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SOFT RINGS RELATED TO FUZZY SET THEORY

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Abstract

We deal with soft rings based on some fuzzy sets, in particular, by using the so called $\in -$soft sets and q-soft sets. Some characterization theorems of soft rings defined on soft sets are given and soft regular rings are hence characterized by special soft sets.

Keywords: Soft rings, Idealistic soft rings, Bi-idealistic soft rings, Quasi-idealistic soft rings, $(\in, \in \vee q)$-fuzzy subrings , $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subrings , regular rings, soft regular rings.

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1. Introduction

In dealing with the complicated problems in economics, engineering and environmental sciences, we are usually unable to apply the classical methods because there are various uncertainties in these problems. There are three theories involved, namely, the theory of probability, the theory of fuzzy sets and the interval mathematics which are considered as the fundamental tools in dealing with uncertainties, however all these theories have their own difficulties. Since uncertainties cannot be simply handled by using traditional mathematics, one has to apply a wider range of existing theories such as probability, intuitionistic fuzzy sets, vague sets, interval mathematics, rough sets and so on to deal with the situation. It is noted that all these theories have their own difficulties which have been pointed out in [13]. Maji et al. [12] and Molodtsov [13] have observed that one reason for these difficulties may be due to the inadequacy of the parametrization

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tools of the theory. In order to overcome these difficulties, Molodtsov [13] introduced the concept of soft set which can be regarded as a new mathematical tool in dealing with uncertainties. Molodtsov also pointed out several directions for the applications of soft sets. In recent years, research on soft set theory has been developed rapidly. Maji et al. [11] described the application of soft set theory to a decision making problem. Chen et al. [3] have recently presented a new definition of soft set parametrization reduction and compared their definition to the related concept of attributes reduction in rough sets. The algebraic structure of set theories dealing with uncertainties has been investigated by some authors and the algebraic theories dealing with uncertainties have also been studied by them. The most appropriate theories for dealing with uncertainties are based on the theory of fuzzy sets established by Zadeh in 1965 (see [18, 19]).

The notion of soft sets for BCK/BCI-algebras was considered by Jun in [5]. He introduced the notions of soft BCK/BCI-algebras and investigated their basic properties [6]. Aktas et al. [1] further studied the basic concept of soft set theory and compared soft sets to fuzzy and rough sets, providing some examples to clarify their differences. It is noteworthy that Feng et al. have started to investigate the structure of soft semirings in [4].

After the concept of fuzzy sets introduced by Zadeh [8] in 1965, there are many papers devoted to fuzzify the classical mathematics into fuzzy mathematics. Because the importance of group theory in mathematics as well as its applications in many disciplines, the notion of fuzzy subgroups was defined by Rosenfeld in 1971. Fuzzy algebraic structures then play a prominent role in mathematics with a wider range of applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. These applications provide sufficient motivation for researchers to review various concepts and results from the realm of abstract algebra to a broader framework of fuzzy setting. Some recent research on algebras can be found in [9, 10, 16, 17, 20-22].

The definition of soft rings has been recently proposed in [8]. Some properties of soft rings were described and isomorphism theorems were established [8]. As a continuation of the above paper, we now continue to study the soft rings by using some special soft sets. The concepts of idealistic soft rings, bi-idealistic soft rings and quasi-idealistic soft rings generated by soft sets are introduced. As a consequence, the relationships between soft rings and their fuzzy subrings (ideals) are described. As a result, the regular rings and soft regular rings are characterized by using special soft sets.

The notions, definitions and terminology used in this paper are standard. For some definitions and notations not given in this paper, the reader is referred to [8] and [18] if necessary.

2. Preliminaries

In this section, for the sake of completeness, we first cite some useful definitions and results.

Throughout this paper, R is a ring.

2.1. Definition ([7]). A subring B of R is called a bi-ideal of R if $BRB \subset B$. A subring Q of R is called a quasi-ideal of R if $RQ \bigcap QR \subset Q$.

It is clear that any left (right) ideal of R is a quasi-ideal of R, and any quasi-ideal of R is a bi-ideal of R.

2.2. Definition ([10]). A fuzzy subset μ in a set X is a function $\mu : X \longrightarrow [0, 1]$. If λ and μ are two fuzzy subsets in X, then the intrinsic product $\lambda * \mu$ is a fuzzy subset in X

defined by

$$(\lambda * \mu)(x) = \sup_{x=\Sigma_{finite} a_i b_i} (\min\{\lambda(a_i), \mu(b_i)\}).$$

2.3. Definition ([10]). A fuzzy subset μ in a set X of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support x and value t, denoted by x_t.

2.4. Definition ([10]). A fuzzy point x_t is said to "belong to" (resp., be quasicoincident with) a fuzzy set μ, written by $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$).

If $x_t \in \mu$ or $x_t q\mu$, then we write $x_t \in \vee q\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we write $x_t \overline{\in} \mu$ (resp., $x_t \overline{q} \mu$). The symbol $\overline{\in \vee q}$ is to mean that $\in \vee q$ does not hold.

2.5. Definition ([14]). A fuzzy set μ in a ring R is said to be a fuzzy subring of R if the following conditions hold for all $x, y \in R$: (1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and (2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

2.6. Definition ([7, 14]). (i) A fuzzy set μ in a ring R is said to be a fuzzy left (right) ideal of R if the following conditions hold for all $x, y \in R$: (1) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$, and (3) $\mu(xy) \geq \mu(y)$ ($\mu(xy) \geq \mu(x)$).

(ii) A fuzzy set μ is said to be a fuzzy ideal of R if it is both a fuzzy left ideal of R and a fuzzy right ideal of R.

(iii) A fuzzy set μ is called a fuzzy bi-ideal of R if it satisfies conditions (1), (2) and (4) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$, for all $x, y, z \in R$.

(iv) A fuzzy set μ is said to be a fuzzy quasi-ideal of R if the conditions (1) and (5) $(\mu * \chi_R) \bigcap (\chi_R * \mu)(x) \leq \mu(x)$ hold for all $x \in R$, where χ_R is the characteristic function of R.

We next cite the following result.

2.7. Proposition ([7, 14]). *A fuzzy set μ in a ring R is a fuzzy subring(ideal, bi-ideal, quasi-ideal) of R if and only if $U(\mu, \alpha) = \{x \in R \mid \mu(x) \geq \alpha)$ is a subring(ideal, bi-ideal, quasi-ideal) of R, respectively.*

3. Soft rings

The concept of soft set was first defined by Molodtsov in 1999 (see [13]).

3.1. Definition. (i) [13] Let $P(U)$ be the power set of U and $A \subset E$, where E is a set of parameters. Then a pair (F, A) is called a soft set over U if F is a mapping given by $F : A \to P(U)$.

(ii) [13] Let (F, A) and (G, B) be two soft sets over U. Then (F, A) is said to be a soft subset of (G, B) if the following conditions are satisfied:

(1) $A \subset B$ and

(2) for all $x \in A$, $F(x) \subset G(x)$.

We now denote the above inclusion relationship by $(F, A)\widetilde{\subset}(G, B)$. Similarly, we call (F, A) a soft superset of (G, B) if (G, B) is a soft subset of (F, A). Denoted the above relationship by $(F, A)\widetilde{\supset}(G, B)$.

