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FIXED POINTS OF MULTIVALUED MAPPING SATISFYING CIRIC TYPE CONTRACTIVE CONDITIONS IN G-METRIC SPACES

Mujahid Abbas [*] [†] , Talat Nazır [*] [§] and B. E. Rhoades[‡]

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Abstract

In this paper, study of necessary conditions for existence of fixed point of multivalued mappings satisfying Ciric type contractive conditions in the setting of generalized metric spaces is initiated. Examples to support our results are presented. Since every symmetric generalized metric reduces to an ordinary metric, we give a new example of a non-symmetric generalized metric to justify the study of fixed point theory in generalized metric spaces.

Keywords: Multivalued mappings, fixed point, non symmetric, generalized metric spaces.

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1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [10] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. ([9, 11, 12]) obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] motivated the study of common fixed point theory in generalized metric spaces. Recently, Saadati et al. [14] proved some fixed

[*]Department of Mathematics, COMSATS Institute of Information Technology, 22060, Abbottabad, Pakistan.
[†]E-mail: (M. Abbas) ABBAS@lums.edu.pk
[§]E-mail: (T. Nazır) talt@ciit.net.pk
[‡]Department of Mathematics, Indiana University, Bloomington, IN 47405-7106.
E-mail: (B. E. Rhoades) rhoades@indiana.edu

point results for contractive mappings in partially ordered $G-$ metric spaces. Abbas et al. [2] obtain some periodic point results in generalized metric spaces.

The aim of this paper is to prove various fixed points results for multivalued mappings taking closed values in generalized metric spaces. It is worth mentioning that our results do not rely on the notion of continuity of the mappings involved therein. Our results extend and unify various comparable results in ([4], [5] and [13]).

Consistent with Mustafa and Sims [10], the following definitions and results will be needed in the sequel.

1.1. Definition. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(a) $G(x, y, z) = 0$ if $x = y = z$;
(b) $0 < G(x, y, z)$ for all $x, y \in X$, with $x \neq y$;
(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;
(d) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry);
(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a $G-$ metric on X and (X, G) is called a $G-$ metric space.

1.2. Definition. A sequence $\{x_n\}$ in a $G-$ metric space X is:

(i) a $G-$ *Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set natural number) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,
(ii) a $G-$ *Convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A $G-$ metric space on X is said to be $G-$ complete if every $G-$ Cauchy sequence in X is $G-$ convergent in X. It is known that $\{x_n\}$ $G-$ converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$ [10].

1.3. Proposition. [10] *Let X be a $G-$ metric space. Then the following are equivalent:*

(1) *$\{x_n\}$ is $G-$ convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

1.4. Definition. A $G-$ metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

1.5. Proposition. *Every $G-$ metric on X will define a metric d_G on X by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \ \forall \ x, y \in X.$$

For a symmetric $G-$ metric space

$$d_G(x, y) = 2G(x, y, y), \ \forall \ x, y \in X.$$

However, if G is not symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \ \forall \ x, y \in X.$$

Now we give an example of non-symmetric $G-$ metric.

1.6. Example. Let $X = \{1, 2, 3\}$, $G : X \times X \times X \to R^+$, be defined as

(x, y, z)	$G(x, y, z)$
$(1,1,1),(2,2,2),(3,3,3)$	0
$(1,1,2),(1,2,1),(2,1,1),$ $(2,2,3),(2,3,2),(3,2,2),$ $(1,1,3),(1,3,1),(3,1,1),$ $(1,2,2),(2,1,2),(2,2,1),$ $(2,3,3),(3,2,3),(3,3,2)$	1
$(1,2,3),(1,3,2),(2,1,3),$ $(2,3,1),(3,1,2),(3,2,1),$ $(1,3,3),(3,1,3),(3,3,1)$	2

Note that G satisfies all of the axioms of a generalized metric but $G(1,1,3) \neq G(1,3,3)$. Therefore G is not a symmetric on X.
Let X be a $G-$ metric space. We denote by $P(X)$ the family of all nonempty subsets of X, and by $P_{cl}(X)$ the family of all nonempty closed subsets of X.
A point x in X is called a fixed point of a multivalued mapping $T : X \to P_{cl}(X)$ provided $x \in Tx$. The collection of all fixed point of T is denoted by $Fix(T)$.

