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# Cofibration Category and Homotopies of Three–Crossed Complexes

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**Abstract**

In this work, we show that category of totally free 2–crossed complexes and that of totally free 3–crossed complexes are cofibration categories in the sense of Baues ([4]). We also explore homotopies for 3–crossed modules and 3–crossed complex morphisms.

## 1. Introduction

Crossed modules were first defined by Whitehead in [15]. They model homotopy connected 2–types. Conduche ([9]), in 1984, described the notion of 2–crossed modules as a model for homotopy connected 4–types. Eventually, Arvasi, Uslu and Kuzpinari ([2]) introduced 3–crossed modules as a model for homotopy connected 4–types. The definition of a homotopy of morphisms of crossed complexes is well known due to Whitehead and this was put in the general context of crossed complexes (of groupoids) by Brown and Higgins in [6]. Also homotopies for 2–crossed complexes can be found in Martin's work [11]. By following Martin's method we give homotopies for 3–crossed complexes.

T.Porter explains cofibration category as follows: The notion of cofibration category was introduced by Hans–Joachim Baues as a variant of the category of cofibrant objects, (for which, see category of fibrant objects and dualise). The axioms are substantially weaker than those of Quillen's model category [13], but add one axiom to those of K. S. Brown. In the first chapter of his book, Algebraic Homotopy, Baues suggests two criteria for an axiom system:

1. The axioms should be sufficiently strong to permit the basic constructions of homotopy theory. 2. The axioms should be as weak (and as simple) as possible, so that the constructions of homotopy theory are available in as many contexts as possible.

Baues in [3] has shown that category of totally free crossed complexes and category of totally free quadratic complexes are cofibration category.

In this article, we obtain similar results. We show that the category of totally free 2–crossed complexes is a cofibration category and we define homotopies for morphisms of 2–crossed complexes. Then we get the following result: Homotopy classes of category of

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totally free 2–crossed complexes is equivalent to the localization of 2–crossed complexes with respect to weak equivalences.

## 2. Preliminaries

**2.1. Definition.** A Baues "cofibration category" is a category $(C, cof, we)$ consisting of a category $C$ and two distinguished classes of maps $cof$ and $we$, called "cofibrations" and "weak equivalences" respectively. A map in $C$ is a trivial cofibration if it is both a weak equivalence and a cofibration. Maps in $C$ are subject to the axioms below:

**BCF1**: All isomorphisms of $C$ are trivial cofibrations. Cofibrations are stable under composition.

**BCF2**: (Two out of three axiom) If $f, g$ are maps of $C$ such that $gf$ is defined, if any two of $f, g, gf$ are weak equivalences, then so is the third.

**BCF3**: (Push out axiom) Given a solid diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A \cup_B Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ B & \xrightarrow[f]{} & Y \end{array}$$

in $C$, with $i$ being a cofibration, then the pushout exists in $C$ and $\bar{i}$ is a cofibration. Moreover:
($a$) if $f$ is a weak equivalence, so is $\overline{f}$,
($b$) if $i$ is a weak equivalence, so is $\bar{i}$.

**BCF4**: (Factorization axiom) Any map of $C$ admits a factorization as a cofibration followed by a weak equivalence.

**BCF5**: (Axiom on fibrant models) For each object $X$ in $C$ there is a trivial cofibration $X \to RX$ where $RX$ is fibrant in $C$. An object $R$ in a cofibration category is fibrant if each trivial cofibration $i : R \to Q$ admits a retraction $r : Q \to R$ such that $ri = 1$. We call $X \to RX$ a fibrant model of $X$; if $X$ is fibrant we take $RX = X$.

**2.2. Definition.** We call $(C, cof, we)$ a"cofibration structure" if all axioms of cofibration category are satisfied except the axiom $BCF3(a)$.

Hence a cofibration structure which satisfies $BCF3(a)$ is a cofibration category. For example, let $(C, cof, fib, we)$ be a model category in the sense of Quillen, then $(C, cof, we)$ is a cofibration structure. An object $X$ in a category $C$ is cofibrant if $* \to X$ is a cofibration where $*$ is the initial object. A full subcategory of a category $C$ consisting of cofibrant objects is denoted by $C_c$.

