the case of $(g, \alpha g')$-continuity. The others are similar.

Let k_0 be an arbitrary fixed index in K and V_{k_0} be any g-α-open set of X'_{k_0}. Then $\prod_{k\neq k_0} X'_k \times V_{k_0}$ is a g-α-open set in $\prod_{k\in K} X'_k$. Since f is $(g, \alpha g')$-continuous, so $f^{-1}(\prod_{k\neq k_0} X'_k \times V_{k_0}) = \prod_{k\neq k_0} X_k \times f_{k_0}^{-1}(V_{k_0})$ is a g-open set in $\prod_{k\in K} X_k$. By Lemma 3.1 , $f_{k_0}^{-1}(V_{k_0})$ is a g-open set in X_{k_0}. This implies that f_{k_0} is $(g, \alpha g')$-continuous. □

3.7. Definition. [7]A space X is said to be *g-compact* (resp., *α-compact*, *β-compact*, *σ-compact*, *π-compact* if every g-open (resp., g-α-open, g-β-open, g-σ-open, g-π-open) cover of X has a finite subcover.

3.8. Theorem. *Let a function $f : X \to X'$ be $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous), and X is g-compact, then X' is α-compact (resp., β-compact, σ-compact, π-compact)*

Proof. We only prove the case of $(g, \alpha g')$-continuity. The others are similar.

Let χ be a cover of $f(x)$ by g-α-open sets in X'. Since f is $(g, \alpha g')$-continuous, then $\{f^{-1}(A) : A \in \chi\}$ is a g-open cover of X. For X is g-compact, so the cover of X has a finite subcover $\{f^{-1}(A) : A \in \chi'\}$ where χ' is a finite subfamily of χ. Then $X' \subset \bigcup_{A\in\chi'} f(f^{-1}(A)) = \bigcup_{A\in\chi'} A$. Therefore X' is α-compact. □

3.9. Definition. [2] A space X is said to be *g-connected* if there are no nonempty disjoint sets $U, V \subset X$ such that $U \cup V = X$.

3.10. Definition. [7]A space (X, g) is said to be *α-connected* (resp., *β-connected*, *σ-connected*, *π-connected*), if $(X, \alpha g)$ (resp.,$(X, \beta g), (X, \sigma g), (X, \pi g)$) is connected.

3.11. Theorem. *Let (X, g) and (X', g') be GTS's and the function $f : X \to X'$ be $(g, \alpha g')$-continuous (resp., $(g, \beta g')$-continuous, $(g, \sigma g')$-continuous, $(g, \pi g')$-continuous), If (X, g) is connected, (X', g') is α-connected (resp., β-connected, σ-connected, π-connected).*

Proof. We only prove the case of $(g, \alpha g')$-continuity. The others are similar.

Suppose there are two nonempty disjoint g'-α-open subsets U', V' of X', such that $U' \cup V' = X'$. For f is $(g, \alpha g')$-continuous, so $f^{-1}(U'), f^{-1}(V')$ are g-open subsets of X. And $f^{-1}(U') \cap f^{-1}(V') = f^{-1}(U' \cap V') = \varnothing$, $f^{-1}(U') \cup f^{-1}(V') = f^{-1}(U' \cup V') = X$. So (X, g) is disconnected. Therefore (X', g') is α-connected. □

References

[1] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96**, 351–357, 2002.

[2] Á. Császár, γ-connected sets, *Acta Math. Hungar.*, **101**, 273–279, 2003.

[3] Á. Császár, Exremally disconnected generalized topologies, *Annales Univ. Sci. Budapest.*, **47**, 91–96, 2004.

[4] Á. Császár, Generalized open sets in generalized topologies, *Acta Math. Hungar.*, **106**, 53–66, 2005.

[5] Á. Császár, Product of generalized topologies, *Acta Math. Hungar.*, **134**, 132–138, 2009.

[6] R. Shen, Remarks on products of generalized topologies, *Acta Math. Hungar.*, **124**, 363–369, 2009.

[7] R. Shen, A note on generalized connectedness, *Acta Math. Hungar.*, **122**, 231–235, 2009.

[8] W. K. Min, Generalized continuous functions defined by genenralized open sets on generalized topological spaces, *Acta Math. Hungar.*, **128**, 299–306, 2010.