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A NOTE ON MULTIPLIERS OF SUBTRACTION ALGEBRAS

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Abstract

In this paper, we introduce the concept of normal ideal of a subtraction algebra and study properties in connection with multiplies of subtraction algebras. The image and inverse image of a normal ideal of a subtraction algebra are proved to be again normal ideals. Also, we characterize the normal ideals of direct products of subtraction algebras. Finally, the concept of a weak congruence is introduced in subtraction algebras and obtain an interconnection between multipliers and weak congruences.

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1. Introduction

In [4] a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. In this paper, we introduce the concept of normal ideal of a subtraction algebra and study properties in connection with multiplies of subtraction algebras. The image and inverse image of a normal ideal of a subtraction algebra are proved to be again normal ideals. Also, we characterize the normal ideals of direct products of subtraction algebras. Finally, the concept of a weak congruence is introduced in subtraction algebras and obtain an interconnection between multipliers and weak congruences.

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2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X;-)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x-(y-x)=x$;
(S2) $x-(x-y)=y-(y-x)$;
(S3) $(x-y)-z=(x-z)-y$.

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a-(a-b)$; the relative complement b' of an element $b \in [0,a]$ is $a-b$; and if $b, c \in [0,a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a-((a-b)\wedge(a-c)) \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

In a subtraction algebra, the following properties are true: for all $x, y, z \in X$,

(p1) $(x-y)-y=x-y$.
(p2) $x-0=x$ and $0-x=0$.
(p3) $(x-y)-x=0$.
(p4) $x-(x-y)\leq y$.
(p5) $(x-y)-(y-x)=x-y$
(p6) $x-(x-(x-y))=x-y$.
(p7) $(x-y)-(z-y)\leq x-z$.
(p8) $x \leq y$ if and only if $x=y-w$ for some $w \in X$.
(p9) $x \leq y$ implies $x-z \leq y-z$ and $z-y \leq z-x$.
(p10) $x, y \leq z$ implies $x-y = x \wedge (z-y)$.
(p11) $(x \wedge y)-(x \wedge z) \leq x \wedge (y-z)$.
(p12) $(x-y)-z=(x-z)-(y-z)$.

A non-empty subset I of a subtraction algebra X is called a *subalgebra* if $x-y \in I$ for all $x, y \in I$. A mapping d from a subtraction algebra X to a subtraction algebra Y is called a *morphism* if $d(x-y)=d(x)-d(y)$ for all $x, y \in X$. A self map d of a subtraction algebra X which is a morphism is called an *endomorphism*.

A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,
(I2) for any $x, y \in X$, $y \in I$ and $x-y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, it is clear that $x \leq y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$. If $x \leq y$ implies $d(x) \leq d(y)$, d is called an *isotone mapping*.

3. Multipliers in subtraction algebras

In what follows, let X denote a subtraction algebra unless otherwise specified.

3.1. Definition. [7] Let $(X,-,0)$ be a subtraction algebra. A self-map f is called a *multiplier* if

$$f(x-y)=f(x)-y$$

for all $x, y \in X$.

3.2. Example. [7] Let $X=\{0,a,b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

Then it is easily checked that f is a multiplier of subtraction algebra X.

3.3. Lemma. [7] *Let f be a multiplier in subtraction algebra X. Then we have $f(0) = 0$.*

3.4. Proposition. *Let X be a subtraction algebra. A multiplier $f : X \to X$ is an identity map if it satisfies $f(x) - y = x - f(y)$ for all $x, y \in X$.*

Proof. Suppose that f satisfy the identity $f(x) - y = x - f(y)$ for all $x, y \in X$. Then $f(x) = f(x - 0) = f(x) - 0 = x - f(0) = x - 0 = x$. Thus f is an identity map. □

3.5. Proposition. [7] *Let f be a multiplier of a subtraction algebra X. Then f is idempotent, that is, $f^2 = f \circ f = f$.*

In general, every multiplier of X need not be identity. However, in the following theorem, we give a set of conditions which are equivalent to be an identity multiplier.

3.6. Theorem. *Let X be a subtraction algebra. A multiplier f of X is an identity map if and only if the following conditions are satisfied for all $x, y \in X$,*
(i) $f(x - y) = f(x) - f(y)$,
(ii) $x - f^2(y) = f(x) - f(y)$.