## 3. Cofibrations in the Category of 2–crossed Complexes

The following definition of 2–crossed module is equivalent to that given by Conduché.
*A 2–crossed module* of groups consists of a complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with (a) actions of $N$ on $M$ and $L$ so that $\partial_2, \partial_1$ are morphisms of $N$–groups, and (b) an $N$–equivariant function

$$\{\ \ ,\ \ \} : M \times M \longrightarrow L,$$

called a Peiffer lifting. This data must satisfy the following axioms:

$$\begin{array}{llrcl}
\mathbf{2CM1}) & & \partial_2\{m,m'\} & = & mm'm^{-1}\left({}^{\partial_1 m}m'^{-1}\right)\\
\mathbf{2CM2}) & & \{\partial_2 l,\partial_2 l'\} & = & [l',l]\\
\mathbf{2CM3}) & (i) & \{mm',m''\} & = & {}^{\partial_1 m}\{m',m''\}\{m,m'm''m'^{-1}\}\\
 & (ii) & \{m,m'm''\} & = & \{m,m'\}^{mm'm^{-1}}\{m,m''\}\\
\mathbf{2CM4}) & & \{m,\partial_2 l\}\{\partial_2 l,m\} & = & {}^{\partial_1 m}ll^{-1}\\
\mathbf{2CM5}) & & {}^{n}\{m,m'\} & = & \{{}^{n}m,{}^{n}m'\}
\end{array}$$

for all $l,l'\in L$, $m,m',m''\in M$ and $n\in N$.

**3.1. Definition.** [12] A 2–crossed complex $C=\{C_n,d_n,\{\quad,\quad\}\}$ is a diagram

$$\cdots \xrightarrow{d_5} C_4 \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

of homomorphisms between groups such that $d_{n-1}d_n=1$ for $n\geq 2$ and such that the following properties are satisfied. The $(\{\quad,\quad\},d_2,d_1)$ is a 2–crossed module. Moreover $C_n$ is a right $\pi$–module.

A map $f:C\to C'$ between 2–crossed complexes is a family of homomorphisms between groups for $n\geq 1$

$$f_n : C_n \to C'_n \quad \text{with} \quad f_{n-1}d_n = d_n f_n$$

such that $(f_3,f_2,f_1)$ is a map between 2–crossed modules.

Let $\mathbf{X_2Comp}$ be the category of 2–crossed complexes and maps, we define the homotopy groups

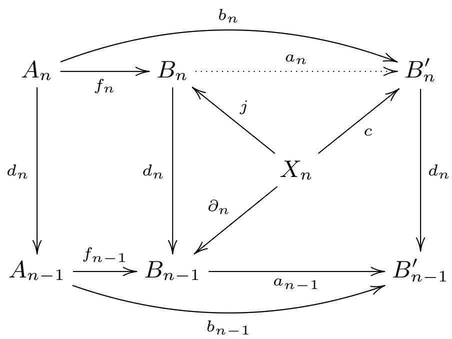
$$\pi_1(C)=\pi=\operatorname{coker}(d_1)$$

$$\pi_n(C)=\frac{\ker d_n}{\operatorname{im} d_{n+1}},\quad n\geq 1.$$

A map $f_n$ is a weak equivalence if $\pi_n(f_n)$ is an isomorphism for $n\geq 1$. We call a 2–crossed complex $C$ totally free if $C_0$ is a free group, $d_1:C_1\to C_0$ is a free pre–crossed module, and $d_2:C_2\to C_1$ is given by a free 2–crossed module. Totally free objects form a full subcategory of 2–crossed complexes. We denote it by $\mathbf{FreeX_2}$. Clearly, $\mathbf{FreeX_2}\subset\mathbf{X_2Comp}$.

**3.2. Definition.** A map $f:A\to B$ in 2–crossed complexes is a cofibration if $f$ is a free extension in each degree $n$ for $n\geq 1$.

Here we say that $f$ is a free extension in degree $n$ with basis $\partial_n : X_n\to B_{n-1}$ where $X_n$ is a set and a map $j:X_n\to B_n$ with $d_n j=\partial_n$ which satisfy the following universal property. Let $B'$ be any 2–crossed complex, $A^n,B^n,B'^n$ be $n$–skeleton of $A,B,B'$ and let $b:A\to B'$, $a^{n-1}:B^{n-1}\to B'^{n-1}$ be 2–crossed complex maps with $a^{n-1}f^{n-1}=b^{n-1}:A^{n-1}\to B'^{n-1}$ and assume a function $c:X_n\to B'_n$ is chosen such that the following diagram of unbroken arrows commutes.




Then there is a unique 2–crossed complex map $a : B^n \to {B'}^n$ for which $a_n$ extends the diagram commutatively. It is clear that cofibrant objects in 2–crossed complexes are exactly the totally free 2–crossed complexes. Then

$$\mathbf{FreeX_2} = \mathbf{X_2Comp_c}$$

where $\mathbf{X_2Comp_c}$ denotes full subcategory of category of 2–crossed complexes consisting of cofibrant objects. The next lemma shows that free extension in each degree exists.

