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THE MEAN REMAINING STRENGTH OF SYSTEMS IN A STRESS-STRENGTH MODEL

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Abstract

In this paper, we study the mean remaining strength of a component in the stress-strength setup. We present the models for the mean remaining strength for systems consisting of n independent components under the k-out-of-n:F, parallel and series configurations. We assume that the strengths of the components are nonidentically distributed random variables and components are designed under the common stress.

Keywords: Stress-strength Model, k-out-of-n:F System, Parallel System, Series System.

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1. Introduction

In the reliability theory, stress-strength models generally concern with the probability $P(X > Y)$ which is the reliability of a component with strength X under the random stress Y. It is assumed that X and Y are independent random variables and the component fails if the stress exceeds the strength of the component. Stress-strength models have received some attention in the literature, see, for example, Bhattacharyya and Johnson [4,5], Greco and Ventura [10], Johnson [16], Hanagal [14], Kotz, Lumelskii and Pensky [17] and for some results on the reliability of systems in this setup, see, Devanji and Rao [6], Ebrahimi [7], Eryılmaz [8,9], Hanagal [15].

If the life of a component is defined in terms of the strength to failure, it can allow the experimenter to decide about the useful operational life and the reliability of the component. Guess et al. [11] and Guess et al. [12] have used the mean residual life function to analyze the tensile strength of medium density fiberboard (MDF) data. In this paper, we study the mean remaining strength (MRS) for the simple stress-strength models and their k-out-of-n:F, parallel and series structures when the stress and strengths

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are independent random variables. In Section 2, the MRS in this setup is obtained and illustrated for exponential and Weibull stress-strength distributions. In Section 3, we present the MRS of the systems for independent and nonidentically distributed (INID) components.

2. MRS of a Component Under the Stress

Let X and Y be two independent random variables with cumulative distribution functions $F_X(x)$ and $G_Y(y)$, for $x,\ y > 0$, respectively. It is assumed that X is the strength to failure of a component subject to a random stress; and the component works if its strength is greater than the applied stress, i.e., $X > Y$. Then, we may estimate the component's survival function under the stress Y. We may also wish to learn for how long, on average, the component can still be safe under the stress. The MRS of the component denoted by Φ^Y can be defined as the expected remaining strength under the stress Y. Hence, we have

$$\Phi^Y = E(X - Y | X > Y). \tag{2.1}$$

If we suppose that a component has survived up to the stress Y which is independent of the strength X, the following gives the survival function of the conditional random variable $X - Y | X > Y$.

$$P(X > Y + x | X > Y) = \frac{\int_0^\infty \overline{F}_X(y+x) dG_Y(y)}{\int_0^\infty \overline{F}_X(y) dG_Y(y)}$$

By conditioning on $Y = y$, the MRS in formula (2.1) can be written as follows:

$$\Phi^Y = \frac{1}{\int_0^\infty \overline{F}_X(y) dG_Y(y)} \int_0^\infty \left[\int_0^\infty \overline{F}_X(y+x) dG_Y(y) \right] dx.$$

If the component survives under the applied stress Y, then Φ^Y shows how much strength is expected for the failure of the component.

2.1. Example. If we assume that X and Y follow exponential distribution with the parameters $\lambda_1,\ \lambda_2 > 0$, respectively. It is clear that the MRS of this stress-strength model will be the same as that of the component.

$$\Phi^Y = \frac{1}{\lambda_1}$$

2.2. Example. In reliability theory, Weibull distribution plays an important role. Weibull [21] has given several examples related to the strength of a material such as, yield strength of steel and fiber strength of cotton. The extension of Weibull distribution to stress-strength model may be necessary in the view of its important role in the modeling data. Let X be the strength of a component of which survival function given as below;

$$\overline{F}_X(x) = \exp\left\{-(\lambda_1 x)^\alpha\right\}, \quad \lambda_1, \alpha > 0.$$

We assume that Y follows a Weibull distribution with the scale parameter $\lambda_2 > 0$ and the same shape parameter $\alpha > 0$. Then, the MRS of the Weibull stress-strength model can be followed by

$$\Phi^Y = \frac{\alpha \lambda_2^{2\alpha} \lambda_1^{-\alpha}}{\lambda_2^{\alpha} \lambda_1^{-\alpha} + 1} \int_0^{\infty} \left(\int_0^{\infty} y^{\alpha-1} \exp(-(\lambda_1(y+x))^{\alpha} - (\lambda_1 y)^{\alpha}) dy \right) dx.$$

Figure 1 shows the MRS of the component for some values of the shape parameter α. For each level of α, as the values of λ_1 increases, the MRS of the component decreases. For the MRS, the smaller the α, the greater the values are, as expected.