3.2. Definition. (i) [13] Two soft sets (F, A) and (G, B) over U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

(ii) [1] The product of two soft sets (F, A) and (G, B) over U is the soft set $(H, A \times B)$, where $H(x, y) = F(x)G(y)$, $(x, y) \in A \times B$. This product is denoted by $(F, A) * (G, B) = (H, A \times B)$.

(iii) [1] If (F, A) and (G, B) are soft sets over U, then we define the soft set $(F, A) \wedge (G, B)$, where $(F, A) \wedge (G, B)$ is defined as $(H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$, $(x, y) \in A \times B$.

Let A be a nonempty set. We now use ρ to denote an arbitrary binary relation between an element of A and an element of the ring R. Then, a set-valued function $F : A \to P(R)$ can be defined by $F(x) = \{y \in R \mid (x, y) \in \rho, x \in A\}$.

3.3. Definition. (i) [8] Let (F, A) be a soft set over R. Then (F, A) is said to be a soft ring over R if and only if $F(x)$ is a subring of R for all $x \in A$. For the sake of convenience, the empty set $\emptyset$ here is regarded as a subring of R.

(ii) [8] Let (F, A) be a soft ring over a ring R. Then (F, A) is said to be an absolute soft ring over R if $F(x) = R$ for all $x \in A$.

3.4. Definition. Given a fuzzy set μ in any ring R and $A \subset [0, 1]$, consider the following two set-valued functions

$$\mathscr{F} : A \to \mathcal{P}(R), \quad t \mapsto \{x \in R \mid x_t \in \mu\}$$

and

$$\mathscr{F}_{\mathrm{q}} : A \to \mathcal{P}(R), \quad t \mapsto \{x \in R \mid x_t \, \mathrm{q} \, \mu\}.$$

Then $(\mathscr{F}, A)$ and $(\mathscr{F}_{\mathrm{q}}, A)$ are called an "$\in$-*soft set*" and "q-*soft set*" over R, respectively.

In the following propositions, we characterize the soft rings over R by fuzzy subrings of R.

The following proposition follows directly from Proposition 2.7.

3.5. Proposition. *Let μ be a fuzzy set in a ring R and $A = [0, 1]$. Then $(\mathscr{F}, A)$ is a soft ring over R if and only if μ is a fuzzy subring of R.*

3.6. Proposition. *Let μ be a fuzzy set in a ring R and $A = [0, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R if and only if μ is a fuzzy subring of R.*

Proof. Assume that $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R. Then for all $t \in A$, $\mathscr{F}_{\mathrm{q}}(t)$ is a subring of R. If there exist $a, b \in R$ such that $\mu(a - b) < \min\{\mu(a), \mu(b)\}$, then we can choose $t \in A$ such that $\mu(a - b) + t \leq 1 < \min\{\mu(a), \mu(b)\} + t$. Hence, $\mu(a) + t > 1$ and $\mu(b) + t > 1$, but $\mu(a - b) + t \leq 1$, i.e., $a, b \in \mathscr{F}_{\mathrm{q}}(t)$. However, we have $a - b \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, which is a contradiction. Thus, $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in R$. In the same way, we can also prove that $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in R$. Therefore, μ is a fuzzy subring of R.

Conversely, suppose that μ is a fuzzy subring of R. Let $t \in A$ and $x, y \in \mathscr{F}_{\mathrm{q}}(t)$. Then $\mu(x - y) + t \geq \min\{\mu(x), \mu(y)\} + t > 1$ and $\mu(xy) + t \geq \min\{\mu(x), \mu(y)\} + t > 1$, and so $x - y \in \mathscr{F}_{\mathrm{q}}(t)$ and $xy \in \mathscr{F}_{\mathrm{q}}(t)$. This proves that $\mathscr{F}_{\mathrm{q}}(t)$ is a subring of R and hence $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R. □

We now introduce a special fuzzy subring of R.

3.7. Definition ([2])**.** We call a fuzzy set μ an $(\in, \in \vee q)$-fuzzy subring of R if for all $t, r \in (0, 1]$ and $x, y \in R$, the following conditions hold:

(A1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x - y)_{\min(t,r)} \in \vee q\mu$,

(A2) $x_t \in \mu$ and $y_r \in \mu$ imply $(xy)_{\min(t,r)} \in \vee q\mu$.

In view of the above definition, we have the following lemma.

3.8. Lemma ([2]). *A fuzzy set μ in a ring R is an $(\in, \in \vee q)$-fuzzy subring of R if and only if for all $x, y \in R$, the following conditions hold:*
(B1) $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$,
(B2) $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$

In the following theorem, we show that the soft rings can be described by the $(\in, \in \vee q)$-fuzzy subrings of R.

3.9. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is a soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy subring of R.*

Proof. Assume that $(\mathscr{F}, A)$ is a soft ring over R. Then $\mathscr{F}(t)$ is a subring of R for all $t \in A$. If there exist $x, y \in R$ such that $\mu(x-y) < \min\{\mu(x), \mu(y), 0.5\}$, then we can choose $t \in (0,1]$ such that $\mu(x-y) < t \leq \min\{\mu(x), \mu(y), 0.5\}$. Thus $0 < t \leq 0.5$, $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(x-y) < t$, that is, $x, y \in \mathscr{F}(t)$, but $x - y \overline{\in} \mathscr{F}(t)$ which is a contradiction. Hence, $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$. In the same way, we can also prove that $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$. By Lemma 3.8, μ is an $(\in, \in \vee q)$-fuzzy subring of R.

Conversely, suppose that μ is an $(\in, \in \vee q)$-fuzzy subring of R. Let $t \in A$. Then, by Lemma 3.8, we can deduce that $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$, for all $x, y \in R$. If $x, y \in \mathscr{F}(t)$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. These imply that $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t$ and so $x - y \in \mathscr{F}(t)$. We can also show that $xy \in \mathscr{F}(t)$. Thus $\mathscr{F}(t)$ is a subring of R and $(\mathscr{F}, A)$ is indeed a soft ring over R. □

The $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R can be defined as the same as an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-bi-ideals of R in [10].

3.10. Definition. A fuzzy set μ is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R if for all $t, r \in (0,1]$ and $x, y \in R$, the following conditions hold:
(C1) $(x-y)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$
(C2) $(xy)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$.

We have the following same conclusion as in [10].

3.11. Lemma. *A fuzzy set μ in a ring R is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R if and only if for all $x, y \in R$, the following conditions hold:*
(D1) $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$,
(D2) $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$.