**3.3. Lemma.** *Let $A^n$ be an $n$–skeleton and assume $f^{n-1} : A^{n-1} \to B^{n-1}$ and a function $\partial_n : X_n \to B_{n-1}$ are given. Then a free extension $f : A^n \to B^n$ with basis $\partial_n$ exists provided that $d_{n-1}\partial_n = 1$.*

Proof is analogue to the case of free extensions in the quadratic complexes in [3].

**3.4. Theorem.** *The category of 2–crossed complexes with cofibrations and weak equivalences is a cofibration structure for which all objects are fibrant.*

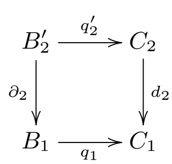
*Proof.* We first check $(BCF4)$. We obtain a factorization

$$f : A \xrightarrow{\ i\ } B \xrightarrow{\ q\ } C$$

of $f : A \to C$ such that $i$ is a cofibration and $q$ is a weak equivalence.

For $n = 1$, $B_1$ is free product of $A_1$ and $F(X_1)$ where $X_1$ is a set and $F(X_1)$ is the free group generated by $X_1$. We choose $X_1$ and $q_1$ such that $B_1 \to C_1 \to \pi_1(C_1)$ is surjective.

For $n = 2$, $B'_2$ is free product of $A'_2$ and $F(X_2)$ where $A'_2$ is the $B_1$–pre–crossed module induced from the $A_1$–pre–crossed module by the morphism of 2–crossed module $A_1 \to B_1$ and $F(X_2)$ is the free pre–crossed $B_1$–module on the set $X_2$. Here we choose a basis $X_2$ and $\partial_2 : B'_2 \to B_1$ such that $\partial_2(B_2)$ is the kernel of $B_1 \to C_1 \to \pi_1(C_1)$. Then we should find $q'_2$ such that the diagram




commutes. The map $q'_2$ is not surjective. Therefore choose a set $X'_2$ such that $B_2$ is free product of $B'_2$ and $F(X'_2)$. Then take $q_2$ as carrying kernel of $\partial_2$ surjectively to kernel of $d_2$.

For $n \geq 3$, $B_n$ is chosen in a similar way. This completes the construction of factorization.

We next check that all objects are fibrant. Let a map $i : A \to B$ be given as a trivial cofibration. We construct inductively a retraction $r : B \to A$ with $ri = 1$ and a homotopy $\alpha : ir \simeq 1_{relA}$ This shows that $i$ is actually a strong deformation retract morphism. Let

$$(\#) \qquad i : A \longrightarrow B_n \xrightarrow[g_n]{} B, \qquad r_n : B_n \dashrightarrow A$$

be given by the subcomplex $B^n$ of $B$ with $(B^n)_k = B_k$ for $k \leq n$ and $(B^n)_k = A_k$ for $k > n$. The map $g_n$ is the inclusion. We choose inductively a retraction $r_n$ and a homotopy $\alpha_n : ir_n \simeq g_n$ relative to $A$. Assume $r_n$ and $\alpha_n$ are defined by

$$(*) \qquad (ir_n)^{-1}g_n = (d\alpha_n)(\alpha_n d).$$

When we compose each side by $d$, we get

$$ir_n d = (g_n d)(d\alpha_n d)^{-1} = d(g_n(\alpha_n d)^{-1}).$$

Since $i$ is a weak equivalence we can choose a map $x : X_{n+1} \to A_{n+1}$ with $dx = r_n d$. Moreover $(ix)^{-1}g_{n+1}(\alpha_n d)^{-1}$ carries $X_{n+1}$ to the cycles of $B$ by $(*)$. Again since $i$ is a weak equivalence we can choose maps $z : X_{n+1} \to A_{n+1}$, $y : X_{n+1} \to B_{n+2}$ such that

$$(iz)(dy) = (ix)^{-1}g_{n+1}(\alpha_n d)^{-1}.$$

We now define the extension $r_{n+1}$ of $r_n$ by $r_{n+1} = xz$ on $X_{n+1}$ and we define the extension $\alpha_{n+1}$ of $\alpha_n$ by $\alpha_{n+1} = y$ on $X_{n+1}$. This completes the induction.

Moreover, we construct push outs in 2–crossed complexes

$$\begin{array}{ccc} B & \xrightarrow{\overline{f}} & B' \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \overline{i}} \\ A & \longrightarrow & A' \end{array}$$

as follows. Let $B_n$ be a free extension of $A$ Then we set $B'_n$ as free extension of $A'$. The basis of $B'$ is given in degree $n$ by the composition

$$f_{n-1}d|_{X_n} : X_n \to A_{n-1} \to B_{n-1}, \quad n \geq 2.$$

The map $\overline{f}$ is the identity on $X_n$.