2. Fixed Point Theorems

Kannan [4] proved a fixed point theorem for a single valued self mapping T of a metric space X satisfying the property

$$d(Tx, Ty) \leq h\{d(x, Tx) + d(y, Ty)\}$$

for all x, y in X and for a fixed $h \in [0, \frac{1}{2})$. Cirić [3] proved a fixed point theorem for a single valued self mapping T of a metric space X satisfying the property

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + e[d(x, Ty) + d(y, Tx)]$$

for all x, y in X and for a fixed $a, b, c, e \geq 0$ with $a + b + c + 2e < 1$. Latif and Beg [5] introduced the notion of a $K-$ multivalued mapping, which is the extension of Kannan mappings, to multivalued mappings. Continuing in this direction, Rus et al. [13] coined the term $R-$ multivalued mapping, which is a generalization of a $K-$ multivalued mapping.
In this section, we obtain some fixed point theorems for a multivalued mapping satisfying Ciric type contractive conditions on generalized metric spaces without using the continuity condition.

2.1. Theorem. *Let X be a complete $G-$ metric space and $T : X \to P_{cl}(X)$. If for each $x, y \in X$, $u_x \in T(x)$ there exist $u_y \in T(y)$ such that*

$$G(u_x, u_y, u_y) \leq h \max\{G(x, y, y), G(x, u_x, u_x), G(y, u_y, u_y), \frac{1}{2}[G(x, u_y, u_y) + G(y, u_x, u_x)]\}, \quad (2.1)$$

where $h \in [0, 1)$, then T has a fixed point.

Proof. Let x_0 be an arbitrary point of X, and $x_1 \in T(x_0)$. Then there exists an $x_2 \in T(x_1)$ such that

$$\begin{aligned} G(x_1, x_2, x_2) &\leq h \max\{G(x_0, x_1, x_1), G(x_0, x_1, x_1), G(x_1, x_2, x_2), \\ &\quad \frac{1}{2}[G(x_0, x_2, x_2) + G(x_1, x_1, x_1)]\} \\ &= h \max\{G(x_0, x_1, x_1), G(x_1, x_2, x_2), \frac{1}{2}[G(x_0, x_2, x_2)]\}. \end{aligned}$$

But, from property (e) of Definition 1.1,

$$\frac{G(x_0,x_2,x_2)}{2} \le \frac{1}{2}[G(x_0,x_1,x_1)+G(x_1,x_2,x_2)] \le \max\{G(x_0,x_1,x_1),G(x_1,x_2,x_2)\},$$

and we now have

$$G(x_1,x_2,x_2) \le h\max\{G(x_0,x_1,x_1),G(x_1,x_2,x_2)\}. \tag{2.2}$$

If $G(x_1,x_2,x_2)=0$, then, by (1) of proposition 1 in [10], $x_1=x_2$. Then $x_2 \in T(x_1) = T(x_2)$, and x_2 is a fixed point of T.
If $G(x_1,x_2,x_2) \neq 0$, then (2.2) becomes

$$G(x_1,x_2,x_2) \le hG(x_0,x_1,x_1).$$

Continuing this process, we obtain a sequence $\{x_n\}$ in X, that is for $x_n \in T(x_{n-1})$, there exists $x_{n+1} \in T(x_n)$ such that

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &\le h\max\{G(x_{n-1},x_n,x_n),G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1}), \\ &\quad (G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n))/2\} \\ &= h\max\{G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1}), \\ &\quad (G(x_{n-1},x_{n+1},x_{n+1}))/2\} \\ &= h\max\{G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1})\}. \end{aligned}$$

Without loss of generality we may assume that $x_n \neq x_{n+1}$ for each n, since, otherwise, it follows that x_{n+1} is a fixed point of T.

Thus we have

$$G(x_n,x_{n+1},x_{n+1}) \le hG(x_{n-1},x_n,x_n) \le ... \le h^nG(x_0,x_1,x_2).$$

For any $m > n \ge 1$, repeated use of property (e) gives

$$\begin{aligned} G(x_n,x_m,x_m) &\le G(x_n,x_{n+1},x_{n+1}) + ... + G(x_{m-1},x_m,x_m) \\ &\le [h^n+h^{n+1}+...+h^{m-1}]G(x_0,x_1,x_1) \le \frac{h^n}{1-h}G(x_0,x_1,x_1), \end{aligned}$$

and so $G(x_n,x_m,x_m) \to 0$ as $n,m \to \infty$. Hence $\{x_n\}$ is a $G-$ Cauchy sequence. By the $G-$ completeness of X, there exist a $u \in X$ such that $\{x_n\}$ converges to u. Let $n \ge N$ be given. Then, for each $x_n \in T(x_{n-1})$, there exists a $u_n \in T(u)$ such that