Proof. The condition for necessary is trivial. For sufficiency, assume that (i) and (ii) hold. Then for $x, y \in X$, we get $x - f(y) = x - f^2(y) = f(x) - f(y) = f(x - y)$. Also, by the definition of the multiplier, we have $f(x - y) = f(x) - y$. Hence

$$f(x - y) = f(x) - y = x - f(y).$$

By Proposition 3.4, f is an identity multiplier of X. □

3.7. Definition. Let X be a subtraction algebra. A non-empty set F of X is called a *normal ideal* if it satisfies the following conditions:
(i) $0 \in F$,
(ii) $x \in F$ and $y \in X$ imply $x - y \in F$.

3.8. Example. Let $X = \{0, a, b, 1\}$ in which "$-$" is defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

It is easy to check that $(X; -, 0)$ is a subtraction algebra. Now consider $F = \{0, a\}$. Then it is easy to check that F is a normal ideal of X.

3.9. Proposition. *Let X be a subtraction algebra. For any $a \in X$, $S_a = \{x - a \mid x \in X\}$ is a subalgebra of X.*

Proof. Let $x - a, y - a \in S_a$. Then $(x - a) - (y - a) = (x - (y - a)) - a \in S_a$. Therefore S_a is a subalgebra of X. □

3.10. Proposition. *Let X be a subtraction algebra. For any $a \in X$, S_a is a normal ideal of X.*

Proof. Clearly, $0 - a = 0 \in S_a$. Let $r \in X$ and $b \in S_a$. Then $b = x - a$ for some $x \in X$. Hence $b - r = (x - a) - r = (x - r) - a \in S_a$. Therefore S_a is a normal ideal of X. □

3.11. Proposition. *Let X be a subtraction algebra. For $u, v \in X$, the set*

$$X(u, v) = \{x \mid (x - u) - v = 0\}$$

is a subalgebra of X.

Proof. Let $x, y \in X(u, v)$. Then we have $(x - u) - v = 0$ and $(y - u) - v = 0$. Hence $((x - y) - u) - v = ((x - u) - y) - v = ((x - u) - v) - y = 0 - y = 0$, which implies $x - y \in X(u, v)$. This completes the proof. □

3.12. Proposition. *Let X be a subtraction algebra. For $u, v \in X$, the set*

$$X(u, v) = \{x \mid (x - u) - v = 0\}$$

is a normal ideal of X, and $u, v \in X(u, v)$.

Proof. Obviously, $0, u, v \in X(u, v)$. Let $x, r \in X$ be such that $x \in X(u, v)$. Then $(x - u) - v = 0$, and so $((x - r) - u) - v = ((x - u) - r) - v = ((x - u) - v) - r = 0 - r = 0$. This implies $x - r \in X(u, v)$. This completes the proof. □

3.13. Proposition. *Let F is a normal ideal of X. For any $w \in X$, the set*

$$F_w = \{x \mid x - w \in F\}$$

is a subalgebra of X.

Proof. Let $x, y \in F_w$. Then $x-w, y-w \in F$. Therefore, $(x-y)-w = (x-w)-(y-w) \in F$, which implies $x - y \in F_w$. This completes the proof. □

3.14. Proposition. *If F is a normal ideal of X, the set F_w is a normal ideal containing F and w.*

Proof. Let $w \in X$. Since $0 - w = 0 \in F$, we have $0 \in F_w$. Let $x, r \in X$ be such that $x \in F_w$. Then $x - w \in F$. Therefore, $(x - r) - w = (x - w) - r \in F$, which implies $x - r \in F_w$. Obviously, F_w contains F and w. This completes the proof. □

Let X_1 and X_2 be two subtraction algebras. Then $X_1 \times X_2$ is also a subtraction algebra with respect to the point-wise operation given by

$$(a, b) - (c, d) = (a - c, b - d)$$

for all $a, c \in X_1$ and $b, d \in X_2$.

3.15. Proposition. *Let X_1 and X_2 be two subtraction algebras. Define a map $f : X_1 \times X_2 \to X_1 \times X_2$ by $f(x, y) = (x, 0)$ for all $(x, y) \in X_1 \times X_2$. Then f is a multiplier of $X_1 \times X_2$ with respect to the point-wise operation.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. The we have

$$\begin{aligned} f((x_1, y_1) - (x_2, y_2)) &= f(x_1 - x_2, y_1 - y_2) \\ &= (x_1 - x_2, 0) \\ &= (x_1 - x_2, 0 - y_2) \\ &= (x_1, 0) - (x_2, y_2) \\ &= f(x_1, y_1) - (x_2, y_2). \end{aligned}$$