3.12. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is a soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R.*

Proof. Let $(\mathscr{F}, A)$ be a soft ring over R. Then $\mathscr{F}(t)$ is a subring of R for all $t \in A$. If there exist $x, y \in R$ such that $\max\{\mu(x-y), 0.5\} < t \leq \min\{\mu(x), \mu(y)\}$, then $t \in A$, $x, y \in \mathscr{F}(t)$, but $x - y \overline{\in} \mathscr{F}(t)$, which is a contradiction. Hence, $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. In the same way, we can prove that $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. Hence, by Lemma 3.11, μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R. Then $x, y \in R, \max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ and $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. If we let $t \in A$ with $x, y \in \mathscr{F}(t)$, then $\mu(x) \geq t > 0.5$, $\mu(y) \geq t > 0.5$, and hence $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\} \geq t, \max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\} \geq t$. Thus, $\mu(x-y) \geq t$ and $\mu(xy) \geq t$, that is, $x - y \in \mathscr{F}(t)$ and $xy \in \mathscr{F}(t)$. These show that $\mathscr{F}(t)$ is a subring of R and $(\mathscr{F}, A)$ is a soft ring over R. □

We next formulate the following theorems by using q-soft sets.

3.13. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R.*

Proof. Assume that $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R. Then $\mathscr{F}_{\mathrm{q}}(t)$ is a subring of R for all $t \in A$. If there exist $x, y \in R$ such that $\max\{\mu(x-y), 0.5\} < \min\{\mu(x), \mu(y)\}$, then we can select $t \in A$ such that $\max\{\mu(x-y), 0.5\} + t \leq 1 < \min\{\mu(x), \mu(y)\} + t$, $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, but $x - y \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, which is a contradiction. Hence, $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. In the same way, we can also prove that $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. It hence follows from Lemma 3.11 that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subring of R. Then $x, y \in R$, $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ and $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. Let $t \in A$ so that $x, y \in \mathscr{F}_{\mathrm{q}}(t)$. Then $\mu(x) + t > 1$, $\mu(y) + t > 1$, hence, $\max\{\mu(x-y), 0.5\} + t \geq \min\{\mu(x), \mu(y)\} + t$, $\max\{\mu(xy), 0.5\} + t \geq \min\{\mu(x), \mu(y)\} + t$. Thus, $\mu(x-y) + t > 1$ and $\mu(xy) + t > 1$, i.e., $x - y \in \mathscr{F}_{\mathrm{q}}(t)$ and $xy \in \mathscr{F}_{\mathrm{q}}(t)$, and so $\mathscr{F}_{\mathrm{q}}(t)$ is a subring of R and $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R. □

3.14. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy subring of R.*

Proof. Let $(\mathscr{F}_{\mathrm{q}}, A)$ be a soft ring over R. Then $\mathscr{F}_{\mathrm{q}}(t)$ is a subring of R, for all $t \in A$. If there exist $x, y \in R$ such that $\mu(x-y) < \min\{\mu(x), \mu(y), 0.5\}$, then we can choose $t \in (0.5, 1]$ such that $\mu(x-y) + t \leq 1 < \min\{\mu(x), \mu(y), 0.5\} + t$. Thus $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, but $x - y \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$. This is a contradiction. Hence, $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$. By using the same arguments, we can prove that $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$. It follows from Lemma 3.8 that μ is an $(\in, \in \vee q)$-fuzzy subring of R.

Conversely, suppose that μ is an $(\in, \in \vee q)$-fuzzy subring of R. Using Lemma 3.8, we have $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$ and $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in R$. Let $t \in A$, $x, y \in F_{\mathrm{q}}(t)$, then $\mu(x) + t > 1$ and $\mu(y) + t > 1$. These imply that $\mu(x-y) + t \geq \min\{\mu(x), \mu(y), 0.5\} + t > 1$, and so $x - y \in \mathscr{F}_{\mathrm{q}}(t)$. We can also similarly prove that $xy \in \mathscr{F}_{\mathrm{q}}(t)$. Thus $\mathscr{F}_{\mathrm{q}}(t)$ is a subring of R and $(\mathscr{F}_{\mathrm{q}}, A)$ is a soft ring over R. □

Same as the definition in [10], we give the following definition.

3.15. Definition. Let $\alpha, \beta \in (0, 1]$ with $\alpha < \beta$. Then a fuzzy set μ is called an (α, β)-fuzzy subring of R if the following conditions are satisfied for any $x, y \in R$:

(E1) $\max\{\mu(x-y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$,

(E2) $\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$.

3.16. Theorem. *Let μ be a fuzzy set in a ring R with $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is a soft ring over R if and only if μ is an (α, β)-fuzzy subring of R.*

Proof. Assume that $(\mathscr{F}, A)$ is a soft ring over R. Then $\mathscr{F}(t)$ is a subring of R for all $t \in A$. If there exist $x, y \in R$ such that $\max\{\mu(x-y), \alpha\} < \min\{\mu(x), \mu(y), \beta\}$, then we can select $t \in (\alpha, \beta]$ such that $\max\{\mu(x-y), \alpha\} < t \leq \min\{\mu(x), \mu(y), \beta\}$. Thus $\mu(x) \geq t$ and $\mu(y) \geq t$, but $\mu(x-y) < t$, that is , $x, y \in \mathscr{F}(t)$, but $x - y \overline{\in} \mathscr{F}(t)$, which is a contradiction. Hence, $\max\{\mu(x-y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$. Similarly, we can prove that $\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$. Consequently, μ is an (α, β)-fuzzy subring of R.

Conversely, suppose that μ is an (α, β)-fuzzy subring of R. For any $t \in A$, if $x, y \in \mathscr{F}(t)$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. These imply that $\max\{\mu(x-y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\} \geq t$ and $\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\} \geq t$. Thus, $\mu(x-y) \geq t$ and $\mu(xy) \geq t$, that is, $x - y \in \mathscr{F}(t)$ and $xy \in \mathscr{F}(t)$, and so $\mathscr{F}(t)$ is a subring of R and $(\mathscr{F}, A)$ is indeed a soft ring over R. □

4. Idealistic soft rings, bi-idealistic soft rings and quasi-idealistic soft rings

We divide this section into three parts. In Subsection 4.1, we describe the idealistic soft rings. In Subsection 4.2, we describe the bi-idealistic soft rings. In Subsection 4.3, we consider the quasi-idealistic soft rings.

4.1. Idealistic soft rings.

4.1. Definition. Let (F, A) be a soft set over a ring R. Then (F, A) is said to be a left (right) idealistic soft ring over R if and only if $F(x)$ is a left (right) ideal of R, for all $x \in A$. We now call (F, A) an idealistic soft ring over R if and only if (F, A) is both a right idealistic soft ring over R and a left idealistic soft ring over R.

For the sake of convenience, we now regard the empty set $\emptyset$ here as an ideal of R.

4.2. Example. Let $Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and (F, A) be a soft set over Z_6, where $A = \{\overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and $F : A \to P(Z_6)$ is defined by $F(x) = \{y \in Z_6 \mid x\rho y \Longleftrightarrow xy = \overline{0}\}$, for all $x \in A$. Then it is clear that $F(\overline{2}) = \{\overline{0}, \overline{3}\}$, $F(\overline{3}) = \{\overline{0}, \overline{2}, \overline{4}\}$, $F(\overline{4}) = \{\overline{0}, \overline{3}\}$ and $F(\overline{5}) = \{\overline{0}\}$ are ideals of Z_6. Clearly, (F, A) is an idealistic soft ring over Z_6.

The proofs of the following propositions are easy (refer to Proposition 2.7 and Proposition 3.6, respectively).