$$\begin{aligned} G(u_n,u_n,u) &\le G(x_n,u_n,u_n)+G(x_n,x_n,u) \\ &\le h\max\{G(x_{n-1},u,u),G(x_{n-1},x_n,x_n),G(u,u_n,u_n), \\ &\quad \frac{1}{2}[G(x_{n-1},u_n,u_n)+G(u,x_n,x_n)]\}+G(x_n,x_n,u) \\ &\le h\max\{G(x_{n-1},u,u),G(x_{n-1},x_n,x_n),G(u_n,u_n,u), \\ &\quad \frac{1}{2}[G(x_{n-1},u,u)+G(u_n,u_n,u)+G(x_n,x_n,u)]\}+G(x_n,x_n,u). \end{aligned}$$

Now, if

$$\begin{aligned} &\max\{G(x_{n-1},u,u),G(x_{n-1},x_n,x_n),G(u_n,u_n,u), \\ &\frac{1}{2}[G(x_{n-1},u,u)+G(u_n,u_n,u)+G(u,x_n,x_n)]\} \\ = \;& G(x_{n-1},u,u), \end{aligned}$$

implies that

$$G(u_n,u_n,u) \le hG(x_{n-1},u,u)+G(x_n,x_n,u).$$

Taking limit as $n \to \infty$, implies $G(u_n, u_n, u) \to 0$, and $u_n \to u$.
If

$$\begin{aligned} &\max\{G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u_n, u_n, u), \\ &\frac{1}{2}[G(x_{n-1}, u, u) + G(u_n, u_n, u) + G(x_n, x_n, u)]\} \\ = \ & G(x_{n-1}, x_n, x_n), \end{aligned}$$

then

$$\begin{aligned} G(u_n, u_n, u) &\leq hG(x_{n-1}, x_n, x_n) + G(x_n, x_n, u) \\ &\leq hG(x_{n-1}, u, u) + 2G(x_n, x_n, u). \end{aligned}$$

On letting limit $n \to \infty$, implies $G(u_n, u_n, u) \to 0$, and $u_n \to u$.
In case

$$\begin{aligned} &\max\{G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u_n, u_n, u), \\ &\frac{1}{2}[G(x_{n-1}, u, u) + G(u_n, u_n, u) + G(x_n, x_n, u)]\} \\ = \ & G(u_n, u_n, u), \end{aligned}$$

then

$$G(u_n, u_n, u) \leq hG(u_n, u_n, u) + G(x_n, x_n, u)$$

which further implies that

$$G(u_n, u_n, u) \leq \frac{1}{1-h} G(x_n, x_n, u).$$

Taking the limit as $n \to \infty$, implies $G(u_n, u_n, u) \to 0$, gives $u_n \to u$.
Finally, if

$$\begin{aligned} &\max\{G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u_n, u_n, u), \\ &\frac{1}{2}[G(x_{n-1}, u, u) + G(u_n, u_n, u) + G(x_n, x_n, u)]\} \\ = \ & \frac{1}{2}[G(x_{n-1}, u, u) + G(u_n, u_n, u) + G(x_n, x_n, u)], \end{aligned}$$

then

$$\begin{aligned} & G(u_n, u_n, u) \\ \leq \ & \frac{h}{2}[G(x_{n-1}, u, u) + G(u_n, u_n, u) + G(x_n, x_n, u)] + G(x_n, x_n, u) \\ \leq \ & \frac{h}{2} G(x_{n-1}, u, u) + \frac{1}{2} G(u, u_n, u_n) + \frac{3}{2} G(x_n, x_n, u), \end{aligned}$$

which further implies

$$G(u_n, u_n, u) \leq hG(x_{n-1}, u, u) + 3G(x_n, x_n, u).$$

Taking the limit as $n \to \infty$, implies that $G(u_n, u_n, u) \to 0$.
Thus $u_n \to u$ as $n \to \infty$. Since $u_n \in T(u)$ and $T(u)$ is closed, it follows that $u \in T(u)$. □

The following corollary generalizes Theorem 3.1 of Rus et al. [13] to $G-$ metric spaces.