Finally we prove $(BCF3b)$. If $i$ is a weak equivalence, then $i$ is a strong deformation retract by $(\#)$. This implies also that $\overline{i}$ is a strong deformation retract. In fact we define the retraction $\overline{r}$ of $\overline{i}$ by $fr$ in $(\#)$. And we define the homotopy $\overline{\alpha} : \overline{ir} \simeq 1_{relA'}$ by $\overline{f}\alpha$ on generators. This shows that $\overline{i}$ is a weak equivalence and $(BCF3b)$ is satisfied. □

The next lemma is given in [4].

**3.5. Lemma.** *Let $C$ be a cofibration structure. Then $C_c$ with cofibrations and weak equivalences as in $C$ is a cofibration category.*

As a result of above theorem and lemmas we give the following result.

**3.6. Corollary.** *The category of totally free 2–crossed complexes is a cofibration category*

## 4. Homotopy of 2–crossed Modules

Recall the notion of homotopy between crossed complexes in [7]. Now similarly we define homotopy for 2–crossed complexes. Let

$$A = L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N \ , \quad A' = L' \xrightarrow{\partial_2} M' \xrightarrow{\partial_1} N'$$

be two 2–crossed modules and $f = (f_0, f_1, f_2)$ be a 2–crossed module morphism. A homotopy on $f$ is a pair $h = (h_1, h_2)$ of maps where $h_1 : N \to M'$ and $h_2 : M \to L'$ satisfying equations below.

$$\begin{aligned} h_1(nn') &= {}^{f_0(n')^{-1}}h_1(n)h_1(n') \\ h_2(mm') &= {}^{((f_1m')(h_1\partial_1 m'))^{-1}}(\{f_1m', {}^{f_0\partial_1 m'^{-1}}(h_1\partial_1 m^{-1})\}h_2(m))h_2(m') \\ \partial_2 h_2({}^n m) &= h_1(n)f_0(n)h_1\partial_1(m^{-1})^{f_0\partial_1(m^{-1})}h_1(n)f_1(m^{-1})\partial_1 h_1(n)f_1(n)h_1\partial_1(m) \\ &\qquad \partial_2 h_2(m)\partial_1 h_1(n^{-1})f_0(n^{-1}) \\ h_2\partial_2({}^n l) &= f_0(n)f_2(l^{-1})\partial_1 h_1(n)f_2(l)\partial_3 h_2(l)\partial_1 h_1(n^{-1})f_0(n^{-1}) \end{aligned}$$

Such a function $h$ is called a quadratic $f$–derivation.

**4.1. Proposition.** *Given a homotopy as above, the formulas*
$f'_0(n) = f_0(n)\partial_1 h_1(n)$
$f'_1(m) = f_1(m)h_1\partial_1(m)\partial_2 h_2(m)$
$f'_2(l) = f_2(l)h_2\partial_2(l)$
*for all $n \in N, m \in M, l \in L$ define a morphism of 2–crossed modules.*

We leave the proof as an exercise.

**4.1. Homotopies of 2–crossed Complexes.** Let $A$ and $A'$ be two 2–crossed complexes and let $f$ be a 2–crossed complex map $A \to A'$. A quadratic $f$–derivation is a sequence of maps $h_i : A_i \to A'_{i+1}$ such that $(h_2, h_1)$ is a quadratic $f$–derivation of 2–crossed modules and all the remaining maps are $A_1$–equivariant for $n = 3$ and $A_1/\partial A_2$–equivariant for $n \geq 4$. We say that two 2–crossed complex maps are homotopic if there exists a quadratic $f$–derivation such that

$$f'_1(a) = f_1(a)\partial_2 h_1(a) \quad \text{and} \quad f'_n(a) = f_n(a)(h_{n-1}\partial(a))(\partial h_n(a)) \quad \text{for} \quad n \geq 2.$$

In [6] Brown and Higgins extended the notion of homotopy to $n$–fold homotopies. In this manner a 0–fold homotopy between two 2–crossed complexes $B$ and $C$ is simply a morphism $B \to C$. For $n \geq 1$ an $n$–fold homotopy $B \to C$ is a pair $(h, f)$, where $f : B \to C$ is a morphism of crossed complexes and $h$ is a map of degree $n$ from $B$ to $C$ i.e., $h : B_k \to C_{k+n}$. 1–fold homotopy is the homotopy we have just defined above.

Moreover, we have equivalence of categories. In [4], a corollary (IV.5.7) for quadratic complexes is given. Since we have homotopy relation for 2-crossed complexes and Corollary 3.6 then we can give analogue lemma for 2–crossed complexes.