4.3. Proposition. *Let μ be a fuzzy set in a ring R and $A = [0, 1]$. Then $(\mathscr{F}, A)$ is a left (right) idealistic soft ring over R if and only if μ is a fuzzy left (right) ideal of R.*

4.4. Proposition. *Let μ be a fuzzy set in a ring R and $A = [0, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) idealistic soft ring over R if and only if μ is a fuzzy left (right) ideal of R.*

We now consider the following special fuzzy left (right) ideals of R.

4.5. Definition. A fuzzy set μ is called an $(\in, \in \vee q)$-fuzzy left (right) ideal of R if for all $t, r \in (0, 1]$ and $x, y \in R$, the following conditions are satisfied:
(F1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x - y)_{\min(t,r)} \in \vee q\mu$,
(F2) $y_t \in \mu$ $(x_t \in \mu)$ imply $(xy)_t \in \vee q\mu$.

By the above definition, we have the following lemma.

4.6. Lemma. *A fuzzy set μ in a ring R is an $(\in, \in \vee q)$-fuzzy left (right) ideal of R if and only if for $x, y \in R$, the following conditions are satisfied:*
(G1) $\mu(x - y) \geq \min\{\mu(x), \mu(y), 0.5\}$,
(G2) $\mu(xy) \geq \min\{\mu(y), 0.5\}$ $(\mu(xy) \geq \min\{\mu(x), 0.5\})$.

Proof. In view of Definition 4.5, we need to prove that conditions $(F1)$ and $(F2)$ are equivalent to conditions $(G1)$ and $(G2)$. Clearly, $(F1) \Longleftrightarrow (G1)$ by Lemma 3.8. We only prove that $(F2) \Longleftrightarrow (G2)$.

To prove that $(F2) \Longrightarrow (G2)$: Assume that there exist $x, y \in R$ with $\mu(xy) < t \leq \min\{\mu(y), 0.5\}$ $(\mu(xy) < t \leq \min\{\mu(x), 0.5\})$. Then $0 < t \leq 0.5$ and $y_t \in \mu$ $(x_t \in \mu)$, but $xy_t\overline{\mu}$. Since $\mu(xy) + t \leq 1$, $(xy)_t\overline{q}\mu$. It follows that $(xy)_t\overline{\in \vee q}\mu$, which is a contradiction. Hence (G2) holds.

To prove that $(G2) \Longrightarrow (F2)$: Let $y_t \in \mu$ $(x_t \in \mu)$. Then $\mu(y) \geq t$ $(\mu(x) \geq t)$. Now $\mu(xy) \geq \min\ \{\mu(y), 0.5\} \geq \min\{t, 0.5\}(\mu(xy) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\})$. If $t > 0.5$, then $\mu(xy) \geq 0.5$. This implies that $\mu(xy) + t > 1$. If $t \leq 0.5$, then $\mu(xy) \geq t$. Therefore, $(xy)_t \in \vee q\mu$. □

The proof of the following theorem is similar to the proof of Theorem 3.9 and is hence omitted.

4.7. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is a left (right) idealistic soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy left (right) ideal of R.*

Same as in [10], we give the following definition.

4.8. Definition. A fuzzy set μ is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of R if for all $t, r \in (0, 1]$ and $x, y \in R$, the following conditions hold :

(H1) $(x-y)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$,

(H2) $(xy)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee \overline{q} \mu$ $(x_t \overline{\in} \vee \overline{q} \mu)$.

The following lemma describes the properties of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideals of R.

4.9. Lemma. *A fuzzy set μ in a ring R is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of R if and only if the following conditions hold for all $x, y \in R$:*

(I1) $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$,

(I2) $\max\{\mu(xy), 0.5\} \geq \mu(y)$ $(\max\{\mu(xy), 0.5\} \geq \mu(x))$.

Proof. It is known that $(H1) \Longleftrightarrow (I1)$, we only need to prove $(H2) \Longleftrightarrow (I2)$.

To prove $(H2) \Longrightarrow (I2)$: If there exist $x, y \in R$ such that $\max\{\mu(xy), 0.5\} < \mu(y)$ $(\max\{\mu(xy), 0.5\} < \mu(x))$, then we can select $t \in (0, 1]$ such that $\max\{\mu(xy), 0.5\} < t \leq \mu(y)(\max\{\mu(xy), 0.5\} < t \leq \mu(x))$, and so $0.5 < t \leq 1$ and $y_t \in \mu$ $(x_t \in \mu)$, but $(xy)_t \overline{\in} \mu$. By H(2), we have $y_t \overline{q} \mu$ $(x_t \overline{q} \mu)$. This implies $\mu(y) + t \leq 1$ $(\mu(x) + t \leq 1)$, a contradiction.

To prove $(I2) \Longrightarrow (H2)$: Let $t \in (0, 1]$ and $(xy)_t \overline{\in} \mu$. Then $\mu(xy) < t$.

(a) If $\mu(xy) \geq \mu(y)$ $(\mu(xy) \geq \mu(x))$, then $\mu(y) < t$ $(\mu(x) < t)$. It follows that $y_t \overline{\in} \mu$ $(x_t \overline{\in} \mu)$. Thus, $y_t \overline{\in} \vee \overline{q} \mu$ $(x_t \overline{\in} \vee \overline{q} \mu)$.

(b) If $\mu(xy) < \mu(y)$ $(\mu(xy) < \mu(x))$, then by (I2), $0.5 \geq \mu(y)(0.5 \geq \mu(x))$. Now, if for $\mu(y) < t$ $(\mu(x) < t)$, then $y_t \overline{\in} \mu$ $(x_t \overline{\in} \mu)$ and if $\mu(y) \geq t$ $(\mu(x) \geq t)$, then $\mu(y) + t \leq 1$ $(\mu(x) + t \leq 1)$. It follows that $y_t \overline{q} \mu$ $(x_t \overline{q} \mu)$. Thus, $y_t \overline{\in} \vee \overline{q} \mu$ $(x_t \overline{\in} \vee \overline{q} \mu)$. □

4.10. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is a left (right) idealistic soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of R.*

Proof. The proof is similar to Theorem 3.12 and is hence omitted. □

We now characterize the left (right) idealistic soft rings over R by using q-soft sets.

4.11. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) idealistic soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of R.*

Proof. Let $(\mathscr{F}_{\mathrm{q}}, A)$ be a left (right) idealistic soft ring over R. Then $\mathscr{F}_{\mathrm{q}}(t)$ is a left (right) ideal of R for every $t \in A$. If there exist $x, y \in R$ such that $\max\{\mu(x-y), 0.5\} < \min\{\mu(x), \mu(y)\}$, then we can select $t \in A$ such that $\max\{\mu(x-y), 0.5\} + t \leq 1 < \min\{\mu(x), \mu(y)\} + t$ and $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, but $x - y \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, this is a contradiction. Hence, $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. If there exist $c, d \in R$ such that $\max\{\mu(cd), 0.5\} < \mu(d)(\max\{\mu(cd), 0.5\} < \mu(c)\})$, then we can select $t \in A$ such that $\max\{\mu(cd), 0.5\} + t \leq 1 < \mu(d) + t(\max\{\mu(cd), 0.5\} + t \leq 1 < \mu(c) + t)$. This leads to $d \in \mathscr{F}_{\mathrm{q}}(t)$, $c \in R$ $(c \in \mathscr{F}_{\mathrm{q}}(t)$, $d \in R)$, but $cd \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, a contradiction. Hence, $\max\{\mu(xy), 0.5\} \geq \mu(y)(\max\{\mu(xy), 0.5\} \geq \mu(x))$ for all $x, y \in R$. It follows that μ is a left (right) $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of R.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of R. Then $x, y \in R, \max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ and $\max\{\mu(xy), 0.5\} \geq \mu(y)(\max\{\mu(xy), 0.5\} \geq \mu(x))$. For $t \in A$, if $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, then $\mu(x) + t > 1$ and $\mu(y) + t > 1$. These lead to