2.2. Corollary. *Let X be a complete $G-$ metric space and $T : X \to P_{cl}(X)$. If for each $x, y \in X$, $u_x \in T(x)$, there exists a $u_y \in T(y)$ such that*

$$\begin{aligned} G(u_x, u_y, u_y) \leq \ & a_1 G(x, y, y) + a_2 G(x, x, y) + a_3 G(x, u_x, u_x) \\ & + a_4 G(x, x, u_x) + a_5 G(y, u_y, u_y) + a_6 G(y, y, u_y), \end{aligned} \tag{2.3}$$

where $a_i \geq 0$ *for* $i = 1, 2, ..., 6$ *and* $a_1 + a_3 + a_5 + 2(a_2 + a_4 + a_6) < 1$, *then* T *has a fixed point.*

Proof. Note that (2.3) implies that

$$\begin{aligned} G(u_x, u_y, u_y) &\leq h \max\{G(x, y, y), G(x, u_x, u_x), G(y, u_y, u_y), \\ &\quad \frac{G(x, x, y)}{2}, \frac{G(x, x, u_x)}{2}, \frac{G(y, y, u_y)}{2}\}, \end{aligned}$$

where $h = a_1 + a_3 + a_5 + 2(a_2 + a_4 + a_6) < 1$.

Which further implies that

$$G(u_x, u_y, u_y) \leq h \max\{G(x, y, y), G(x, u_x, u_x), G(y, u_y, u_y)\},$$

and the result follows from Theorem 2.1. □

2.3. Example. Let $X = [0, \infty)$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ be a symmetric $G-$metric on X. Define $T : X \to P_{cl}(X)$ as

$$Tx = [0, \frac{x}{6}].$$

Now for case $x = y$, $u_x \in Tx$. Take $u_y = 0$, then

$$\begin{aligned} & G(u_x, u_y, u_y) \\ = \ & u_x \leq \frac{x}{6} \\ \leq \ & \frac{2}{12}(0) + \frac{3}{12}(\frac{5x}{6}) + \frac{3}{12}(x) \\ \leq \ & \frac{1}{12}(x - y) + \frac{1}{12}(x - y) + \frac{2}{12}(x - u_x) + \frac{1}{12}(x - u_x) + \frac{2}{12}(y - u_y) + \frac{1}{12}(y - u_y) \\ = \ & a_1 G(x, y, y) + a_2 G(x, x, y) + a_3 G(x, u_x, u_x) \\ & + a_4 G(x, x, u_x) + a_5 G(y, u_y, u_y) + a_6 G(y, y, u_y). \end{aligned}$$

Thus (2.3) is satisfied with $a_1 + a_3 + a_5 + 2(a_2 + a_4 + a_6) = \frac{11}{12}$.

Now when $x < y$, $u_x \in Tx$. Take $u_y = 0$, then

$$\begin{aligned} & G(u_x, u_y, u_y) \\ = \ & u_x \leq \frac{x}{6} \\ \leq \ & \frac{2}{12}(0) + \frac{3}{12}(\frac{5x}{6}) + \frac{3}{12}(x) \\ \leq \ & \frac{1}{12}(y - x) + \frac{1}{12}(y - x) + \frac{2}{12}(x - u_x) + \frac{1}{12}(x - u_x) + \frac{2}{12}(y - u_y) + \frac{1}{12}(y - u_y) \\ = \ & a_1 G(x, y, y) + a_2 G(x, x, y) + a_3 G(x, u_x, u_x) \\ & + a_4 G(x, x, u_x) + a_5 G(y, u_y, u_y) + a_6 G(y, y, u_y). \end{aligned}$$

Thus (2.3) is satisfied with $a_1 + a_3 + a_5 + 2(a_2 + a_4 + a_6) = \frac{11}{12}$.

Finally for, $y < x$, $u_x \in Tx$. Take $u_y = \frac{y}{6}$, then

$$\begin{aligned}
& G(u_x, u_y, u_y) \\
= \; & |u_x - u_y| \le u_x + u_y \le \frac{1}{6}(x+y) \\
\le \; & \frac{2}{12}(x-y) + \frac{3}{12}(\frac{5x}{6}) + \frac{3}{12}(\frac{5y}{6}) \\
\le \; & \frac{1}{12}(x-y) + \frac{1}{12}(x-y) + \frac{2}{12}(x-u_x) + \frac{1}{12}(x-u_x) + \frac{2}{12}(y-u_y) + \frac{1}{12}(y-u_y) \\
= \; & a_1 G(x,y,y) + a_2 G(x,x,y) + a_3 G(x,u_x,u_x) \\
& + a_4 G(x,x,u_x) + a_5 G(y,u_y,u_y) + a_6 G(y,y,u_y).
\end{aligned}$$

Thus (2.3) is satisfied with $a_1 + a_3 + a_5 + 2(a_2 + a_4 + a_6) = \frac{11}{12}$.
Hence all conditions of Corollary 2.2 are satisfied. Moreover, T has a fixed point.

