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WEAKLY Φ-CONTINUOUS FUNCTIONS IN GRILL TOPOLOGICAL SPACES

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Abstract

In this paper, we introduce and investigate the notion of a weakly Φ-continuous function in grill topological spaces and using this function we obtain a decomposition of continuity. Also, we investigate its relationship with other related functions.

Keywords: Grill topological space, Weak Φ-continuity, Decomposition of continuity

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1. Introduction

The idea of grills on a topological space was first introduced by Choquet [4]. The concept of grills has proved to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces, and the theory of compactifications and extension problems of different kinds (see [2], [3], [15] for details). In [14], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [6] defined new classes of sets and give a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notion of a weakly Φ-continuous function of a topological space into a grill topological space. By using weak Φ-continuity, we obtain a decomposition of continuity which is analogous to the decomposition of continuity due to Levine [9].

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2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$. A subcollection $\mathcal{G}$ of $\mathcal{P}(X)$ is called a grill [4] on X if $\mathcal{G}$ satisfies the following conditions:

(1) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,
(2) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x.

2.1. Definition. [14] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the *operator associated with the grill* $\mathcal{G}$ *and the topology* τ.

2.2. Proposition. [14] *Let* (X, τ) *be a topological space and* $\mathcal{G}$ *a grill on* X*. Then for all* $A, B \subseteq X$*:*

(1) $A \subseteq B$ *implies that* $\Phi(A) \subseteq \Phi(B)$*,*
(2) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$*,*
(3) $\Phi(\Phi(A)) \subseteq \Phi(A) = \mathrm{Cl}(\Phi(A)) \subseteq \mathrm{Cl}(A)$*.*

Let G be a grill on a space X. Then we define a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ is a Kuratowski closure operator. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\text{-}\mathrm{Cl}(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill, $\mathcal{G}$ on X, then we call it a *grill topological space* and denote it by $(X, \tau, \mathcal{G})$.

The concept of ideals in topological spaces is treated in the classic text of Kuratowski [8] and Vaidyanathaswamy [16]. Janković and Hamlett [7] investigated further properties of ideal spaces. An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal topological space or simply an ideal space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function of* A *with respect to* $\mathcal{I}$ *and* τ [8]. We simply write A^* in case there is no chance for confusion. A Kuratowski closure operator $\mathrm{Cl}^*(\cdot)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology finer than τ, is defined by $\mathrm{Cl}^*(A) = A \cup A^*$ [7].

The following lemma will be useful in the sequel.

2.3. Lemma. [12] *Let* (X, τ) *be a topological space. Then the following hold.*

(1) $\mathcal{G}$ *is a grill on* X *if and only if* $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$ *is an ideal on* X*,*
(2) *The operators* Cl^* *on* $(X, \tau, \mathcal{I})$*, where* $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$ *and* Ψ *on* $(X, \tau, \mathcal{G})$ *are equal.* □

3. Φ-continuous functions

3.1. Definition. [6] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A subset A in X is said to be Φ-*open* if $A \subseteq \mathrm{Int}(\Phi(A))$. The complement of a Φ-open set is said to be Φ-*closed*.

3.2. Lemma. *If a subset* A *of a grill topological space* $(X, \tau, \mathcal{G})$ *is* Φ*-closed, then* $\Phi(\mathrm{Int}(A)) \subseteq A$*.*

Proof. Suppose that A is Φ-closed. Then, we have $X - A \subseteq \mathrm{Int}(\Phi(X - A)) \subseteq \mathrm{Int}(\mathrm{Cl}(X - A)) = X - \mathrm{Cl}(\mathrm{Int}(A))$. Therefore $\Phi(\mathrm{Int}(A)) \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \subseteq A$. □

We denote by $\Phi O(X, \tau) = \{A \subseteq X : A \subseteq \mathrm{Int}(\Phi(A))\}$ or simply write $\Phi O(X)$ for $\Phi O(X, \tau)$ when there is no chance for confusion.

3.3. Definition. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be

(1) $\mathcal{G}$-*dense-in-itself* (rep. $\mathcal{G}$-*perfect*) if $A \subseteq \Phi(A)$ (resp. $A = \Phi(A)$),
(2) $\mathcal{G}$-*preopen* [6] if $A \subseteq \mathrm{Int}(\Psi(A))$,
(3) *preopen* [11] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$.