$\max\{\mu(x-y), 0.5\} + t \geq \min\{\mu(x), \mu(y)\} + t > 1$, i.e., $x - y \in \mathscr{F}_{\mathrm{q}}(t)$. Let $x \in \mathscr{F}_{\mathrm{q}}(t)$, $z \in R$. Then $\max\{\mu(zx), 0.5\} + t \geq \mu(x) + t > 1(\max\{\mu(xz), 0.5\} + t \geq \mu(x) + t > 1)$. Thus, $\mu(zx) + t > 1(\mu(xz) + t > 1)$, i.e., $zx \in \mathscr{F}_{\mathrm{q}}(t)$ $(xz \in \mathscr{F}_{\mathrm{q}}(t))$, and hence $\mathscr{F}_{\mathrm{q}}(t)$ is a left (right) ideal of R and $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) idealistic soft ring over R. □

4.12. Theorem. *Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) idealistic soft ring over R if and only if μ is a $(\in, \in \vee q)$-fuzzy left (right) ideal of R.*

Proof. Assume that $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) idealistic soft ring over R. Then $\mathscr{F}_{\mathrm{q}}(\alpha)$ is a left (right) ideal of R, for all $\alpha \in A$. If there exist $x, y \in R$ such that $\mu(x - y) < \min\{\mu(x), \mu(y), 0.5\}$, then we can select $t \in A$ such that $\mu(x-y) + t \leq 1 < \min\{\mu(x), \mu(y), 0.5\} + t$. Thus $\mu(x) + t > 1$, $\mu(y) + t > 1$, $\mu(x-y) + t \leq 1$. i.e., $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, but $x - y \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$. This is a contradiction. Hence we have $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$. If there exist $a, b \in R$ such that $\mu(ab) < \min\{\mu(b), 0.5\}$ $(\mu(ab) < \min\{\mu(a), 0.5\})$, then we can select $t \in A$ such that $\mu(ab) + t \leq 1 < \min\{\mu(b), 0.5\} + t$ $(\mu(ab) + t \leq 1 < \min\{\mu(a), 0.5\} + t)$, then $b \in \mathscr{F}_{\mathrm{q}}(t)$, $a \in R$ $(a \in \mathscr{F}_{\mathrm{q}}(t), b \in R)$, but $ab \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, which is absurd. Hence, we have $\mu(xy) \geq \min\{\mu(y), 0.5\}$ $(\mu(xy) \geq \min\{\mu(x), 0.5\})$, and this proves that μ is an $(\in, \in \vee q)$-fuzzy left (right)ideal of R.

Conversely, let μ be an $(\in, \in \vee q)$-fuzzy left (right) ideal of R. If $t \in A$, then, by Lemma 4.6, we can get $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$ and $\mu(xy) \geq \min\{\mu(y), 0.5\}$ $(\mu(xy) \geq \min\{\mu(x), 0.5\})$ for all $x, y \in R$. If $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, then $\mu(y) + t > 1$ and $\mu(x) + t > 1$. These imply that $\mu(x-y) + t \geq \min\{\mu(x), \mu(y), 0.5\} + t > 1$, and so $x - y \in \mathscr{F}_{\mathrm{q}}(t)$. If $x \in \mathscr{F}_{\mathrm{q}}(t)$ and $z \in R$, then $\mathscr{F}_{\mathrm{q}}(x) + t > 1$. This leads to $\mu(zx) + t \geq \min\{\mu(x), 0.5\} + t > 1$ $(\mu(xz) + t \geq \min\{\mu(x), 0.5\} + t > 1)$. Hence, $zx \in \mathscr{F}_{\mathrm{q}}(t)$ $(xz \in \mathscr{F}_{\mathrm{q}}(t))$. Thus $\mathscr{F}_{\mathrm{q}}(t)$ is a left (right) ideal of R and (F_{q}, A) is a left (right) idealistic soft ring over R. □

We now introduce the concept of (α, β)-fuzzy left (right) ideals of R.

4.13. Definition. Let $\alpha, \beta \in (0, 1]$ with $\alpha < \beta$. Then a fuzzy set μ is called an (α, β)-fuzzy left (right) ideal of R if for $x, y \in R$, the following conditions are satisfied:

(J1) $\max\{\mu(x-y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$,

(J2) $\max\{\mu(xy), \alpha\} \geq \min\{\mu(y), \beta\}(\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \beta\})$.

The following proposition follows from Theorem 3.16.

4.14. Theorem. *Let μ be a fuzzy set in a ring R and $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is a left (right) idealistic soft ring over R if and only if μ is an (α, β)-fuzzy left (right) ideal of R.*

4.2. Bi-idealistic soft rings.

4.15. Definition. Let (F, A) be a soft set over a ring R. Then (F, A) is said to be a bi-idealistic soft ring over R if and only if $F(x)$ is a bi-ideal of R for all $x \in A$. For the sake of convenience, we now regard the empty set $\emptyset$ as a bi-ideal of R.

4.16. Example. In Example 4.2, (F, A) is a bi-idealistic soft ring over Z_6.

The proofs of following propositions are straightforward and are omitted.

4.17. Proposition. *Let μ be a fuzzy set in a ring R and $A = [0, 1]$. Then $(\mathscr{F}, A)$ is a bi-idealistic soft ring over R if and only if μ is a fuzzy bi-ideal of R.*

4.18. Proposition. *Let μ be a fuzzy set in a ring R and $A = [0, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a bi-idealistic soft ring over R if and only if μ is a fuzzy bi-ideal of R.*

As the same as [10], we also have the following definitions and lemmas.

4.19. Definition. A fuzzy set μ is said to be an $(\in, \in \vee q)$-fuzzy bi-ideal of R if for all $t, r \in (0,1]$ and $x, y, z \in R$, the following conditions hold:
(K1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x-y)_{\min(t,r)} \in \vee q\mu$,
(K2) $x_t \in \mu$ and $y_r \in \mu$ imply $(xy)_{\min(t,r)} \in \vee q\mu$,
(K3) $x_t \in \mu$ and $z_r \in \mu$ imply $(xyz)_{\min(t,r)} \in \vee q\mu$.

4.20. Lemma. *A fuzzy set μ in a ring R is an $(\in, \in \vee q)$-fuzzy bi-ideal of R if and only if the following conditions hold for any $x, y, z \in R$:*
(L1) $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$,
(L2) $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$,
(L3) $\mu(xyz) \geq \min\{\mu(x), \mu(z), 0.5\}$.