**4.2. Lemma.** *Homotopy classes of the category of totally free 2–crossed complexes is equivalent to the localization of 2–crossed complexes with respect to weak equivalences which can be pictured as a functor $M$,*

$$\mathbf{M} : \mathbf{Ho}(\mathbf{X_2Comp}) \longrightarrow \mathbf{FreeX_2}/\simeq$$

For the proof, see [4]; for the localization of a category with respect to a class of morphisms see [13].

## 5. Cofibrations in the Category of 3–crossed Complexes

We follow conventions of [2] for the definition of 3–crossed modules.

**5.1. Definition.** A 3–crossed module consists of a complex of groups

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with an action of $N$ on $K, L, M$ and an action of $M$ on $K, L$ and an action of $L$ on $K$ so that $\partial_3, \partial_2, \partial_1$ are morphisms of $N, M$–groups. And the $M, N$–equivariant liftings

$$\begin{array}{ll} \{\ ,\ \}_{(1)(0)} : L \times L \longrightarrow K, & \{\ ,\ \}_{(0)(2)} : L \times L \longrightarrow K, \\ \{\ ,\ \}_{(2)(1)} : L \times L \longrightarrow K, & \{\ ,\ \}_{(1,0)(2)} : M \times L \longrightarrow K, \\ \{\ ,\ \}_{(2,0)(1)} : M \times L \longrightarrow K, & \{\ ,\ \}_{(0)(2,1)} : L \times M \longrightarrow K, \\ \{\ ,\ \} : M \times M \longrightarrow L & \end{array}$$

are called 3–dimensional Peiffer liftings. This data must satisfy the axioms ($3CM1-3CM18$) given in [2].

Here we give the definition of 3–crossed complex of groups.

**5.2. Definition.** A 3–crossed complex $C = \{C_n, d_n, \{\ ,\ \}_{(2)(1)}, \{\ ,\ \}\}$ is a diagram of homomorphisms between groups

$$\cdots \xrightarrow{d_5} C_4 \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

such that $d_{n-1}d_n = 1$ for $n \geq 2$ and $(\{\ ,\ \}_{(2)(1)}, \{\ ,\ \}, d_3, d_2, d_1)$ is a 3–crossed module with $\pi = coker d_1$; hence $\ker d_2$ is a $\pi$–module. Moreover; $C_n$ is a right $\pi$–module. A map $f : C \to C'$ between 3–crossed complexes is a family of homomorphisms between groups for $n \geq 1$

$$f_n : C_n \to C'_n \quad \text{with} \quad f_{n-1}d_n = d_n f_n$$

such that $(f_4, f_3, f_2, f_1)$ is a map between 3–crossed modules. Let $\mathbf{X_3Comp}$ be the category of 3–crossed complexes and maps, we define the homotopy groups

$$\pi_1(C) = \pi = \operatorname{coker}(d_1)$$

$$\pi_n(C) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}, \quad n \geq 2.$$

We call a 3–crossed complex $C$ totally free if $C_1$ is a free group, $d_2 : C_1 \to C_0$ is a free pre–crossed module, and $d_3, d_2, d_1$ are given by a free 3–crossed module. Let $\mathbf{FreeX_3} \subset \mathbf{X_3Comp}$ be the full subcategory consisting of totally free 3–crossed complexes. In the second section, we introduced cofibrations in the category of 2–complexes by use of the universal properties of free extensions. In the same way we now define cofibrations in the category of 3–crossed complexes.

**5.3. Definition.** A map $f : A \to B$ in 3–crossed complexes is a cofibration if $f$ is a free extension in each degree $n$, $n \geq 1$.

Here we define a free extension in each degree $n$ literally in the same way as in 2–crossed complexes.

The cofibrant objects in 3–crossed complexes are exactly the totally free 3–crossed complexes; hence we get the notation

$$\mathbf{FreeX_3} = \mathbf{X_3Comp_c}$$

**5.4. Lemma.** *Let $A^n$ be an $n$–skeleton of a 3–crossed complex $A$, let $f^{n-1} : A^{n-1} \to B^{n-1}$ be a morphism in 3–crossed complex and let $\partial_n : X_n \to B_{n-1}$ be a function. Then a free extension $f : A^n \to B^n$ with basis $\partial_n$ exists provided that $d_{n-1}\partial_n = 1$.*