2.4. Corollary. *Let X be a complete $G-$ metric space and $T : X \to P_{cl}(X)$. If for each $x, y \in X$, $u_x \in T(x)$, there exist $u_y \in T(y)$ such that*

$$G(u_x, u_y, u_y) \le \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y), \tag{2.4}$$

where $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma < 1$, then T has a fixed point.

2.5. Example. Let $X = \{0,1\}$ and a nonsymmetric $G-$ metric from X to R^+ be define as

$$\begin{aligned}
G(0,0,0) &= G(1,1,1) = 0, \\
G(0,0,1) &= G(0,1,0) = G(1,0,0) = 0.5, \\
G(0,1,1) &= G(1,0,1) = G(1,1,0) = 1.
\end{aligned}$$

Define $T : X \to P_{cl}(X)$ as

$$T(0) = T(1) = \{0,1\}.$$

Now if $x = 0$, $y = 0$, $u_x \in T(0)$. Then two cases arise.
When $u_x = 0$, take $u_y = 0 \in T(y)$, then

$$\begin{aligned}
G(u_x, u_y, u_y) &= G(0,0,0) = 0 \\
&= \frac{1}{8}(0) + \frac{3}{8}(0) + \frac{3}{8}(0) \\
&= \alpha G(0,0,0) + \beta G(0,0,0) + \gamma G(0,0,0) \\
&= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y).
\end{aligned}$$

When $u_x = 1$, take $u_y = 1 \in T(y)$, then

$$\begin{aligned}
G(u_x, u_y, u_y) &= G(1,1,1) = 0 \\
&< \frac{1}{8}(0) + \frac{3}{8}(1) + \frac{3}{8}(1) \\
&= \alpha G(0,0,0) + \beta G(0,1,1) + \gamma G(0,1,1) \\
&= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y).
\end{aligned}$$

Thus (2.4) is satisfied with $\alpha + \beta + \gamma = \frac{7}{8}$.

For case $x = 0$, $y = 1$, $u_x \in T(0)$. Then for $u_x = 0$, take $u_y = 0 \in T(y)$, then

$$\begin{aligned} G(u_x, u_y, u_y) &= 0 \\ &< \frac{1}{8}(1) + \frac{3}{8}(0) + \frac{3}{8}(0.5) \\ &= \alpha G(0,1,1) + \beta G(0,0,0) + \gamma G(1,0,0) \\ &= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y). \end{aligned}$$

And when $u_x = 1$, take $u_y = 0 \in T(y)$, then

$$\begin{aligned} G(u_x, u_y, u_y) &= G(1,0,0) = 0.5 \\ &< \frac{1}{8}(1) + \frac{3}{8}(1) + \frac{3}{8}(0.5) \\ &= \alpha G(0,1,1) + \beta G(0,1,1) + \gamma G(1,0,0) \\ &= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y). \end{aligned}$$

Thus (2.4) is satisfied with $\alpha + \beta + \gamma = \frac{7}{8}$.
For case $x = 1$, $y = 0$, $u_x \in T(1)$. Then for $u_x = 0$, take $u_y = 0$, we have

$$\begin{aligned} G(u_x, u_y, u_y) &= 0 \\ &< \frac{1}{8}(0.5) + \frac{3}{8}(0.5) + \frac{3}{8}(0) \\ &= \alpha G(1,0,0) + \beta G(1,0,0) + \gamma G(0,0,0) \\ &= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y). \end{aligned}$$

and when $u_x = 1$, again by taking $u_y = 1$, we have

$$\begin{aligned} G(u_x, u_y, u_y) &= G(1,1,1) = 0 \\ &< \frac{1}{8}(0.5) + \frac{3}{8}(0) + \frac{3}{8}(1) \\ &= \alpha G(1,0,0) + \beta G(1,1,1) + \gamma G(0,1,1) \\ &= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y). \end{aligned}$$

Thus (2.4) is satisfied with $\alpha + \beta + \gamma = \frac{7}{8}$.
Finally for $x = 1$, $y = 1$, $u_x \in T(1)$, then for the case $u_x = 0$, take $u_y = 0 \in T(1)$, we have

$$\begin{aligned} G(u_x, u_y, u_y) &= 0 \\ &< \frac{1}{8}(0) + \frac{3}{8}(0.5) + \frac{3}{8}(0.5) \\ &= \alpha G(1,1,1) + \beta G(1,0,0) + \gamma G(1,0,0) \\ &= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y). \end{aligned}$$

