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ON THE SPECTRAL NORMS OF TOEPLITZ MATRICES WITH FIBONACCI AND LUCAS NUMBERS

Süleyman Solak*† and Mustafa Bahşi‡

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Abstract

This paper is concerned with the work of the authors' [M.Akbulak and D. Bozkurt, on the norms of Toeplitz matrices involving Fibonacci and Lucas numbers, Hacettepe Journal of Mathematics and Statistics, 37(2), (2008), 89-95] on the spectral norms of the matrices: $A = [F_{i-j}]$ and $B = [L_{i-j}]$, where F and L denote the Fibonacci and Lucas numbers, respectively. Akbulak and Bozkurt have found the inequalities for the spectral norms of $n \times n$ matrices A and B, as for us, we are finding the equalities for the spectral norms of matrices A and B.

Keywords: Spectral norm, Toeplitz matrix, Fibonacci number, Lucas Number.

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1. Introduction and Preliminaries

The matrix $T = [t_{ij}]_{i,j=0}^{n-1}$ is called Toeplitz matrix such that $t_{ij} = t_{j-i}$. In Section 2, we calculate the spectral norms of Toeplitz matrices

(1) $$A = [F_{j-i}]_{i,j=0}^{n-1}$$

and

(2) $$B = [L_{j-i}]_{i,j=0}^{n-1}$$

where F_k and L_k denote k-th the Fibonacci and Lucas numbers, respectively.

Now we start with some preliminaries. Let A be any $n \times n$ matrix. The spectral norm of the matix A is defined as $\|A\|_2 = \sqrt{\max_{1 \le i \le n} |\lambda_i(A^H A)|}$ where A^H is the conjugate transpose of matrix A. For a square matrix A, the square roots of the eigenvalues

*N.E. University, A.K. Education Faculty, 42090, Meram, Konya-TURKEY. E-mail: ssolak42@yahoo.com

†Corresponding Author.

‡Aksaray University, Education Faculty, Aksaray-TURKEY. E-mail: mhvbahsi@yahoo.com

of $A^H A$ are called singular values of A. Generally, we denote the singular values as $\sigma_n = \left\{\sqrt{\lambda_i} : \lambda_i \text{ is eigenvalue of matrix } A^H A\right\}$. Moreover, the spectral norm of matrix A is the maximum singular value of matrix A. The equation $\det(A - \lambda I) = 0$ is known as the characteristic equation of matrix A and the left-hand side known as the characteristic polynomial of matrix A. The solutions of characteristic equation are known as the eigenvalues of matrix.

Fibonacci and Lucas numbers are the numbers in the following sequences, respectively:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \text{ and } 2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots$$

in addition, these numbers are defined backwards by

$$0, 1, -1, 2, -3, 5, -8, 13, -21, \ldots \text{ and } 2, -1, 3, -4, 7, -11, -18, 29, -47, \ldots$$

2. Main Results

2.1. Theorem. *Let the matrix A be as in (1). Then the singular values of A are*

$$\sigma_{1,2} = \begin{cases} F_n, & \text{if } n \text{ is even} \\ \sqrt{F_n^2 - 1}, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \sigma_m = 0, \text{ where } m=3,4,\ldots,n.$$

Proof. From matrix multiplication

$$AA^H = \left[\sum_{k=0}^{n-1} F_{k-i}F_{k-j}\right]_{i,j=0}^{n-1}.$$

By using mathematical induction principle on n, we have

$$\sum_{k=0}^{n-1} F_{k-i}F_{k-j} = \begin{cases} F_{n-1}F_{n-(i+j)} + F_{-i}F_{-j}, & \text{if } n \text{ is odd} \\ F_nF_{n-(i+j+1)}, & \text{if } n \text{ is even} \end{cases}.$$

Since the singular values of matrix A are the square roots of the eigenvalues of matrix AA^H, we must find the roots of characteristic equation $\left|\lambda I - AA^H\right| = 0$, for this there are two cases.