*Proof.* If $X$ is a set, $F(X)$ will denote the free group on $X$.
For $n = 1$ we set free product of groups;

$$B_1 = A_1 * F(X_1).$$

For $n = 2$ we consider the free pre–crossed module

$$\overline{d_2} : \overline{B_2} = F((A_2 \cup X_2) \times B_1) \to B_1$$

with basis $(f_1 d_2, \partial_2) : A_2 \cup X_2 \to B_1$ where $A_2 \cup X_2$ is disjoint union. The inclusion $i : A_2 \to \overline{B_2}$, however, is not a map between pre–crossed modules. Let $U$ be the normal subgroup of $\overline{B_2}$ generated by the relations

$$i(x)i(y)i(xy)^{-1} \simeq 1$$

$$i(x^{\alpha})({}^{f_1\alpha} i(x))^{-1} \simeq 1$$

for $x, y \in A_2, \alpha \in A_1$. Then $\overline{d_2}$ induces the pre–crossed module $d_2 : B_2 = \overline{B_2}/U \to B_1$. One readily checks that $d_2$ has the universal property of free extensions.
For $n = 3$, we consider the free pre–crossed module

$$\overline{d_3} : \overline{B_3} = F((A_3 \cup X_3) \times B_2) \to B_2$$

with basis $(f_2 d_3, \partial_3) : A_3 \cup X_3 \to B_2$ where $A_3 \cup X_3$ is disjoint union. The map $j : A_3 \to \overline{B_3}$, is not a map between pre–crossed modules. Let $V$ be the normal subgroup of $\overline{B_3}$ generated by the relations

$$j(x)j(y)j(xy)^{-1} \simeq 1$$

$$j(x^{\alpha})({}^{f_1\alpha} j(x))^{-1} \simeq 1$$

$$j\{\ ,\ \}(v)(\overline{\{\ ,\ \}} f_2(v) \times f_2(v))^{-1} \simeq 1$$

where $x, y \in A_3, \alpha \in A_1, v \in A_2 \times A_2$, and $\overline{\{\ ,\ \}} : B_2 \times B_2 \to \overline{B_3}$. Then $\overline{d_3}$ induces the pre–crossed module $d_3 : B_3 = \overline{B_3}/V \to B_2$.
For $n = 4$, the 3–crossed module $B^4$ is as follows. Let

$$\overline{B_4} \to B_3 \to B_2 \to B_1$$

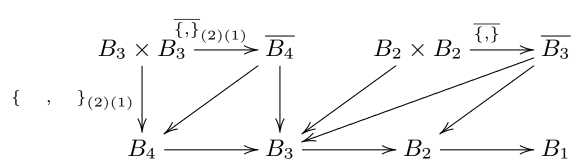
be the free 3–crossed module with Peiffer map $\overline{\{\ ,\ \}}_{(2)(1)} : B_3 \times B_3 \to \overline{B_4}$ with basis $(f_3 d_4, \partial_4) : A_4 \cup X_4 \to B_3$. The inclusion $k : A_4 \to \overline{B_4}$, however, is not a map between 3–crossed modules. Let $Y$ be the normal subgroup of $\overline{B_4}$ generated by the relations

$$k(x)k(y)k(xy)^{-1} \simeq 1$$

$$k(x^{\alpha})({}^{f_1\alpha} k(x))^{-1} \simeq 1$$

$$k\{\ ,\ \}_{(2)(1)}(p)(\overline{\{\ ,\ \}}_{(2)(1)} f_3(p) \times f_3(p))^{-1} \simeq 1$$

where $x, y \in A_4, \quad \alpha \in A_1, \quad p \in A_3 \times A_3$. Then above diagram induces the commutative diagram




where $B_4 = \overline{B_4}/Y$ is the quotient group with induced action of $B_1$ and diagram is a well defined 3–crossed module. The bottom row with $\{\ ,\ \}_{(2)(1)}$ is a well defined 3–crossed module and one readily checks the universal property of free extensions is satisfied. Finally, for $n \geq 5$, $B_n$ is the direct sum of a free $R$–module generated by $X_n$ and $K$. Hence $K$ is the tensor product of $A_n$ and group ring of $\pi_1(A)$ □

**5.5. Theorem.** *The category of 3–crossed complexes with cofibrations and weak equivalences is a cofibration structure for which all objects are fibrant.*

We use the same arguments as in the proof of parallel theorem in 2–crossed complexes. Moreover we prove that all objects are fibrant by showing that $i : A \to B$ is a strong deformation retract morphism. A retraction $r$ and a homotopy $\alpha : ir \simeq 1$ is obtained by the same formula. We construct push outs in $\mathbf{X_3Comp}$

$$\begin{array}{ccc} B & \xrightarrow{\overline{f}} & B' \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \overline{i}} \\ A & \longrightarrow & A' \end{array}$$

as follows. Let $B_n$ be a free extension of $A$ Then we set $B'_n$ as free extension of $A'$. The basis of $B'$ is given in degree $n$ by the composition

$$f_{n-1}d|_{X_n} : X_n \to A_{n-1} \to B_{n-1}, \quad n \geq 2.$$

As a result of above theorem and lemmas we have the following corollary.

