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DYNAMICS OF A NONLINEAR RATIONAL DIFFERENCE EQUATION

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Abstract

In this paper, we investigate the dynamical properties of the following nonlinear difference equation:

$$x_{n+1} = \frac{x_n^a x_{n-2} x_{n-3} + x_n x_{n-2} x_{n-3}^a + 1}{x_n^a x_{n-3} + x_n x_{n-3}^a + 1} \quad , \quad n = 0, 1, \ldots$$

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1. Introduction

Recently, there has been a great interest in studying the qualitative behavior of rational difference equations. Berenhaut et al.[4] has showed that the unique positive equilibrium $\bar{y} = 1$ of the difference equation:

$$y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k} y_{n-m}},\ n = 0, 1, \ldots$$

is globally asymptotically stable.

Chen et al.[5] investigated the dynamical properties of the following fourth-order nonlinear difference equation:

$$x_{n+1} = \frac{x_{n-2}^a + x_{n-3}}{x_{n-2}^a x_{n-3} + 1},\ n = 0, 1, \ldots$$

with nonnegative initial conditions and $a \in [0, 1)$.

Das [6] investigate the qualitative behavior of the following fourth-order difference equation:

$$x_{n+1} = \frac{x_{n-1} x_{n-2}^a + x_{n-1} x_{n-3}^a + 1}{x_{n-2}^a + x_{n-3}^a + 1},\ n = 0, 1, \ldots$$

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where $a \in (0,\infty)$ and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty)$. For more work, see [1, 2, 3, 7, 8, 9, 10].

To be motivated by the above studies, in this paper, we consider the following nonlinear difference equation:

$$(1.1) \qquad x_{n+1} = \frac{x_n^a x_{n-2} x_{n-3} + x_n x_{n-2} x_{n-3}^a + 1}{x_n^a x_{n-3} + x_n x_{n-3}^a + 1},\ n = 0, 1, \ldots$$

where $a \in (0,\infty)$ and the initial conditions are arbitrary positive real numbers. It is easy to see that the positive equilibrium $\bar{x} = 1$ of Eq.(1.1) satisfies $\bar{x} = (2\bar{x}^{a+2}+1)/(2\bar{x}^{a+1}+1)$.

In the following, we state some main definitions used in this paper.

1.1. Definition. A positive semi-cycle of a solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) consists of a "string" of terms $\{x_\ell, x_{\ell+1}, ..., x_m\}$ all greater than or equal to the equilibrium $\bar{x}$,

$$\begin{aligned}
&\textit{with } \ell \geq -3 \textit{ and } m < \infty \textit{ such that}\\
&\textit{either } \ell = -3 \textit{ or } \ell > -3 \textit{ and } x_{\ell-1} < \bar{x}\\
&\textit{and}\\
&\textit{either } m = \infty \textit{ or } m < \infty \textit{ and } x_{m+1} < \bar{x}
\end{aligned}$$

A negative semi-cycle of a solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) consists of a "string" of terms $\{x_\ell, x_{\ell+1}, ..., x_m\}$ all less than $\bar{x}$,

$$\begin{aligned}
&\textit{with } \ell \geq -3 \textit{ and } m < \infty \textit{ such that}\\
&\textit{either } \ell = -3 \textit{ or } \ell > -3 \textit{ and } x_{\ell-1} \geq \bar{x}\\
&\textit{and}\\
&\textit{either } m = \infty \textit{ or } m < \infty \textit{ and } x_{m+1} \geq \bar{x}
\end{aligned}$$

The length of a semi-cycle is the number of the total terms contained in it.

1.2. Definition. A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) is said to be eventually trivial if x_n is eventually equal to $\bar{x} = 1$; Otherwise the solution is said to be nontrivial. A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) is said to be eventually positive (negative) if x_n is eventually greater (less) than $\bar{x} = 1$.

2. Three Lemmas

Before to draw a qualitatively clear picture for the positive solutions of Eq.(1.1), we first establish three basic lemmas which will play a key role in the proof of our main results.