Case I: If n is odd, since $AA^H = \left[F_{n-1}F_{n-(i+j)} + F_{-i}F_{-j}\right]_{i,j=0}^{n-1}$, in this case the characteristic equation:

$$\left|\lambda I - AA^H\right| = \begin{vmatrix} \lambda - F_{n-1}F_n & -F_{n-1}^2 & \cdots & -F_{n-1}F_1 \\ -F_{n-1}^2 & \lambda - F_{n-1}F_{n-2} - F_{-1}F_{-1} & \cdots & -F_{n-1}F_0 - F_{-1}F_{1-n} \\ \vdots & \vdots & & \vdots \\ -F_{n-1}F_1 & -F_{n-1}F_0 - F_{1-n}F_{-1} & \cdots & \lambda - F_{n-1}F_{-n+2} - F_{1-n}F_{1-n} \end{vmatrix} = 0.$$

Let $e\left[(i,j), r, k\right]$ be an elementary row operation, where $e\left[(i,j), r, k\right]$ is addition of k times of addition of ith and jth rows to rth row. Firstly, we apply $e\left[(i+1, i+2), i, -1\right]$, $(i = 1, 2, \ldots, n-2)$. Secondly, we add proper times of first $n-2$ rows to $(n-1)$th row and then to nth row, so we have

$$\begin{aligned} \left|\lambda I - AA^H\right| &= \begin{vmatrix} \lambda & -\lambda & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -\lambda & -\lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - F_n^2 + 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda - F_n^2 + 1 \end{vmatrix} = 0 \\ &= \lambda^{n-2}\left(\lambda - F_n^2 + 1\right)^2 = 0. \end{aligned}$$

Hence, the singular values of the matrix A are

$$\sigma_{1,2} = F_n^2 - 1, \quad \sigma_m = 0, \text{ where } m = 3, 4, \ldots, n.$$

Case II: If n is even, since $AA^H = \left[F_n F_{n-(i+j+1)}\right]_{i,j=0}^{n-1}$, the characteristic equation:

$$\left|\lambda I - AA^H\right| = \begin{vmatrix} \lambda - F_n F_{n-1} & -F_n F_{n-2} & \cdots & -F_n F_1 & -F_n F_0 \\ -F_n F_{n-2} & \lambda - F_n F_{n-3} & \cdots & -F_n F_0 & -F_n F_{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ -F_n F_1 & -F_n F_0 & \cdots & \lambda - F_n F_{3-n} & -F_n F_{2-n} \\ -F_n F_0 & -F_n F_{-1} & \cdots & -F_n F_{2-n} & \lambda - F_n F_{1-n} \end{vmatrix} = 0.$$

If we apply elemanter row operations in *Case I* to the determinant given above, we have

$$\begin{aligned} \left|\lambda I - AA^H\right| &= \begin{vmatrix} \lambda & -\lambda & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -\lambda & -\lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - F_n^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda - F_n^2 \end{vmatrix} = 0 \\ &= \lambda^{n-2}\left(\lambda - F_n^2\right)^2 = 0. \end{aligned}$$

In that case, the singular values of the matrix A are

$$\sigma_{1,2} = F_n^2, \quad \sigma_m = 0, \text{ where } m = 3, 4, \ldots, n.$$

Thus the proof is completed. □

2.2. Corollary. *Let the matrix A be as in (1), then* $\|A\|_2 = \begin{cases} F_n, & \text{if } n \text{ is even} \\ \sqrt{F_n^2 - 1}, & \text{if } n \text{ is odd} \end{cases}$.

Proof. The proof is trivial from Theorem 2.1. □

2.3. Theorem. *Let the matrix B be as in (2). Then the singular values of B are*

$$\sigma_{1,2} = \begin{cases} L_n \pm 1, & \text{if } n \text{ is odd} \\ \sqrt{F_n^2 - 1}, & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad \sigma_m = 0, \text{ where } m = 3, 4, \ldots, n.$$

Proof. From matrix multiplication

$$BB^H = \left[\sum_{k=0}^{n-1} L_{k-i} L_{k-j}\right]_{i,j=0}^{n-1}.$$

By using mathematical induction principle on n, we have

$$\sum_{k=0}^{n-1} L_{k-i}L_{k-j} = \begin{cases} F_{n-(i+j+1)}L_{n-1} + F_{n-(i+j+2)}L_{n+2} - 5F_{-i}F_{-j}, \text{ if } n \text{ is odd} \\ 5F_nF_{n-(i+j+1)}, \text{ if } n \text{ is even} \end{cases}.$$

Firstly, we must find the roots of characteristic equation $\left|\lambda I - BB^H\right| = 0$, for this there are two cases.