3.4. Theorem. [6] *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then*

(1) *Every Φ-open set A is $\mathcal{G}$-preopen.*
(2) *Every $\mathcal{G}$-preopen set A is preopen.* □

3.5. Theorem. *For a subset A of a grill topological space $(X, \tau, \mathcal{G})$, the following conditions are equivalent:*

(1) *A is Φ-open;*
(2) *A is $\mathcal{G}$-preopen and $\mathcal{G}$-dense-in-itself.*

Proof. (1) $\Longrightarrow$ (2) By Theorem 3.4 every Φ-open set is $\mathcal{G}$-preopen. On the other hand $A \subseteq \mathrm{Int}(\Phi(A)) \subseteq \Phi(A)$, which show that A is $\mathcal{G}$-dense-in-itself.

(2) $\Longrightarrow$ (1) By assumption, $A \subseteq \mathrm{Int}(\Psi(A)) = \mathrm{Int}(\Phi(A) \cup A) = \mathrm{Int}(\Phi(A))$ and hence A is Φ-open. □

The following examples show that $\mathcal{G}$-preopen and $\mathcal{G}$-dense-in-itself are independent concepts.

3.6. Example. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and the grill $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$. Then $A = \{a, c\}$ is a $\mathcal{G}$-preopen set which is not $\mathcal{G}$-dense-in-itself.

3.7. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and the grill $\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}$. Then $B = \{a, b\}$ is a $\mathcal{G}$-dense-in-itself set which is not preopen and hence it is not $\mathcal{G}$-preopen.

3.8. Remark. It should be noted that:

(1) It is shown in Example 2.1 of [6] that Φ-openness and openness are independent of each other.

(2) In [6], it is shown that Φ-openness $\Longrightarrow$ $\mathcal{G}$-openness $\Longrightarrow$ preopenness and the converses are not true in general.

3.9. Definition. [6] A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be Φ-*continuous* if for each open set V in Y, $f^{-1}(V)$ is Φ-open in $(X, \tau, \mathcal{G})$.

3.10. Theorem. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is Φ-continuous;*
(2) *The inverse image of each closed set of Y is Φ-closed;*
(3) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $W \in \Phi O(X)$ containing x such that $f(W) \subseteq V$;*
(4) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, $\Phi(f^{-1}(V))$ is a neighborhood of x.*

Proof. (1) $\Longleftrightarrow$ (2) Obvious.

(1) $\Longrightarrow$ (3) Since $V \in \sigma$ contains $f(x)$, by (1), $f^{-1}(V) \in \Phi O(X)$. By putting $W = f^{-1}(V)$, we have $x \in W$ and $f(W) \subseteq V$.

(3) $\Longrightarrow$ (4) Since $V \in \sigma$ contains $f(x)$, by (3), there exists $W \in \Phi O(X)$ containing x such that $f(W) \subseteq V$. Thus, $x \in W \subseteq \mathrm{Int}(\Phi(W)) \subseteq \mathrm{Int}(\Phi(f^{-1}(V))) \subseteq \Phi(f^{-1}(V))$. Hence $\Phi(f^{-1}(V))$ is a neighborhood of x.

(4) $\Longrightarrow$ (1) Let V be any open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V \in \sigma$. By (4), there exists an open set of X such that $x \in U \subseteq \Phi(f^{-1}(V))$. Therefore, $x \in U \subseteq \mathrm{Int}(\Phi(f^{-1}(V)))$. This shows that $f^{-1}(V) \subseteq \mathrm{Int}(\Phi(f^{-1}(V)))$. Therefore, f is Φ-continuous. □

A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\mathcal{G}$-*dense-continuous* (resp. $\mathcal{G}$-*precontinuous* [6], *precontinuous* [11]) if the inverse image of every open set is $\mathcal{G}$-dense-in-itself (resp. $\mathcal{G}$-preopen, preopen).

Thus we have the following decomposition of Φ-continuity.

3.11. Theorem. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is Φ-continuous;*
(2) *f is $\mathcal{G}$-precontinuous and $\mathcal{G}$-dense-continuous.* □

The following two examples show that $\mathcal{G}$-precontinuity and $\mathcal{G}$-dense-continuity are independent of each other.

3.12. Example. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $\sigma = \{\phi, X, \{a\}, \{c\}\{a, c\}\}$. Let $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$ be a grill on X. The identity function $f : (X, \tau, \mathcal{G}) \to (X, \sigma)$ is $\mathcal{G}$-precontinuous but it is not $\mathcal{G}$-dense-continuous.

(i) Let $V = X \in \sigma$, then $f^{-1}(V)$ is $\mathcal{G}$-preopen.
(ii) Let $V = \{a\} \in \sigma$, then $f^{-1}(V)$ is $\mathcal{G}$-preopen.
(iii) Let $V = \{c\} \in \sigma$, $f^{-1}(V)$ is $\mathcal{G}$-preopen.
(iv) Let $V = \{a, c\} \in \sigma$, then $f^{-1}(V)$ is $\mathcal{G}$-preopen set which is not $\mathcal{G}$-dense-in-itself.