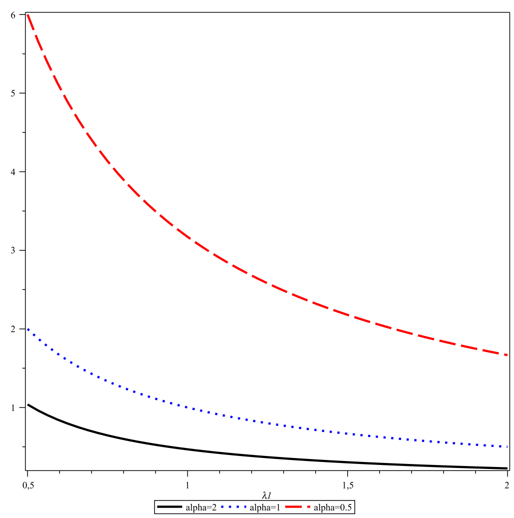

FIGURE 1. The MRS curves for Weibull distributed stress-strength model ($\lambda_2 = 0.5$).

3. MRS of a k-out-of-n:F system

A k-out-of-n:F system works if and only if $n - k + 1$ of the n components work and fails if k or more components fail. Recently papers have appeared investigating the mean residual life function of k-out-of-n systems. Examples of such results for independent and identically distributed components and references can be found in Asadi and Bairamov [1], Asadi and Goliforushani [2], and additional results can be found in Gurler and Bairamov [13], Sadegh [19].

Let $X_1, X_2, \ldots, X_n$ be the independent strengths of the components. We assume that the random variable Y is the common stress and it is independent of $X_1, X_2, \ldots, X_n$. We assume that the strengths are INID random variables with distribution function $F_{X_i}(x)$ and Y follows a distribution given by $G_Y(y)$. We also assume that $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are the ordered strengths of the components. Since the failure of the k-out-of-n:F system depends on the failure of the kth weakest component, the strength of the system is represented by $X_{k:n}$, $k = 1, 2, \ldots, n$.

3.1. Theorem. *If $\Phi_{k:n}^Y$ is the MRS of the k-out-of-n:F system under the condition that all the components function under the stress Y, then for $x, y > 0$,*

$$\Phi_{k:n}^{Y} = \frac{1}{\int_0^\infty \prod_{i=1}^{n} \overline{F}_i(y)dG_Y(y)} \int_0^\infty \left(\sum_{i=0}^{k-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} \left[\overline{F}_{j_l}(y) - \overline{F}_{j_l}(x+y) \right] \times \prod_{l=i+1}^{n} \overline{F}_{j_l}(x+y) dG_Y(y) \right) dx,$$

where the summation extends over all permutations $j_1, \ldots, j_n$ *of* $1, \ldots, n$ *for which* $j_1 < \ldots < j_i$ *and* $j_{i+1} < \ldots < j_n$.

Proof. If $S_{k:n}^{Y}(x|y)$ denotes the survival function of the conditional random variable $X_{k:n} - Y|X_{1:n} > Y$, then,

$$S_{k:n}^{Y}(x|y) = P\left(X_{k:n} - Y|X_{1:n} > Y\right) = \frac{\int_0^\infty P(X_{k:n} > y + x, X_{1:n} > y)dG_Y(y)}{\int_0^\infty P(X_{1:n} > y)dG_Y(y)}$$

$$= \frac{\int_0^\infty \left(\sum_{i=0}^{k-1} \sum_{j_1,\ldots,j_n} \prod_{l=1}^{i} \left[\overline{F}_{j_l}(y) - \overline{F}_{j_l}(x+y) \right] \prod_{l=i+1}^{n} \overline{F}_{j_l}(x+y) \right) dG_Y(y)}{\int_0^\infty \prod_{i=1}^{n} \overline{F}_i(y)dG_Y(y)}. \tag{3.1}$$

See, Gurler and Bairamov [13] and Sadegh [19] for the results on the mean residual life of systems consist of INID components. Then, the MRS can be obtained for the k-out-of-n:F system in the stress-strength setup by noting that

$$\Phi_{k:n}^{Y} = E(X_{k:n} - Y|X_{1:n} > Y) = \int_0^\infty S_{k:n}^{Y}(x|y)dx.$$

□

3.2. Remark. From the theory of order statistics for INID random variables, it can be observed that the MRS of the k-out-of-n:F system under the stress Y, $\Phi_{k:n}^{Y}$ can be written as

$$\Phi_{k:n}^{Y} = \frac{1}{\int_0^\infty \prod_{i=1}^{n} \overline{F}_i(y)dG_Y(y)} \int_0^\infty \left(\sum_{i=0}^{k-1} \frac{1}{i!(n-i)!} \sum_{(j_1,\ldots,j_n)\in S} \prod_{l=1}^{i} \left[\overline{F}_{j_l}(y) - \overline{F}_{j_l}(x+y) \right] \times \prod_{l=i+1}^{n} \overline{F}_{j_l}(x+y) dG_Y(y) \right) dx,$$

where S extends over all $n!$ permutations $j_1, \ldots, j_n$ of $\{1, 2, ..., n\}$.

