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A RESULT ON GENERALIZED DERIVATIONS IN PRIME RINGS

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Abstract

Let R be a prime ring, H a generalized derivation of R, L a noncentral Lie ideal of R, and $0 \neq a \in R$. Suppose that $au^s(H(u))^n u^t = 0$ for all $u \in L$, where $s, t \geq 0$ and $n > 0$ are fixed integers. If $s = 0$, then $H(x) = bx$ for all $x \in R$, where $b \in U$, the right Utumi quotient ring of R, with $ab = 0$ unless R satisfies s_4, the standard identity in four variables. If $s > 0$, then $H = 0$ unless R satisfies s_4.

Keywords: prime ring, derivation, generalized derivation, extended centroid, right Utumi quotient ring.

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1. Introduction

Throughout this paper, R is always a prime ring with extended centended C, right Utumi quotient ring U, and two-sided Martindale quotient ring Q. The definitions and properties of these objects can be found in [3, Chapter 2]. Denote s_4 as the standard identity in four variables.

By a generalized derivation on R one usually means an additive map $H : R \to R$ such that $H(xy) = H(x)y + xd(y)$, for some derivation d of R. Obviously any derivation is a generalized derivation. Another basic example of generalized derivations is the following: $H(x) = ax + xb$ for $a, b \in R$. Hvala [12] initiated the study of generalized derivations on prime rings. Lee proved the following essential result: every generalized derivation H on a dense left ideal of R can be uniquely extended to U and assume the form $H(x) = bx + d(x)$ for some $b \in U$ and a derivation d on U [16, Theorem 3]. In recent years, a number of articles discussed generalized derivations in the context of prime and semiprime rings (see [1, 5, 9, 10, 11, 18, 19, 21, 22]).

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Dhara and Sharma [6] proved that, if $a \in R$ such that $au^s d(u)^n u^t = 0$ for all $u \in L$, a noncommutative Lie ideal of R, where d a derivation of R, $s \geq 0, t \geq 0, n \geq 1$ are fixed integers, then either $a = 0$ or $d = 0$ unless char $R = 2$ and R satisfies s_4. Dhara and Filippis [5] proved that, if $u^s H(u) u^t = 0$ for all $u \in L$, where L a noncommutative Lie ideal of R, H a generalized derivation of R, and $s, t \geq 0$ are fixed integers, then $H = 0$ unless char $R = 2$ and R satisfies s_4. Recently, the second author [22] investigated the situation when $au^s H(u) u^t = 0$ for all $u \in L$, where L a noncentral Lie ideal of R.

In the present paper we shall generalize the above results in a full general situation. More precisely, we shall prove the following main result of this paper.

1.1. Theorem. *Let R be a prime ring, H a generalized derivation of R, L a noncentral Lie ideal of R, and $0 \neq a \in R$. Suppose that $au^s(H(u))^n u^t = 0$ for all $u \in L$, where $s, t \geq 0$ and $n > 0$ are fixed integers. If $s = 0$, then $H(x) = bx$ for all $x \in R$, where $b \in U$ with $ab = 0$ unless R satisfies s_4. If $s > 0$, then $H = 0$ unless R satisfies s_4.*

2. The proof of the main result

We begin with the following result, which will be used in the proof of our main result.

2.1. Lemma. *Let R be a prime ring with $\dim_C RC > 4$. Let $0 \neq a \in R$ and $b \in U$ such that*

$$a[x, y]^s (b[x, y])^n [x, y]^t = 0$$

for all $x, y \in R$, where $s, t \geq 0$ and $n > 0$ are fixed integers. If $s = 0$, then $ab = 0$. If $s > 0$, then $b = 0$.

Proof. Suppose first that $b \in C$, by assumption we have

$$ab^n [x, y]^{s+n+t} = 0$$

for all $x, y \in R$. It is easy to check that either $ab^n = 0$ or R is commutative (see the proof of [17, Theorem 1] or [6, Theorem 2.2]). Hence $b = 0$ as $a \neq 0$ and $\dim_C RC > 4$.