**5.6. Corollary.** *The category of totally free 3–crossed complexes is a cofibration category.*

## 6. Homotopy of 3–crossed Modules

Let $A = K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ , $A' = K' \xrightarrow{\partial_3} L' \xrightarrow{\partial_2} M' \xrightarrow{\partial_1} N'$ be two 3–crossed modules and $f = (f_0, f_1, f_2, f_3)$ be a 3–crossed module morphism. A homotopy on $f$ is a pair $h = (h_1, h_2, h_3)$ of maps where $h_1 : N \to M'$, $h_2 : M \to L'$, and $h_3 : L \to K'$satisfying equations below.

$$\begin{aligned}
h_1(nn') &= {}^{f_0(n')^{-1}}h_1(n)h_1(n') \\
h_2(mm') &= {}^{((f_1m')(h_1\partial_1m'))^{-1}}(\{f_1m', {}^{f_0\partial_1m'^{-1}}(h_1\partial_1m^{-1})\}h_2(m))h_2(m') \\
\partial_2h_2({}^nm) &= h_1(n)f_0(n)h_1\partial_1(m^{-1})^{f_0\partial_1(m^{-1})}h_1(n)f_1(m^{-1})\partial_1h_1(n)f_1(n)h_1\partial_1(m) \\
&\qquad \partial_2h_2(m)\partial_1h_1(n^{-1})f_0(n^{-1}) \\
\partial_3h_3(ll') &= h_2\partial_2(ll')^{-1}f_2(l'^{-1})h_2\partial_2(l)\partial_3h_3(l)f_2(l')h_2\partial_2(l')\partial_3h_3(l') \\
h_3\partial_3(kk') &= f_3(k')^{-1}h_3\partial_3(k)f_3(k')h_3\partial_3(k') \\
\partial_3h_3({}^nl) &= h_2\partial_2({}^nl)^{-1}f_0(n)f_2(l^{-1})\partial_1h_1(n)f_2(l)h_2\partial_2(l)\partial_3h_3(l)\partial_1h_1(n^{-1})f_0(n^{-1}) \\
h_3\partial_3({}^nk) &= f_0(n)f_3(k^{-1})\partial_1h_1(n)f_3(k)\partial_3h_3(k)\partial_1h_1(n^{-1})f_0(n^{-1})
\end{aligned}$$

Such a function $h$ is called a 2–quadratic $f$–derivation.

**6.1. Proposition.** *Given a homotopy as above, the formulas*
$f'_0(n) = f_0(n)\partial_1h_1(n)$
$f'_1(m) = f_1(m)h_1\partial_1(m)\partial_2h_2(m)$
$f'_2(l) = f_2(l)h_2\partial_2(l)\partial_3h_3(l)$
$f'_3(k) = f_3(k)h_3\partial_3(k)$
*for all $n \in N, m \in M, l \in L$ define a morphism of 3–crossed modules.*

*Proof.*

$$\begin{aligned}
f'_0(nn') &= f_0(nn')\partial_1h_1(nn') \\
&= f_0(n)f_0(n')\partial_1({}^{f_0(n')^{-1}}h_1(n)h_1(n')) \\
&= f_0(n)f_0(n')f_0(n')^{-1}\partial_1h_1(n)f_0(n')\partial_1h_1(n')) \\
&= f_0(n)\partial_1h_1(n)f_0(n')\partial_1h_1(n')) \\
&= f'_0(n)f'_0(n')
\end{aligned}$$

then $f_0'$ is a group homomorphism.

$$\begin{aligned} f_1'(mm') &= f_1(mm')h_1\partial_1(mm')\partial_2h_2(mm') \\ &= f_1(m)f_1(m')h_1\partial_1(mm')\partial_2h_2(mm') \\ &= f_1(m)f_1(m')^{f_0\partial_1(m')}h_1\partial_1(m)h_1\partial_1(m')\partial_2h_2(mm') \\ &= f_1'(m)\partial_2h_2(m^{-1})h_1\partial_1(m^{-1})f_1(m')^{f_0\partial_1(m')}h_1\partial_1(m)h_1\partial_1(m')\partial_2h_2(mm'). \qquad (1) \end{aligned}$$