And if $u_x = 1$, take $u_y = 1 \in T(1)$, implies

$$\begin{aligned} G(u_x, u_y, u_y) &= G(1,1,1) = 0 \\ &= \frac{1}{8}(0) + \frac{3}{8}(0) + \frac{3}{8}(0) \\ &= \alpha G(1,1,1) + \beta G(1,1,1) + \gamma G(1,1,1) \\ &= \alpha G(x,y,y) + \beta G(x,u_x,u_x) + \gamma G(y,u_y,u_y). \end{aligned}$$

Thus (2.4) is satisfied with $\alpha + \beta + \gamma = \frac{7}{8}$. Hence all the conditions of Corollary 2.4 are satisfied and $Fix(T) \neq 0$.

The following corollary generalizes Theorem 4.1 of Latif and Beg [5] to $G-$ metric Spaces.

2.6. Corollary. *Let X be a complete $G-$ metric space and $T : X \to P_{cl}(X)$. If for each $x, y \in X$, $u_x \in T(x)$, there exist $u_y \in T(y)$ such that*

$$G(u_x, u_y, u_y) \leq h[G(x, u_x, u_x) + G(y, u_y, u_y)],$$

where $0 \leq h < 1$, then T has a fixed point.

2.7. Corollary. *Let X be a complete $G-$ metric space and $T : X \to P_{cl}(X)$. If for each $x, y \in X$, $u_x \in T(x)$, there exist $u_y \in T(y)$ such that*

$$G(u_x, u_y, u_y) \leq \lambda G(x, y, y),$$

where $0 \leq \lambda < 1$, then T has a fixed point.

References

[1] Abbas, M. and Rhoades, B. E. *Common fixed point results for non-commuting mappings without continuity in generalized metric spaces*, Appl. Math. and Computation **215** , 262–269, 2009.

[2] Abbas, M., Rhoades, B. E. and Nazir T. *Some periodic point results in generalized metric spaces*, Appl. Math. and Computation **217** , 4094-4099, 2010.

[3] Cirić, Lj.*Generalized contractions and fixed-point theorems*, Publ. Inst. Math. **12** (26), 19-26, 1971.

[4] Kannan, R. *Some results on fixed points, Bull. Calcutta*, Math. Soc. **60**, 71–76, 1968.

[5] Latif, A. and Beg, I. *Geometric fixed points for single and multivalued mappings*, Demons. Math., **30** (4), 791–800, 1997.

[6] Matkowski, J. *Fixed point theorems for mappings with a contractive iterate at a point*, Proceedings of the American Mathematical Society, **62**(2), 344-348, 1977.

[7] Rus, I. A., Petrusel A. and Sintamarian A. *Data dependence of fixed point set of some multivalued weakly Picard operators*, Nonlinear Analysis **52**, 1944-1959, 2003.

[8] Saadati, R., Vaezpour, S. M., Vetro, P. and Rhoades, B. E. *Fixed point theorems in generalized partially ordered $G-$ metric spaces*, Mathematical and Computer Modelling, in press, 2010.

[9] Mustafa, Z. and Sims, B. *Some remarks concerning $D-$ metric spaces*, Proc. Int. Conf. on Fixed Point Theory and Applications, Valencia (Spain), 189–198, 2003.

[10] Mustafa, Z. and Sims, B. *A new approach to generalized metric spaces*, J. of Nonlinear and Convex Analysis, **7** (2), 289–297, 2006.

[11] Mustafa, Z., Obiedat, H. and Awawdehand, F. *Some fixed point theorem for mapping on complete $G-$ metric spaces*, Fixed Point Theory Appl., **2008**, Article ID 189870, 12 pages.

[12] Mustafa, Z. and Sims, B. *Fixed point theorems for contractive mapping in complete $G-$ metric spaces*, Fixed Point Theory Appl., Article ID 917175, 10 pages, 2009.

[13] Rus, I. A. , Petrusel, A. and Sintamarian, A. *Data dependence of fixed point set of some multivalued weakly Picard operators*, Nonlinear Analysis **52**, 1944–1959, 2003.

[14] Saadati, R., Vaezpour, S. M., Vetro, P. and Rhoades, B. E. *Fixed point theorems in generalized partially ordered $G-$ metric spaces*, Math. and Comp. Modelling, **52** (5-6), 797–801, 2010.