Case I: If n is odd, since $BB^H = \left[F_{n-(i+j+1)}L_{n-1} + F_{n-(i+j+2)}L_{n+2} - 5F_{-i}F_{-j}\right]_{i,j=0}^{n-1}$, in this case the characteristic equation:

$$\left|\lambda I - BB^H\right| = \begin{vmatrix} \lambda - F_{n-1}L_{n-1} - F_{n-2}L_{n+2} & \cdots & -F_0L_{n-1} - F_{-1}L_{n+2} \\ -F_{n-2}L_{n-1} - F_{n-3}L_{n+2} & \cdots & -F_{-1}L_{n-1} - F_{-2}L_{n+2} + 5F_{-1}F_{1-n} \\ \vdots & & \vdots \\ -F_0L_{n-1} - F_{-1}L_{n+2} & \cdots & \lambda - F_{1-n}L_{n-1} - F_{-n}L_{n+2} + 5F_{1-n}F_{1-n} \end{vmatrix} = 0.$$

If we apply elementary row operations in *Case I* of Theorem 2.1 to the determinant given above, we have

$$\begin{aligned} \left|\lambda I - BB^H\right| &= \begin{vmatrix} \lambda & -\lambda & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -\lambda & -\lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - a_1 & 2F_{n-3}L_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2F_{n-1}L_n & \lambda - a_2 \end{vmatrix} = 0 \\ &= \lambda^{n-2}\left[\lambda^2 - \left((L_n - 1)^2 + (L_n + 1)^2\right)\lambda + (L_n^2 - 1)^2\right] = 0 \end{aligned}$$

where $a_1 = (L_n - 1)^2 - (2F_{n-2} - 2)L_n$ and $a_2 = (L_n + 1)^2 + (2F_{n-2} - 2)L_n$. Hence, the singular values of the matrix B are

$$\sigma_{1,2} = L_n \pm 1, \ \sigma_m = 0, \text{ where } m = 3, 4, \ldots, n.$$

Case II: If n is even, since $BB^H = \left[5F_nF_{n-(i+j+1)}\right]_{i,j=0}^{n-1}$, in this case the characteristic equation:

$$\left|\lambda I - BB^H\right| = \begin{vmatrix} \lambda - 5F_nF_{n-1} & -5F_nF_{n-2} & \cdots & -5F_nF_0 \\ -5F_nF_{n-2} & \lambda - 5F_nF_{n-3} & \cdots & -5F_nF_{-1} \\ \vdots & \vdots & & \vdots \\ -5F_nF_0 & -5F_nF_{-1} & \cdots & \lambda - 5F_nF_{1-n} \end{vmatrix} = 0.$$

If we apply elementary row operations in *Case I* of Theorem 2.1 to the determinant given above, we have

$$\left|\lambda I - BB^H\right| = \begin{vmatrix} \lambda & -\lambda & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -\lambda & -\lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda - L_n^2 + 4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda - L_n^2 + 4 \end{vmatrix} = 0$$

$$= \lambda^{n-2}\left(\lambda - L_n^2 + 4\right)^2 = 0.$$

Hence, the singular values of the matrix B are

$$\sigma_{1,2} = \sqrt{L_n^2 - 4},\ \sigma_m = 0,\ \text{where } m = 3, 4, \ldots, n.$$

Thus the proof is completed. □

2.4. Corollary. *Let the matrix B be as in (2), then* $\|B\|_2 = \begin{cases} L_n + 1, & \text{if } n \text{ is odd} \\ \sqrt{L_n^2 - 4}, & \text{if } n \text{ is even} \end{cases}$.

Proof. The proof is trivial from Theorem 2.3. □

References

[1] Akbulak, M., and Bozkurt, D. *On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers*, Hacettepe Journal of Mathematics and Statistic, **37** (2), 89-95, 2008.