On the other hand

$$\begin{aligned} \partial_2h_2(mm') &= h_1\partial_1(m')^{-1}f_1(m')^{-1}\partial_2(\{f_1m', {}^{f_0\partial_1m'^{-1}}(h_1\partial_1m^{-1})\}h_2(m))f_1(m')h_1\partial_1(m')\partial_2h_2(m') \\ &= h_1\partial_1(m')^{-1}f_1(m')^{-1}\langle f_1m', {}^{f_0\partial_1m'^{-1}}(h_1\partial_1m^{-1})\rangle\partial_2h_2(m)f_1(m')h_1\partial_1(m')\partial_2h_2(m') \\ &= h_1\partial_1(m')^{-1}f_1(m')^{-1}f_1m'{}^{f_0\partial_1m'^{-1}}(h_1\partial_1m^{-1})f_1(m')^{-1}h_1\partial_1(m)\partial_2h_2(m)f_1'(m') \\ &= h_1\partial_1(m')^{-1f_0\partial_1m'^{-1}}(h_1\partial_1m^{-1})f_1(m')^{-1}h_1\partial_1(m)\partial_2h_2(m)f_1'(m'). \qquad (2) \end{aligned}$$

From (1) and (2) we get

$$f_1'(mm') = f_1'(m)f_1'(m')$$

$$\begin{aligned} f_2'(ll') &= f_2(ll')h_2\partial_2(ll')\partial_3h_3(ll') \\ &= f_2(l)f_2(l')h_2\partial_2(ll')\partial_3h_3(ll') \\ &= f_2'(l)\partial_3h_3(l^{-1})h_2\partial_2(l^{-1})f_2(l')h_2\partial_2(ll')\partial_3h_3(ll') \\ &= f_2'(l)f_2'(l') \end{aligned}$$

$$\begin{aligned} f_3'(kk') &= f_3(kk')h_3\partial_3(kk') \\ &= f_3(k)f_3(k')h_3\partial_3(kk') \\ &= f_3'(k)h_3\partial_3(k^{-1})f_3(k')h_3\partial_3(kk') \\ &= f_3'(k)f_3'(k') \end{aligned}$$

Then $f = (f_0, f_1, f_2, f_3)$ is a homomorphism. Now we show that $f$ is a morphism of 3–crossed modules.

$$\begin{aligned} f_1'({}^nm) &= f_1({}^nm)h_1\partial_1({}^nm)\partial_2h_2({}^nm) \\ &= {}^{f_0(n)}f_1(m)h_1(n\partial_1mn^{-1})\partial_2h_2({}^nm) \\ &= f_0(n)f_1(m)f_0(n)^{-1f_0(n)}({}^{f_0\partial_1(m)^{-1}}(h_1(n))h_1\partial_1(m))h_1(n)^{-1}\partial_2h_2({}^nm) \\ &= f_0(n)f_1(m)f_0(n)^{-1}f_0(n)^{f_0\partial_1(m)^{-1}}(h_1(n))h_1\partial_1(m)f_0(n)^{-1}h_1(n)^{-1}\partial_2h_2({}^nm) \\ &= {}^{f_0'(n)}f_1'(m) \end{aligned}$$

$$\begin{aligned} f_2'({}^nl) &= f_2({}^nl)h_2\partial_2({}^nl)\partial_3h_3({}^nl) \\ &= {}^{f_0(n)}f_2(l)h_2\partial_2({}^nl)\partial_3h_3({}^nl) \\ &= f_0(n)f_2(l)f_0(n)^{-1}h_2\partial_2({}^nl)\partial_3h_3({}^nl) \\ &= {}^{f_0'(n)}f_2'(l) \end{aligned}$$

$$\begin{aligned} f'_3({}^n k) &= f_3({}^n k) h_3 \partial_3({}^n k) \\ &= {}^{f_0(n)} f_3(k) h_3 \partial_3({}^n k) \\ &= f_0(n) f_3(k) f_0(n)^{-1} h_3 \partial_3({}^n k) \\ &= {}^{f'_0(n)} f'_3(k) \end{aligned}$$

□

**6.1. Homotopies of 3–crossed Complexes.** Let $A$ and $A'$ be two 3–crossed complexes and let $f$ be a 3–crossed complex map $A \to A'$. A 2–quadratic $f$–derivation is a sequence of maps $h_i : A_i \to A'_{i+1}$ such that $(h_3, h_2, h_1)$ is a 2–quadratic $f$–derivation of 3–crossed modules and all the remaining maps are $A_1$–equivariant for $n = 4$ and $A_1/\partial A_2$–equivariant for $n \geq 5$. We say that two 3–crossed complex maps are homotopic if there exists a 2–quadratic $f$–derivation such that

$$f'_1(a) = f_1(a)\partial_2 h_1(a) \quad \text{and} \quad f'_n(a) = f_n(a)(h_{n-1}\partial(a))(\partial h_n(a)) \quad \text{for} \quad n \geq 2.$$

Here we can write a lemma by the light of the Lemma 3.6.

**6.2. Lemma.** *Homotopy classes of category of totally free* 3*–crossed complexes is equivalent to the localization of* 3*–crossed complexes with respect to weak equivalences.*

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