Suppose next that $b \notin C$. Since R and U satisfy the same generalized polynomial identity [4, Theorem 2], we have

$$a[x, y]^s (b[x, y])^n [x, y]^t = 0 \qquad (2.1)$$

for all $x, y \in U$. In case C is infinite, the GPI (2.1) is also satisfied by $U \otimes_C \bar{C}$ where $\bar{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \bar{C}$ are prime and centrally closed [7], we may replace R by U or $U \otimes_C \bar{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed such that $a[x, y]^s (b[x, y])^n [x, y]^t = 0$ for all $x, y \in R$.

If $s = 0$ and $ab \neq 0$, then $a(b[X, Y])^n [X, Y]^t$ is a nonzero GPI on R as it has nonzero monomial $a(bXY)^n (XY)^t$. By Martindale's theorem in [20] R is a primitive ring having nonzero socle and the commuting division D is a finite dimensional central division algebra over C. Since C is either finite or algebraically closed, D must coincide with C. Thus R is isomorphic to a dense subring of $\mathrm{End}_C V$ for some vector space V over C. Since $\dim_C RC > 4$, it is obvious that $\dim_C V \geq 3$. We will show that, for any given $v \in V$, v and bv are C-dependent. Assume on the contrary that v and bv are C-independent and set $W = Cv + Cbv$. Since $\dim_C V \geq 3$, there exists $u \in V$ such that v, bv, u are also C-independent. If $abv \neq 0$, by the density of R in $\mathrm{End}_C V$ there exist two elements r_1 and r_2 in R such that

$$r_1 v = 0,\ r_1 bv = 0,\ r_1 u = v; \quad r_2 v = u,\ r_2 bv = u,\ r_2 u = 0$$

and so

$$[r_1, r_2] v = v \quad \text{and} \quad [r_1, r_2] bv = v.$$

Hence,

$$0 = a(b[r_1, r_2])^n [r_1, r_2]^t v = abv,$$

a contradiction.

Suppose that $abv = 0$. Since $ab \neq 0$, there exists $w \in V$ such that $abw \neq 0$ and so $ab(v - w) \neq 0$. By the previous argument we have that there exist $\beta, \gamma \in C$ such that

$$bw = \beta w \text{ and } b(w - v) = \gamma(w - v).$$

This yields that $(\beta - \gamma)w \in W$. Now $\beta = \gamma$ implies the contradiction that $bv = \beta v$. Thus $\beta \neq \gamma$ and so $w \in W$. But if $u \in V$ with $abu = 0$, then $ab(w + u) \neq 0$. So $w + u \in W$ forcing $u \in W$. Thus $V = W$ and so $\dim_C V = 2$, a contradiction.

If $s \geq 1$, it is easy to see that $a[X, Y]^s (b[X, Y])^n [X, Y]^t$ is a nonzero GPI on R. By the previous argument R is isomorphic to a dense subring of $\mathrm{End}_C V$ with $\dim_C V \geq 3$. We will show that, for any given $v \in V$, v and bv are C-dependent. Assume on the contrary that v and bv are C-independent and set $W = Cv + Cbv$. Since $\dim_C V \geq 3$, there exists $u \in V$ such that v, bv, u are also C-independent. If $av \neq 0$, by the density of R in $\mathrm{End}_C V$ there exist two elements r_1 and r_2 in R such that

$$r_1 v = 0,\ r_1 bv = 0,\ r_1 u = v; \quad r_2 v = u,\ r_2 bv = u,\ r_2 u = 0$$

and so

$$[r_1, r_2]v = v \quad \text{and} \quad [r_1, r_2]bv = v.$$

Hence,

$$0 = a[r_1, r_2]^s (b[r_1, r_2])^n [r_1, r_2]^t v = av,$$

a contradiction.

Suppose that $av = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $a(v - w) \neq 0$. By the previous argument we have that there exist $\beta, \gamma \in C$ such that

$$bw = \beta w \text{ and } b(w - v) = \gamma(w - v).$$

This yields that $(\beta - \gamma)w \in W$. Now $\beta = \gamma$ implies the contradiction that $bv = \beta v$. Thus $\beta \neq \gamma$ and so $w \in W$. But if $u \in V$ with $au = 0$, then $a(w + u) \neq 0$. So $w + u \in W$ forcing $u \in W$. Thus $V = W$ and so $\dim_C V = 2$, a contradiction.