2.1. Lemma. *A positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) is eventually equal to 1 if and only if*

$$(2.1) \qquad (x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0$$

Proof. Assume that (2.1) holds. Then according to Eq.(**??**), it is easy to see that the following conclusions hold:

(i) if $x_{-2} = 1$, then $x_n = 1$ for $n \geq 40$
(ii) if $x_{-1} = 1$, then $x_n = 1$ for $n \geq 40$
(ii) if $x_0 = 1$, then $x_n = 1$ for $n \geq 40$

Conversely, assume that

(2.2) $\quad (x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0$

Then one can show that

(2.3) $\quad x_n \neq 1 \; for \; any \; n\geq 1$

Assume the contrary that for some $N\geq 1$,

(2.4) $\quad x_N = 1 \; and \; that \; x_n \neq 1 \; for \; -2\leq n\leq N-1$

It is easy to see that

(2.5) $$1 = x_N = \frac{x^a_{N-1}x_{N-3}x_{N-4}+x_{N-1}x_{N-3}x^a_{N-4}+1}{x^a_{N-1}x_{N-4}+x_{N-1}x^a_{N-4}+1}$$

which implies $(x^a_{N-1}x_{N-4}+x_{N-1}x^a_{N-4})(x_{N-3}-1)=0$. Obviously, this contradicts (2.3). □

2.2. Remark. If the initial conditions do not satisfy Eq.(1.1), then, for any solution x_n of Eq.(1.1), $x_n\neq 1$ for $n\geq -3$. Here, the solution is a nontrivial one.

2.3. Lemma. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of Eq.(1.1). Then the following conclusions are true for $n\geq 0$:*

(a) $(x_{n+1}-1)(x_{n-2}-1)>0$
(b) $(x_{n+1}-x_{n-2})(x_{n-2}-1)<0$

Proof. It follows in light of Eq.(1.1) that

$$x_{n+1}-1 = \frac{(x^a_n x_{n-3}+x_n x^a_{n-3})(x_{n-2}-1)}{x^a_n x_{n-3}+x_n x^a_{n-3}+1},\; n=0,1,\dots$$

$$x_{n+1}-x_{n-2} = \frac{(1-x_{n-2})}{x^a_n x_{n-3}+x_n x^a_{n-3}+1},\; n=0,1,\dots$$

from which inequalities (a) and (b) follow. □

2.4. Lemma.

(i) *If $x_{-2},x_{-1},x_0>1$, then $\{x_n\}_{n=-3}^{\infty}$ has a positive semi-cycle with an infinite number of terms and it monotonically tends to the positive equilibrium point $\bar{x}=1$.*
(ii) *If $x_{-2},x_{-1},x_0<1$, then $\{x_n\}_{n=-3}^{\infty}$ has a negative semi-cycle with an infinite number of terms and it monotonically tends to the positive equilibrium point $\bar{x}=1$.*

Proof. (i) If $x_{-2},x_{-1},x_0>1$, from Lemma 2.3.(a) and (b), for $n\geq -3$

$$1<x_{3k-2}<\dots<x_4<x_1<x_{-2}$$
$$1<x_{3k-1}<\dots<x_5<x_2<x_{-1}$$
$$1<x_{3k}<\dots<x_6<x_3<x_0,\; k=0,1,\dots$$

Clearly, $\{x_n\}_{n=-3}^{\infty}$ has a positive semi-cycle with an infinite number of terms and monotonically decreasing for $n\geq 0$. So the limit

(2.6) $$\lim_{n\to\infty} x_n = L$$

exists and finite. Taking the limits on both sides of Eq.(1.1), we have

$$L = \frac{2L^{a+2}+1}{2L^{a+1}+1}$$

we can easily see that $\{x_n\}_{n=-3}^{\infty}$ tends to the positive equilibrium point $\bar{x} = 1$.