3.1. MRS of Parallel Systems. A special case for $k = n$ is equivalent to a parallel system. A technical system has a parallel structure if at least 1 of the n components work for the entire system and fails if all components fail. Hence, a parallel system works until the failure of the nth ordered strength of the component. In the sense of the mean residual life of systems such as parallel and series, some papers have appeared and have investigated its properties. See for example, Bairamov et al. [3], Sadegh [18], Shen, Xie and Tang [20].

In a stress-strength setup, there may be parallel configurations of components which have a common stress. For example, if we arrange different electric circuits in parallel with a common voltage passing through them, the resistance of n electric circuits depends on the circuit which allows the maximum voltage. If the voltage is less than the maximum strength of the n electric circuits, then the system will function. This type of stress-strength models may find wide applications in engineering systems.

Let $X_1, X_2, \ldots, X_n$ be the strengths of the components and independent of Y in the stress-strength setup. Let also $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the ordered strengths of the components. We assume that the strengths are INID random variables with distribution function $F_{X_i}(x)$ and Y follows a distribution given by $G_Y(y)$ in the range of Y. If we denote the reliability function of the system by $S^Y_{n:n}$, we have,

$$\begin{aligned} S^Y_{n:n}(x) &= P(X_{n:n} > Y) \\ &= \int_0^\infty \left[1 - \prod_{i=1}^n F_{X_i}(y)\right] dG_Y(y), \quad \text{for } y > 0. \end{aligned} \tag{3.2}$$

3.3. Lemma. *If $\Phi^Y_{n:n}$ is the MRS of the parallel system under the condition that the strongest component functions under the stress Y, then for $x, y > 0$,*

$$\Phi^Y_{n:n} = \frac{1}{\int_0^\infty \left[1 - \prod_{i=1}^n F_{X_i}(y)\right] dG_Y(y)} \int_0^\infty \left[\int_0^\infty \left[1 - \prod_{i=1}^n F_{X_i}(x+y)\right] dG_Y(y)\right] dx.$$

Proof. Let $(X_{n:n} - Y | X_{n:n} > Y)$ define the remaining strength of the parallel system under the common stress. Then, the conditional survival function denoted by $S^Y_{n:n}(x|y)$ can be shown as

$$\begin{aligned} S^Y_{n:n}(x|y) &= P(X_{n:n} - Y > x | X_{n:n} > Y) \\ &= \frac{1}{\int_0^\infty \left[1 - \prod_{i=1}^n F_{X_i}(y)\right] dG_Y(y)} \int_0^\infty \left[1 - \prod_{i=1}^n F_{X_i}(x+y)\right] dG_Y(y). \end{aligned} \tag{3.3}$$

Thus, the proof is completed from (3.3) noting that

$$\Phi^Y_{n:n} = \int_0^\infty S^Y_{n:n}(x|y) dx. \tag{3.4}$$

□

3.2. MRS of Series Systems. A series system works if all of the components work for the entire system and fails if one or more components fail. Most of systems have basic subsystems of which levels are arranged in a series configurations. For example a helicopter can be considered as a system consisting of several subsystems in series and a failure of any subsystem will result in failure for the entire system.

Let us consider a series system with n components with random strengths $X_1, X_2, \ldots, X_n$. Suppose that they are subjected to the common stress Y which is independent of the strengths and the X_i's have marginal distributions given by $F_{X_i}(x)$ for $i = 1, \ldots, n$ and Y follows a distribution given by $G_Y(y)$ in the range of Y. When $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$

are the ordered strengths of the n components, then $X_{1:n}$ represents the strength to failure of the series system. If we denote the survival function of the series system by $S^{Y}_{1:n}$ in the stress-strength setup, then

$$\begin{aligned} S^{Y}_{1:n}(x) &= P(X_{1:n} > Y) \\ &= \int_0^{\infty} \prod_{i=1}^{n} \overline{F}_{X_i}(y) dG_Y(y), \quad \text{for } y > 0. \end{aligned} \tag{3.5}$$

3.4. Lemma. *Let $\Phi^{Y}_{1:n}$ be the MRS of the series system under the condition that all of the components are still in good condition under the stress Y, then for $x, y > 0$,*

$$\Phi^{Y}_{1:n} = \frac{1}{\int_0^{\infty} \prod_{i=1}^{n} \overline{F}_{X_i}(y)\ dG_Y(y)} \int_0^{\infty} \left[\int_0^{\infty} \prod_{i=1}^{n} \overline{F}_{X_i}(x+y) dG_Y(y) \right] dx.$$

Proof. For the proof see, Theorem 3.1. □

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