Hence, in any case, for all $v \in V$, v and bv are linearly C-dependent. Thus, standard arguments show that $b \in C$, which contradicts our hypothesis. □

We are in a position to give

The proof of Theorem 1.1. We assume that R does not satisfy s_4. That is, $\dim_C RC > 4$. By a theorem of Lanski and Montgomery [15, Theorem 13] we have $0 \neq [I, R] \subseteq L$, where I is a nonzero ideal of R. Hence we may assume without loss of generality that $L = [I, I]$. By [16, Theorem 3] we may assume that $H(x) = bx + d(x)$ for all $x \in U$, where $b \in U$ and d a derivation of U. Thus

$$a[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))^n [x_1, x_2]^t = 0$$

for all $x_1, x_2 \in I$. Since I and U satisfy the same differential identities [4], we have

$$a[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))^n [x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. Assume first that d is Q-inner, i.e., there exists $b, c \in U$ such that $H(x) = bx + xc$ for all $x \in U$. So

$$f(x_1, x_2) = a[x_1, x_2]^s (b[x_1, x_2] + [x_1, x_2]c)^n [x_1, x_2]^t = 0 \qquad (2.2)$$

for all $x_1, x_2 \in U$. In case C is infinite, the GPI (2.2) is also satisfied by $U \otimes_C \bar{C}$ where $\bar{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \bar{C}$ are prime and centrally closed

[7], we may replace R by U or $U \otimes_C \bar{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed such that $f(x_1, x_2) = 0$ for all $x_1, x_2 \in R$.

Suppose first that $c \notin C$. Then $f(X_1, X_2)$ is a nonzero GPI for R as it has a nonzero monomial $a(X_1X_2)^s(X_1X_2c)^n(X_1X_2)^t$. By Martindale's theorem in [20] R is a primitive ring having nonzero socle and the commuting division D is a finite dimensional central division algebra over C. Since C is either finite or algebraically closed, D must coincide with C. Thus R is isomorphic to a dense subring of $\mathrm{End}_C V$ for some vector space V over C. Since $\dim_C R > 4$, it is obvious that $\dim_C V \geq 3$. We will show that, for any given $v \in V$, v and cv are C-dependent. Assume on the contrary that v and cv are C-independent and set $W = Cv + Ccv$. Since $\dim_C V \geq 3$, there exists $u \in V$ such that v, cv, u are also C-independent. If $av \neq 0$, by the density of R in $\mathrm{End}_C V$ there exist two elements r_1 and r_2 in R such that

$$r_1v = 0,\ r_1cv = u,\ r_1u = v \quad \text{and} \quad r_2v = u,\ r_2cv = 0,\ r_2u = bv - v$$

and so

$$[r_1, r_2]v = v \quad \text{and} \quad [r_1, r_2]cv = -bv + v.$$

Hence,

$$0 = a[r_1, r_2]^s(b[r_1, r_2] + [r_1, r_2]c)^n[r_1, r_2]^t v = av,$$

a contradiction.

Suppose that $av = 0$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and so $a(v - w) \neq 0$. By the previous argument we have that there exist $\beta, \gamma \in C$ such that

$$cw = \beta w \text{ and } c(w - v) = \gamma(w - v).$$

This yields that $(\beta - \gamma)w \in W$. Now $\beta = \gamma$ implies the contradiction that $cv = \beta v$. Thus $\beta \neq \gamma$ and so $w \in W$. But if $u \in V$ with $au = 0$, then $a(w + u) \neq 0$. So $w + u \in W$ forcing $u \in W$. Thus $V = W$ and so $\dim_C V = 2$, a contradiction.

Hence, in any case, for all $v \in V$, v and cv are linearly C-dependent. Thus, standard arguments show that $c \in C$ which contradicts our hypothesis.

Suppose next that $c \in C$. By our assumption we have

$$a[x_1, x_2]^s((b + c)[x_1, x_2])^n[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. Then the result follows from Lemma 2.1.