Therefore f is a multiplier of the direct product $X_1 \times X_2$. □

3.16. Theorem. *If F_1 and F_2 are normal ideals of X_1 and X_2 respectively, then $F_1 \times F_2$ is a normal ideal of the product algebra $X_1 \times X_2$.*

Proof. Let F_1 and F_2 be normal ideals of X_1 and X_2 respectively. Since $0 \in F_1$ and $0 \in F_2$, we have $(0,0) \in F_1 \times F_2$. Let $(x,y) \in X_1 \times X_2$ and $(x_1,y_1) \in F_1 \times F_2$. Also, since F_1 and F_2 are normal ideals of X_1 and X_2 respectively, we get $x_1 - x \in F_1$ and $y_1 - y \in F_2$. Hence $(x_1,y_1) - (x,y) = (x_1 - x, y_1 - y) \in F_1 \times F_2$. Therefore, $F_1 \times F_2$ is a normal ideal of $X_1 \times X_2$. □

3.17. Theorem. *Let f be a multiplier of subtraction X. For any normal ideal F of X, both $f(F)$ and $f^{-1}(F)$ are normal ideals of X.*

Proof. Clearly, $0 = f(0)$. Let $x \in X$ and $a \in f(F)$. Then $a = f(s)$ for some $s \in F$. Now $a - x = f(s) - x = f(s - x) \in f(F)$ because $s - x \in F$. Therefore $f(F)$ is a normal ideal of X. Since F is a normal ideal of X, we obtain $f(0) = 0 \in F$. Hence $0 = f^{-1}(F)$. Let $x \in X$ and $a \in f^{-1}(F)$. Then $f(a) \in F$. Since F is a normal ideal, we get $f(a - x) = f(a) - x \in F$. Hence $a - x \in f^{-1}(F)$. Therefore $f^{-1}(F)$ is a normal ideal of X. □

3.18. Definition. [4] Let f be a multiplier of a subtraction algebra X. Define the kernel of the multiplier f by

$$Kerf = \{x \in X \mid f(x) = 0\}.$$

3.19. Proposition. *For any multiplier f of a subtraction algebra X, $Kerf$ is a normal ideal of X.*

Proof. Clearly, $0 \in Kerf$. Let $a \in Kerf$ and $x \in X$. Then $f(a-x) = f(a)-x = 0-x = 0$. Hence $a - x \in Kerf$, which implies that $Kerf$ is a normal ideal of X. □

3.20. Definition. Let f be a multiplier of a subtraction algebra. An element $a \in X$ is called a *fixed element* if $f(a) = a$.

Let us denote the set of all fixed elements of X by $Fix_f(X) = \{x \in X \mid f(x) = x\}$ and the image of X under the multiplier f by $Im(f)$.

3.21. Lemma. *Let f be a multiplier of subtraction algebra X. Then $Im(f) = Fix_f(X)$.*

Proof. Let $x \in Fix_f(X)$. Then $x = f(x) \in Im(f)$. Hence $Fix_f(X) \subseteq Im(f)$. Now let $a \in Im(f)$. Then we get $a = f(b)$ for some $b \in X$. Thus $f(a) = f(f(b)) = f(b) = a$, which implies $Im(f) \subseteq Fix_f(X)$. Therefore, $Im(f) = Fix_f(X)$. This completes the proof. □

3.22. Theorem. *Let f be a multiplier of a subtraction algebra X. then we have*
(i) $Fix_f(X)$ is a normal ideal of X.
(ii) $Im(f)$ is a normal ideal of X.

Proof. (i) Since $f(0) = 0$, we have $0 \in Fix_f(X)$. Let $x \in X$ and $a \in Fix_f(X)$. Then $f(a) = a$ Now $f(a - x) = f(a) - x = a - x$. Hence $a - x \in Fix_f(X)$. Therefore, $Fix_f(X)$ is a normal ideal of X.

(ii) Obviously, $0 = f(0)$. Let $x \in X$ and $a \in Im(f)$. Then $a = f(b)$ for some $b \in X$. Now $a - x = f(b) - x = f(b - x) \in f(X)$. Therefore, $Im(f)$ is a normal ideal of X. □

Let us recall from [4] that the composition of two multipliers f and g of a subtraction algebra X is a multiplier of X where $(f \circ g)(x) = f(g(x))$ for all $x \in X$.

3.23. Theorem. *Let f and g be two multipliers of X such that $f \circ g = g \circ f$. Then the following conditions are equivalent.*

(i) $f = g$.

(ii) $f(X) = g(X)$.