(ii) If $x_{-2}, x_{-1}, x_0 < 1$, from Lemma 2.3.(a) and (b), for $n \geq -2$

$$x_{-2} < x_1 < x_4 < ... < x_{3k-2} < 1$$
$$x_{-1} < x_2 < x_5 < ... < x_{3k-1} < 1$$
$$x_0 < x_3 < x_6 < ... < x_{3k} < 1 \; k = 0, 1, \ldots$$

Therefore, $\{x_n\}_{n=-3}^{\infty}$ has a negative semi-cycle with an infinite number of terms and monotonically increasing for $n \geq 0$. So the limit

$$\lim_{n\to\infty} x_n = M \tag{2.7}$$

exists and finite. Taking the limits on both sides of Eq.(1.1), we have

$$M = \frac{2M^{a+2} + 1}{2M^{a+1} + 1}$$

So, $\{x_n\}_{n=-3}^{\infty}$ tends to the positive equilibrium point $\bar{x} = 1$. □

3. Main Results and their proofs

First we analyze the structure of the semi-cycles of nontrivial solutions of Eq.(1.1). Here we confine us to consider the situation of the strictly oscillatory solution of Eq.(1.1).

3.1. Theorem. *Let $\{x_n\}_{n=-3}^{\infty}$ be a strictly oscillatory solution of Eq.(1.1). Then the rule for the lengths of positive and negative semi-cycles of this solution to successively occur is $\ldots 2^+, 1^-, 2^+, 1^-, 2^+, 1^-, \ldots$ or $\ldots 2^-, 1^+, 2^-, 1^+, 2^-, 1^+, \ldots$.*

Proof. By Lemma 2.3.(a) and (b), one can see that the length of a positive semi-cycle is not larger than 2 and the length of a negative semi-cycle is at most 2. Based on the strictly oscillatory character of the solution, we see, for some $p \geq 0$, that one of the following two cases must occur:

Case1. $x_{p-2} > 1,\ x_{p-1} < 1$ *and* $x_p > 1$

Case2. $x_{p-2} > 1,\ x_{p-1} < 1$ *and* $x_p < 1$

If Case 1. Occurs, it follows from Lemma 2.3.(a) that

$$x_{p+1} > 1, x_{p+2} < 1, x_{p+3} > 1, x_{p+4} > 1, x_{p+5} < 1, x_{p+6} > 1, x_{p+7} > 1, x_{p+8} < 1, \ldots$$

It means that the rule of the lengths of positive and negative semi-cycles of the solution of Eq.(1.1) to occur successively is $\ldots 2^+, 1^-, 2^+, 1^-, 2^+, 1^-, \ldots$.

If Case 2. Occurs, it follows from Lemma 2.3.(a) that

$$x_{p+1} > 1, x_{p+2} < 1, x_{p+3} < 1, x_{p+4} > 1, x_{p+5} < 1, x_{p+6} < 1, x_{p+7} > 1, x_{p+8} < 1, x_{p+9} < 1, \ldots$$

It means that the rule of the lengths of positive and negative semi-cycles of the solution of Eq.(1.1) to occur successively is $\ldots 2^-, 1^+, 2^-, 1^+, 2^-, 1^+ \ldots$

Therefore, the proof is complete. □

Now we present the global asymptotically stable results for Eq.(1.1).

3.2. Theorem. *Assume that $a \in (0, \infty)$. Then the positive equilibrium of Eq.(1.1) is globally asymptotically stable.*

Proof. We should prove that the positive equilibrium point $\bar{x}$ of Eq.(1.1) is both locally asymptotically stable and globally attractive. The linearized equation of Eq.(1.1) about the positive equilibrium point $\bar{x} = 1$ is

$$y_{n+1} = 0.y_n + \frac{2}{3}.y_{n-2} + 0.y_{n-3} \quad , \quad n = 0, 1, ...$$

By virtue of [7, Remark 1.3.7], $\bar{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1.1) converges to 1 as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = 1 \tag{3.1}$$

If the solution is nonoscillatory about the positive equilibrium point $\bar{x}$ of Eq.(1.1), then from Lemma 2.1 and Lemma 2.4, the solution is either equal to 1 or eventually positive or negative one which has an infinite number of terms and monotonically tends to the positive equilibrium point $\bar{x}$ of Eq.(1.1), and so Eq.(3.1) holds.Therefore, it suffices to prove that Eq.(3.1) holds for the solution to be strictly oscillatory.