4.21. Definition. A fuzzy set μ is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of R if for all $t, r \in (0,1]$, the following conditions hold for $x, y \in R$:
(M1) $(x-y)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$,
(M2) $(xy)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$,
(M3) $(xyz)_{\min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $z_r \overline{\in} \vee \overline{q}\mu$.

It is easy to see that the $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of R has the following properties:

4.22. Lemma. *A fuzzy set μ in R is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of R if and only if for all $x, y \in R$, the following conditions hold:*
(N1) $\max\{\mu(x-y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$,
(N2) $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$.
(N3) $\max\{\mu(xyz), 0.5\} \geq \min\{\mu(x), \mu(z)\}$.

4.23. Definition. Let $\alpha, \beta \in (0,1]$ with $\alpha < \beta$. Then a fuzzy set μ is called an (α, β)-fuzzy bi-ideal of R if $x, y, z \in R$, the following conditions hold:
(O1) $\max\{\mu(x-y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$,
(O2) $\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$,
(O3) $\max\{\mu(xyz), \alpha\} \geq \min\{\mu(x), \mu(z), \beta\}$.

In the following theorem, the properties of the bi-idealistic soft rings will be described. The proofs are similar to Theorem 3.9, Theorem 3.12, Theorem 3.13, Theorem 3.14 and Theorem 3.16, respectively.

4.24. Theorem. *(i) Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is a bi-idealistic soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of R.*

(ii) Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is a bi-idealistic soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of R.

(iii) Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a bi-idealistic soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of R.

(iv) Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a bi-idealistic soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy bi-deal of R.

(v) Let μ be a fuzzy set in a ring R and $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is a bi-idealistic soft ring over R if and only if μ is an (α, β)-fuzzy bi-ideal of R.

4.3. Quasi-idealistic soft rings.

4.25. Definition. Let (F, A) be a soft set over a ring R. Then (F, A) is said to be a quasi-idealistic soft ring over R if and only if $F(x)$ is a quasi-ideal of R for all $x \in A$.

For the sake of convenience, we now regard the empty set $\emptyset$ here as a quasi-ideal of R.

4.26. Example. Let $R = M_2(\mathbb{R})$, $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} |a \neq 0 \right\}$ and $F(x) = \{y \in R \mid xy = 0\}$. Then (F, A) is a soft ring over R. $\forall x \in A$, $F(x) = \left\{ \begin{pmatrix} 0 & 0 \\ x_1 & x_2 \end{pmatrix} |x_1, x_2 \in \mathbb{R} \right\}$. Let $\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in R$. Then

$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x_1 & x_2 \end{pmatrix} = \begin{pmatrix} x_1y_2 & x_2y_2 \\ x_1y_4 & x_2y_4 \end{pmatrix}$ and

$\begin{pmatrix} 0 & 0 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x_1y_1 + x_2y_3 & x_1y_2 + x_2y_4 \end{pmatrix}$. Then

$F(x)R \cap RF(x) = \left\{ \begin{pmatrix} 0 & 0 \\ m & n \end{pmatrix} |m, n \in \mathbb{R} \right\}$. Because

$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ m & n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and

$F(x)R \cap RF(x) \subset F(x)$, (F, A) is a quasi-idealistic soft ring over R.

Since any left (right) ideal of a ring R is a quasi-ideal of R and any quasi-ideal of R is a bi-ideal of R, by Proposition 2.7 and Proposition 3.6 we can easily deduce the following proposition.

4.27. Proposition. *(i) Any left (right) idealistic soft ring over R is a quasi-idealistic soft ring over R.*

(ii) Any quasi-idealistic soft ring over R is a bi-idealistic soft ring over R.

(iii) Let μ be a fuzzy set in a ring R and $(\mathscr{F}, A)$ a soft set over R with $A = [0, 1]$. Then (F, A) is a quasi-idealistic soft ring over R if and only if μ is a fuzzy quasi-ideal of R.

(iv) Let μ be a fuzzy set in a ring R and $(\mathscr{F}_{\mathrm{q}}, A)$ a soft set over R with $A = [0, 1]$. Then (F_q, A) is a quasi-idealistic soft ring over R if and only if μ is a fuzzy quasi-ideal of R.

Same as in [10], we have the following definitions and lemmas.

4.28. Definition. A fuzzy set μ is said to be an $(\in, \in \vee q)$-fuzzy quasi-ideal of R if for all $t, r \in (0, 1]$, the following conditions hold for $x, y \in R$:

(P1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x - y)_{\min(t,r)} \in \vee q\mu$,

(P2) $x_t \in (\mu * \chi_R) \bigcap (\chi_R * \mu)$implies $x_t \in \vee q\mu$.

The following lemma follows from the definition.

4.29. Lemma. *A fuzzy set μ in a ring R is an $(\in, \in \vee q)$-fuzzy quasi-ideal of R if and only if the following conditions hold for $x, y \in R$:*

(Q1) $\mu(x - y) \geq \min\{\mu(x), \mu(y), 0.5\}$,

(Q2) $\mu(x) \geq \min\{((\mu * \chi_R) \bigcap (\chi_R * \mu))(x), 0.5)\}$.

4.30. Definition. A fuzzy set μ is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of R if for all $t, r \in (0, 1]$ and $x, y \in R$, the following conditions hold:

(R1) $(x - y)_{\min(t,r)} \overline{\in} \mu$, implies $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$,

(R2) $x_t \overline{\in} \mu$, implies $x_t \overline{\in} \vee \overline{q}(\mu * \chi_R) \bigcap (\chi_R * \mu)$.

The proof of the following lemma is easy and is hence omitted.

4.31. Lemma. *A fuzzy set μ in a ring R is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of R if and only if for all $x, y \in R$, the following conditions hold:*

(S1) $\max\{\mu(x - y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$,

(S2) $\max\{\mu(x), 0.5\} \geq ((\mu * \chi_R) \bigcap (\chi_R * \mu))(x)$.

4.32. Definition. Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Then a fuzzy set μ is called an (α,β)-fuzzy quasi-ideal of R if for $x, y \in R$, the following conditions hold:
(T1) $\max\{\mu(x-y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$,
(T2) $\max\{\mu(x), \alpha\} \geq \min\{((\mu * \chi_R) \bigcap (\chi_R * \mu))(x), \beta\}$.

The proofs of the following theorem follow from Theorem 3.9, Theorem 3.12, Theorem 3.13, Theorem 3.14 and Theorem 3.16, respectively.

4.33. Theorem. *(i) Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is a quasi-idealistic soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy quasi-ideal of R.*

(ii) Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is a quasi-idealistic soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of R.

(iii) Let μ be a fuzzy set in a ring R and $A = (0, 0.5]$. Then $(\mathscr{F}_q, A)$ is a quasi-idealistic soft ring over R if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of R.

(iv) Let μ be a fuzzy set in a ring R and $A = (0.5, 1]$. Then $(\mathscr{F}_q, A)$ is a quasi-idealistic soft ring over R if and only if μ is an $(\in, \in \vee q)$-fuzzy quasi-deal of R.

(v) Let μ be a fuzzy set in a ring R and $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is an quasi-idealistic soft ring over R if and only if μ is an (α, β)-fuzzy quasi-ideal of R.