(iii) $Fix_f(X) = Fix_g(X)$.

Proof. (i)$\Rightarrow$ (ii): It is obvious.

(ii) $\Rightarrow$ (iii): Assume that $f(X) = g(X)$. Let $x \in Fix_f(X)$. Then $x = f(x) \in f(X) = g(X)$. Hence $x = g(y)$ for some $y \in X$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in Fix_g(X)$. Therefore, $Fix_f \subseteq Fix_g(X)$. Similarly, we can obtain $Fix_g(X) \subseteq Fix_f(X)$. Thus $Fix_f(X) = Fix_g(X)$.

(iii) $\Rightarrow$ (i): Assume that $Fix_f(X) = Fix_g(X)$. Let $x \in X$. Since $f(x) \in Fix_f(X) = Fix_g(X)$, we have $g(f(x)) = f(x)$. Also, we obtain $g(x) \in Fix_g(X) = Fix_f(X)$. Hence we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

Therefore, f and g are equal in the sense of mappings. □

3.24. Definition. Let X be a subtraction algebra. An equivalence relation θ on X is called a *weak congruence* if $(x, y) \in \theta$ implies $(x - a, y - a)$ for any $a \in X$.

Clearly, every congruence on X is a weak congruence on X. In the following theorem, we have an example for a weak congruence in terms of multipliers.

3.25. Theorem. *Let f be a multiplier of a subtraction algebra X. Define a binary operation θ_f on X as follows:*

$$(x, y) \in \theta_f \text{ if and only if } \quad f(x) = f(y) \quad \text{for all } x, y \in X.$$

Then θ_f is a weak congruence on X.

Proof. Clearly, θ_f is an equivalence relation on X. Let $(x, y) \in \theta_f$. Then we have $f(x) = f(y)$. Now for any $a \in X$, we have

$$f(x - a) = f(x) - a = f(y) - a = f(y - a).$$

Hence $(x - a, y - a) \in \theta_f$. □

3.26. Lemma. *Let f be a multiplier of a subtraction algebra X. Then*

(i) $f(x) = x$ for all $x \in f(X)$.

(ii) If $(x, y) \in \theta_f$ and $x, y \in f(X)$, $x = y$.

Proof. (i) Let $x \in f(X)$. Then $x = f(a)$ for some $a \in X$. Now $f(x) = f^2(x) = f(f(x)) = f(a) = x$.

(ii) Let $(x, y) \in \theta_f$ and $x, y \in f(X)$. Then by (i), $x = f(x) = f(y) = y$. □

3.27. Theorem. *Let X be a subtraction algebra and let F be a normal ideal of X. Then there exists multiplier f of X such that $f(X) = F$ if and only if $F \cap \theta_f(x)$ is a single-ton set for all $x \in X$, where θ_f is the congruence class of x with respect to θ_f.*

Proof. Let f be a multiplier of X such that $f(X) = F$. Then clearly $\theta_f$is a weak congruence on X. Let $x \in X$ be an arbitrary element. Since $f(x) = f^2(x)$, we get $(x, f(x)) = \theta_f$. Hence $f(x) \in \theta_f(x)$. Also, $f(x) \in f(X) = F$, which implies $f(x) \in F \cap \theta_f(x)$. Therefore $F \cap \theta_f(x)$ is non-empty. Let a, b be two element of $F \cap \theta_f(x)$. Then by Lemma 3.26, we get $a = b$. Hence $F \cap \theta_f(x)$ is a single-ton set. Conversely, assume that $F \cap \theta_f(x)$ is a single-ton set for all $x \in X$. Let x_0 be the single element of $F \cap \theta_f(x)$. Now define a self map as follows,

$$f : X \to X \text{ by } f(x) = x_0$$

for all $x \in X$. By the definition of the map f, we get $f(a) \in F$ and $f(f(a)) = f(a)$. Since F is normal, we get $f(a) - b \in F$, and so

$$\begin{aligned} f(f(a)) = f(a) &\Rightarrow (f(a), a) \in \theta_f \\ &\Rightarrow (f(a-b), a-b) \in \theta_f \\ &\Rightarrow f(a) - b \in \theta_f(a-b) \\ &\Rightarrow f(a) - b \in F \cap \theta_f(a-b) \quad (f(a) - b \in F) \end{aligned}$$

Since $f(a-b) \in F \cap \theta_f(a-b)$ and $F \cap \theta_f(a-b)$ is a single-ton set, we get $f(a-b) = f(a)-b$. Therefore f is a multiplier of X. □

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