Consider now $\{x_n\}$ to be strictly oscillatory about the positive equilibrium point $\bar{x}$ of Eq.(1.1). By virtue of Theorem 3.1, one understands that the rule for the lengths of positive and negative semi-cycles which occur successively is $\ldots 2^+, 1^-, 2^+, 1^-, 2^+, 1^-, \ldots$ or $\ldots, 2^-, 1^+, 2^-, 1^+, 2^-, 1^+, \ldots$

Now, we investigate the case where the rule for the lengths of positive and negative semi-cycles which occur successively is $\ldots 2^+, 1^-, 2^+, 1^-, \ldots$

For simplicity, we denote by $\{x_t, x_{t+1}\}^+$ the terms of a positive semi-cycle of length two, followed by $\{x_{t+2}\}^-$ the terms of a negative semi-cycle with length one,followed by $\{x_{t+3}, x_{t+4}\}^+$ the terms of a positive semi-cycle of length two, followed by $\{x_{t+5}\}^-$ the terms of a negative semi-cycle with length one,and so on. Namely, the rule for the lengths of positive and negative semi-cycles to occur successively can be periodically expressed as follows for $n = 0, 1, \ldots$:

$$\{x_{t+6n}, x_{t+6n+1}\}^+, \ \{x_{t+6n+2}\}^-, \ \{x_{t+6n+3}, x_{t+6n+4}\}^+, \ \{x_{t+6n+5}\}^-$$

then the following results can be easily observed:

$$1 < x_{t+6n+4} < x_{t+6n+1} \tag{3.2}$$

$$1 < x_{t+6n+6} < x_{t+6n+3} < x_{t+6n} \tag{3.3}$$

$$x_{t+6n+2} < x_{t+6n+5} < 1 \tag{3.4}$$

It follows from 3.2 that $\{x_{t+6n+1}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So the limit

$$\lim_{n\to\infty} x_{t+6n+1} = L$$

exists and finite. Accordingly, by view of 3.2, we obtain

$$\lim_{n\to\infty} x_{t+6n+4} = L$$

Also, it is easy to see from 3.3 that $\{x_{t+6n}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So the limit

$$\lim_{n\to\infty} x_{t+6n} = M$$

exists and finite. By view of 3.4, we obtain

$$\lim_{n\to\infty} x_{t+6n+3} = \lim_{n\to\infty} x_{t+6n+6} = M$$

Lastly, from 3.4 that $\{x_{t+6n+2}\}_{n=0}^{\infty}$ is increasing with upper bound 1. So the limit

$$\lim_{n\to\infty} x_{t+6n+2} = N$$

exists and finite. By view of 3.4, we obtain

$$\lim_{n\to\infty} x_{t+6n+5} = N$$

Taking the limits on both sides of

$$x_{t+6n+6} = \frac{x^a_{t+6n+5}x_{t+6n+3}x_{t+6n+2} + x_{t+6n+5}x_{t+6n+3}x^a_{t+6n+2} + 1}{x^a_{t+6n+5}x_{t+6n+2} + x_{t+6n+5}x^a_{t+6n+2} + 1}$$

one has, $M = (2MN^{a+1} + 1)/(2N^{a+1} + 1)$, which gives rise to $M = 1$.
Similarly, taking the limits on both sides of

$$x_{t+6n+5} = \frac{x^a_{t+6n+4}x_{t+6n+2}x_{t+6n+1} + x_{t+6n+4}x_{t+6n+2}x^a_{t+6n+1} + 1}{x^a_{t+6n+4}x_{t+6n+1} + x_{t+6n+4}x^a_{t+6n+1} + 1}$$

one has, $N = (2NL^{a+1} + 1)/(2L^{a+1} + 1)$, which gives rise to $N = 1$.
Lastly, taking the limits on both sides of

$$x_{t+6n+4} = \frac{x^a_{t+6n+3}x_{t+6n+1}x_{t+6n} + x_{t+6n+3}x_{t+6n+1}x^a_{t+6n} + 1}{x^a_{t+6n+3}x_{t+6n} + x_{t+6n+3}x^a_{t+6n} + 1}$$

one has, $L = (2LM^{a+1} + 1)/(2M^{a+1} + 1)$, which gives rise to $L = 1$.
So we can see that

$$\lim_{n\to\infty} x_{t+6n+k} = 1,\ k = 0, 1, \ldots, 6$$

For $\ldots, 2^-, 1^+, 2^-, 1^+, 2^-, 1^+, \ldots$ can be similarly shown. □

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