5. Soft regular rings

5.1. Definition ([7])**.** A ring R is called regular if for each element a of R, there exists an element x such that $a = axa$.

5.2. Definition. A soft ring (F, A) over R is called regular if for $\forall x \in A$, $F(x)$ is regular.

5.3. Example. In example 4.2, (F, A) is a regular soft ring.

5.4. Definition. A ring R is called soft regular if every soft ring (F, A) over R is a regular soft ring.

5.5. Example. Let $R = Z_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Then every subring of R is regular, and so every soft ring (F, A) over R is a regular soft ring. Thus, R is soft regular.

We now characterize the regular rings by using soft sets.

5.6. Theorem. *A ring R is regular if and only if $(F, A) * (G, B) = (F, A) \wedge (G, B)$ for every right idealistic soft ring (F, A) over R and every left idealistic soft ring (G, B) over R.*

Proof. Assume that R is a regular ring. Let (F, A) and (G, B) be any right idealistic soft ring over R and any left idealistic soft ring over R, respectively. Then for all $x \in A$ and for all $y \in B$, $F(x)$ is a right ideal of R and $G(y)$ is a left ideal of R. Let $a \in F(x) \cap G(y)$. Then there exists $r \in R$ such that $a = ara \in F(x)G(y)$. Thus $F(x) \cap G(y) \subset F(x)G(y)$. On the other hand, if $a \in F(x)G(y)$, then $a = bc, b \in F(x), c \in G(y)$, and hence, there exist $r, s \in R$ such that $a = brbcsc$. In this case, we have $a \in F(x) \cap G(y)$ and $F(x)G(y) \subset F(x) \cap G(y)$. It hence follows that $F(x)G(y) = F(x) \cap G(y)$ for every $(x, y) \in A \times B$, that is, $(F, A) * (G, B) = (F, A) \wedge (G, B)$.

Conversely, if we let a be an element of R such that $F(x) = aR$ for all $x \in A$ and $G(y) = Ra$ for all $y \in B$, then (F, A) is a right idealistic soft ring over R and (G, B) is a left idealistic soft ring over R. Since $(F, A) * (G, B) = (F, A) \wedge (G, B)$, $aR \cap Ra = aRRa$ and hence $a \in aR \cap Ra$. Thus $a \in aRRa \subset aRa$. This shows that R is regular. $\square$

5.7. Lemma ([15])**.** *A ring R is regular if and only if $Q = QRQ$ for every quasi-ideal Q of R.*

5.8. Theorem. *If R is a ring and $(F,A) \wedge (G,B) \wedge (F,A) = (F,A) * (G,B) * (F,A)$ for every quasi-idealistic soft ring (F,A) over R, where (G,B) is an absolute soft ring over R, then R is regular.*

Proof. Let Q be any quasi-ideal of R and $F(x) = Q$ for all $x \in A$. Then (F,A) is a quasi-idealistic soft ring over R. Since $(F,A) \wedge (G,B) \wedge (F,A) = (F,A) * (G,B) * (F,A)$, $Q = QRQ$. It follows that R is regular. □

5.9. Corollary. *If R is a ring and $(F,A) \wedge (G,B) \wedge (F,A) = (F,A) * (G,B) * (F,A)$ for every bi-idealistic soft ring (F,A) over R, where (G,B) is an absolute soft ring over R, then R is regular.*

5.10. Theorem. *If R is a regular ring and (G,B) is an absolute soft ring over R, then $(F,A) \wedge (G,B) \wedge (H,C) = (F,A) * (G,B) * (H,C)$ for every right idealistic soft ring (F,A) over R and every left idealistic soft ring (H,C) over R.*

Proof. For all $y \in B$ with $G(y) = R$, let (F,A) and (H,C) be any right idealistic soft ring over R and left idealistic soft ring over R, respectively. Then, for all $x \in A$ and $z \in C$, we have $F(x)RH(z) \subset F(x)RR \subset F(x)$ and $F(x)RH(z) \subset RRH(z) \subset H(z)$. Hence, we deduce that $F(x)RH(z) \subset F(x) \cap H(z)$ and $(F,A) * (G,B) * (H,C) \widetilde{\subset} (F,A) \wedge (G,B) \wedge (H,C)$.

On the other hand, let $x \in A$ and $z \in C$. Since R is regular, $\forall a \in F(x) \cap R \cap H(z) \subset R$, there exists $r \in R$ such that $a = ara \in F(x)RH(z)$. Hence, we deduce that $F(x) \cap R \cap H(z) \subset F(x)RH(z)$ and $(F,A) \wedge (G,B) \wedge (H,C) \widetilde{\subset} (F,A) * (G,B) * (H,C)$. Thus, $(F,A) \wedge (G,B) \wedge (H,C) = (F,A) * (G,B) * (H,C)$. □

5.11. Lemma ([15]). *A ring R is regular if and only if $I \cap Q = QIQ$ holds for every ideal I of R and every quasi-ideal Q of R.*

5.12. Theorem. *If $(F,A) \wedge (G,B) \wedge (F,A) = (F,A) * (G,B) * (F,A)$ holds for every quasi-idealistic soft ring (F,A) over R and every idealistic soft ring (G,B) over R, then the ring R is regular.*

Proof. Assume that I is an ideal of R and Q is a quasi-ideal of R. If $F(x) = Q$ for all $x \in A$ and $G(y) = I$ for all $y \in B$, then (F,A) is a quasi-idealistic soft ring over R and (G,B) is an idealistic soft ring over R. Since $(F,A) \wedge (G,B) \wedge (F,A) = (F,A) * (G,B) * (F,A)$, $I \cap Q = QIQ$. This shows that R is regular. □

5.13. Corollary. *If $(F,A) \wedge (G,B) \wedge (F,A) = (F,A) * (G,B) * (F,A)$ holds for every bi-idealistic soft ring (F,A) and every idealistic soft ring (G,B) over a ring R, then R is a regular ring.*

5.14. Theorem. *If R is a regular ring, then $(F,A) \wedge (G,B) \wedge (H,C) = (F,A) * (G,B) * (H,C)$ holds for every right idealistic soft ring (F,A) over R, every idealistic soft ring (G,B) over R and every left idealistic soft ring (H,C) over R.*

Proof. Let (F,A), (G,B) and (H,C) be a right idealistic soft ring over R, an idealistic soft ring over R and a left idealistic soft ring over R, respectively. Then for all $x \in A$, for all $y \in B$, and for all $z \in C$, we have $F(x)G(y)H(z) \subset F(x)RR \subset F(x)$, $F(x)G(y)H(z) \subset RRH(z) \subset H(z)$ and $F(x)G(y)H(z) \subset RG(y)R \subset G(y)$. Hence, $F(x)G(y)H(z) \subset F(x) \cap G(y) \cap H(z)$.

On the other hand, if for all $a \in F(x) \cap G(y) \cap H(z) \subset R$, there exists $r \in R$ such that $a = ara = arara \in F(x)G(y)H(z)$, then this leads to $F(x)G(y)H(z) \supset F(x) \cap G(y) \cap H(z)$. Hence, $F(x)G(y)H(z) = F(x) \cap G(y) \cap H(z)$ and so $(F,A) \wedge (G,B) \wedge (H,C) = (F,A) * (G,B) * (H,C)$. □

5.15. Theorem. *For a ring R, the following conditions are equivalent:*

(1) R is regular.