Assume next that d is not Q-inner. Then

$$a[x_1, x_2]^s(b[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)])^n[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. In view of the powerful Kharchenko's theorem [14] we have

$$a[x_1, x_2]^s(b[x_1, x_2] + [x_3, x_2] + [x_1, x_4])[x_1, x_2]^t = 0$$

for all $x_1, x_2, x_3, x_4 \in U$. Setting $x_3 = ix_1$ and $x_4 = 0$, where $i = 1, 2$, we have

$$a[x_1, x_2]^s((b + i)[x_1, x_2])^n[x_1, x_2]^t = 0 \tag{2.3}$$

for all $x_1, x_2 \in R$. If $s = 0$, we get from Lemma 2.1 that $a(b + i) = 0$. It follows that $a = 0$, contradicting our assumption. If $s > 0$, we get from Lemma 2.1 that $b + i = 0$, a contradiction. The proof of the result is complete. □

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References

[1] Albas, B., Argac, N. and Fillippis, V. D. *Generalized derivations with Engel conditions on one-sided ideals*, Comm. Algebra **36**, 2063-2071, 2008.

[2] Albert A. A. and Muckenhoupt, B. *On matrices of trace zero* Michigan J. Math. **1**, 1-3, 1957.

[3] Beidar, K. I., Martindale W. S. and Mikhalev, A. V. *Rings with Generalized Identities*, Marcel Dekker, New York-Basel-Hong Kong, 1996.

[4] Chuang, C. L. *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103**, 723-728, 1988.

[5] Dhara, B. and De Filippis, V. *Notes on generalized derivations on Lie ideals in prime rings*, Bull. Korean Math. Soc. **46** (3), 599-605, 2009.

[6] Dhara, B and Sharma, R. K. *Derivations with annihilator conditions in prime rings*, Publ. Math. Debrecen **71** (1), 11-20, 2007.

[7] Erickson, T. S., Martindale, W S. and Osborn, J. M. *Prime nonassociative algebras*, Pacific J. Math. **60** (1), 49-63, 1975.

[8] Faith, C. and Utumi, Y. *On a new proof of Litoff's theorem*, Acta Math. Acad. Sci. Hung. **14**, 369-371, 1963.

[9] De Filippis, V. *An Engel condition with generalized derivations on multilinear polynomials*, Israel J. Math. **162**, 93-108, 2007.

[10] De Filippis, V. *Posner's second theorem and an annihilator condition with generalized derivations*, Turk J. Math. **32**, 197-211, 2008.

[11] De Filippis, V. *Generalized derivations in prime rings and noncommutative Banach algebras*, Bull. Korean Math. Soc. **45**, 621-629, 2008.

[12] Hvala, B. *Generalized derivations in rings*, Comm. Algebra **26**, 1147-1166, 1998.

[13] Jacobson, N. *Structure of Rings*, Amer. Math. Soc. Colloq. Pub., **37**, Amer. Math. Soc., Providence, RI, 1964.

[14] Kharchenko, V. K. *Differential identities of prime rings*, Algebra and Logic **17** 155-168, 1978.

[15] Lanski, C. and Montgomery, S. *Lie structure of prime rings of characteristic 2*, Pacific J. Math. **42**, 117-136, 1972.

[16] Lee, T. K. *Generalized derivations of left faithful rings*, Comm. Algebra **27**, 4057-4073, 1999.

[17] Lee, T. K. and Lin, J. S. *A result on derivations*, Proc. Amer. Math. Soc. **124**, 1687-1691, 1996.

[18] Lee, T. K. Lee and Shiue, W. K. *Identities with generalized derivations*, Comm. Algebra **29**, 4437-4450, 2001.

[19] Lin, J. S. and Liu, C. K. *Generalized derivations with invertible or nilpotent on multilinear polynomials values*, Comm. Algebra **34**, 633-640, 2006.

[20] Martindale 3rd, W. S. *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12**, 576-584, 1969.

[21] Wang, Y. *Generalized derivations with power-central values on multilinear polynomials*, Algebra Colloq. **13**, 405-410, 2006.

[22] Wang, Y. *Annihilator conditions with generalized derivations in prime rings*, Bull. Korean Math. Soc. **48**, 917-922, 2011.