(2) $(F,A)\wedge(G,B)\widetilde{\subset}(F,A)*(G,B)$ for every right idealistic soft ring (F,A) over R and every bi-idealistic soft ring (G,B) over R.

(3) $(F,A)\wedge(G,B)\widetilde{\subset}(F,A)*(G,B)$ for every right idealistic soft ring (F,A) over R and every quasi-idealistic soft ring (G,B) over R.

(4) $(F,A)\wedge(H,C)\widetilde{\subset}(F,A)*(H,C)$ for every bi-idealistic soft ring (F,A) over R and every left idealistic soft ring (H,C) over R.

(5) $(F,A)\wedge(H,C)\widetilde{\subset}(F,A)*(H,C)$ for every quasi-idealistic soft ring (F,A) over R and every left idealistic soft ring (H,C) over R.

(6) $(F,A)\wedge(G,B)\wedge(H,C)\widetilde{\subset}(F,A)*(G,B)*(H,C)$ for every right idealistic soft ring (F,A) over R, every bi-idealistic soft ring (G,B) over R and every left idealistic soft ring (H,C) over R.

(7) $(F,A)\wedge(G,B)\wedge(H,C)\widetilde{\subset}(F,A)*(G,B)*(H,C)$ for every right idealistic soft ring (F,A) over R, every quasi-idealistic soft ring (G,B) over R and every left idealistic soft ring (H,C) over R.

Proof. Assume that (1) holds. Let $x\in A$, $y\in B$. Since R is regular, for all $a\in F(x)\cap G(y)\subset R$, there exists $r\in R$ such that $a=ara=(ar)a\in F(x)G(y)$. This leads to $(F,A)\wedge(G,B)\widetilde{\subset}(F,A)*(G,B)$ and so (2) holds. Thus,(1) implies (2).

It can be similarly proved that (1) implies (4). Since any quasi-idealistic soft ring over R is a bi-idealistic soft ring over R, (2) also implies (3), and (4) implies (5).

Assume that (3) holds. Since any left idealistic soft ring over R is a quasi-idealistic soft ring, by Theorem 5.6, R is regular, and so (3) implies (1).

Similarly, we can prove (5) implies (1).

Assume that (1) holds. Let (F,A), (G,B) and (H,C) be any right idealistic soft ring over R, any bi-idealistic soft ring over R and any left idealistic soft ring over R. Let $x\in A$, $y\in B$ and $z\in C$. Since R is regular, for all $a\in F(x)\cap G(y)\cap H(z)\subset R$, there exists $r\in R$ such that $a=ara=arara=(ar)a(ra)\in F(x)G(y)G(z)$. Hence, $(F,A)\wedge(G,B)\wedge(H,C)\widetilde{\subset}(F,A)*(G,B)*(H,C)$ and (6) holds. Thus (1) implies (6).

It is clear that (6) implies (7).

Finally, we assume that (7) holds. Let (F,A) and (H,C) be any right idealistic soft ring over R and any left idealistic soft ring over R, respectively. If (G,B) is an absolute soft ring over R, then (G,B) is a quasi-idealistic soft ring over R. This implies that $(F,A)\wedge(G,B)\wedge(H,C)\widetilde{\subset}(F,A)*(G,B)*(H,C)$. Let $x\in A$, $z\in C$. Then $F(x)\cap H(z)=F(x)\cap R\cap H(z)\subset F(x)RH(z)\subset F(x)H(z)$ and $(F,A)\wedge(H,C)\widetilde{\subset}(F,A)*(H,C)$. Hence, it follows that R is regular and so (7) implies (1). □

Finally, we state the following theorem of regular rings to be soft regular.

5.16. Theorem. *If a ring R is soft regular, then R is regular.*

Proof. If (F,A) is an absolute soft ring over R, then $F(x)=R$ is a regular ring. □

5.17. Corollary. *If a ring R is soft regular, then $(F,A)*(G,B)=(F,A)\wedge(G,B)$ for every right idealistic soft ring (F,A) over R and every left idealistic soft ring (G,B) over R.*

5.18. Corollary. *If R is a soft regular ring and (G,B) is an absolute soft ring over R, then $(F,A)\wedge(G,B)\wedge(H,C)=(F,A)*(G,B)*(H,C)$ for every right idealistic soft ring (F,A) over R and every left idealistic soft ring (H,C) over R.*

5.19. Corollary. *If R is a soft regular ring, then $(F,A)\wedge(G,B)\wedge(H,C)=(F,A)*(G,B)*(H,C)$ for every right idealistic soft ring (F,A) over R, every idealistic soft ring (G,B) over R and every left idealistic soft ring (H,C) over R.*

5.20. Corollary. *If a ring R is soft regular, then the following conditions hold:*

(1) $(F,A) \wedge (G,B) \widetilde{\subset} (F,A) * (G,B)$ for every right idealistic soft ring (F,A) over R and every bi-idealistic soft ring (G,B) over R.

(2) $(F,A) \wedge (G,B) \widetilde{\subset} (F,A) * (G,B)$ for every right idealistic soft ring (F,A) over R and every quasi-idealistic soft ring (G,B) over R.

(3) $(F,A) \wedge (H,C) \widetilde{\subset} (F,A) * (H,C)$ for every bi-idealistic soft ring (F,A) over R and every left idealistic soft ring (H,C) over R.

(4) $(F,A) \wedge (H,C) \widetilde{\subset} (F,A) * (H,C)$ for every quasi-idealistic soft ring (F,A) over R and every left idealistic soft ring (H,C) over R.

(5) $(F,A) \wedge (G,B) \wedge (H,C) \widetilde{\subset} (F,A) * (G,B) * (H,C)$ for for every right idealistic soft ring (F,A) over R, every bi-idealistic soft ring (G,B) over R and every left idealistic soft ring (H,C) over R.

(6) $(F,A) \wedge (G,B) \wedge (H,C) \widetilde{\subset} (F,A) * (G,B) * (H,C)$ for every right idealistic soft ring (F,A) over R, every quasi-idealistic soft ring (G,B) over R and every left idealistic soft ring (H,C) over R.

In order to answer when will a regular ring be soft regular, we give the following lemma.

5.21. Lemma. *If a ring R is regular, then every idealistic soft ring over R is regular.*

Proof. Let (F,A) be any idealistic soft ring over R. Then $\forall x \in A$, $F(x)$ is an ideal of R. If $a \in F(x)$, then $a \in R$ and there exists an element $r \in R$ such that $a = ara = arara = a(rar)a \in aF(x)a$. Thus $F(x)$ is regular and (F,A) is a regular soft ring over R. □

By using the above lemma, we obtain the following theorem for regular rings to be soft regular.

5.22. Theorem. *If a ring R is regular and every soft ring (F,A) over R is an idealistic soft ring, then R is soft regular.*

Proof. If (F,A) is a soft ring over R, then (F,A) is an idealistic soft ring over R. Hence, by Lemma 5.21, (F,A) is regular and consequently R is soft regular. □

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