By (i), (ii, (iii) and (iv), f is a $\mathcal{G}$-precontinuous function which is not $\mathcal{G}$-dense-continuous.

3.13. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{c, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}$ be a grill on X. Define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = a$, $f(c) = b$ and $f(d) = b$. Then f is $\mathcal{G}$-dense-continuous but it is not $\mathcal{G}$-precontinuous.

(i) Let $V = X \in \sigma$, then $f^{-1}(V) = X$ is $\mathcal{G}$-dense-in-itself.
(ii) Let $V = \{a\} \in \sigma$, then $f^{-1}(V) = \{b\}$ is $\mathcal{G}$-dense-in-itself.
(iii) Let $V = \{c\} \in \sigma$, $f^{-1}(V) = \{a\}$ is $\mathcal{G}$-dense-in-itself.
(iv) Let $V = \{a, c\} \in \sigma$, then $f^{-1}(V) = \{a, b\}$ is $\mathcal{G}$-dense-in-itself set which is not $\mathcal{G}$-preopen.

By (i), (ii, (iii) and (iv), f is a $\mathcal{G}$-dense-continuous function which is not $\mathcal{G}$-precontinuous.

4. Weakly Φ-continuous functions

Let (X, τ) be a topological space and $(Y, \sigma, \mathcal{G})$ a grill topological space. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly continuous* [9] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq \mathrm{Cl}(V)$.

4.1. Definition. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is said to be *weakly Φ-continuous* if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq \Psi(V)$.

Every weakly Φ-continuous function is weakly continuous but the converse is not true (see Example 5.1).

4.2. Definition. [1] A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is said to be *weakly* $\mathcal{I}$*-continuous* if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq \mathrm{Cl}^*(V)$.

4.3. Remark. Let (X, τ), (Y, σ) be two topological spaces and $\mathcal{G}$ a grill on Y. If $\mathcal{I} = \mathcal{P}(Y) - \mathcal{G}$, then by Lemma 2.3, a weakly Φ-continuous function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ coincides with the weakly $\mathcal{I}$-continuous function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$.

4.4. Definition. A grill topological space $(X, \tau, \mathcal{G})$ is called an $A\mathcal{G}$*-space* if $\mathrm{Cl}(A) \subseteq \Phi(A)$ for every open set $A \subseteq X$.

4.5. Remark. An ideal topological space $(X, \tau, \mathcal{I})$ is called an FI^*-space [1] if $\mathrm{Cl}(A) \subseteq A^*$ for every $A \in \tau$. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$ the ideal on X. If $(X, \tau, \mathcal{G})$ is an $A\mathcal{G}$-space, then $(X, \tau, \mathcal{I})$ is the FI^*-space.

4.6. Theorem. *For a grill topological space* $(Y, \sigma, \mathcal{G})$*, the following properties are equivalent:*

(1) $(Y, \sigma, \mathcal{G})$ *is an* $A\mathcal{G}$*-space;*
(2) $\sigma \setminus \{\phi\} \subseteq \mathcal{G}$*;*
(3) $\Phi(V) = \mathrm{Cl}(V) = \Psi(V)$ *for every* $V \in \sigma$*.*

Proof. (1) $\Longrightarrow$ (2) Let $(Y, \sigma, \mathcal{G})$ be an $A\mathcal{G}$-space. Suppose that there exists $U \in \sigma \setminus \{\phi\}$ such that $U \notin \mathcal{G}$. Then, there exist $x \in U$ such that $U \cap U = U \notin \mathcal{G}$. Therefore, $x \notin \Phi(U)$ and $x \in \mathrm{Cl}(U) \setminus \Phi(U)$. This shows that $\mathrm{Cl}(U) \nsubseteq \Phi(U)$. This is contrary that $(Y, \sigma, \mathcal{G})$ is an $A\mathcal{G}$-space.

(2) $\Longrightarrow$ (3) For any $V \in \sigma$, in case $V = \phi$ (3) is obvious. In case $V \neq \phi$, $V \in \mathcal{G}$. Let $x \in \mathrm{Cl}(V)$. Then $\phi \neq U \cap V \in \sigma$ for every $U \in \sigma(x)$. By (2), $U \cap V \in \mathcal{G}$ and hence $x \in \Phi(V)$. Therefore, $\mathrm{Cl}(V) \subseteq \Phi(V)$ for any $V \in \sigma$. Moreover, we have $\mathrm{Cl}(V) \subseteq \Phi(V) \subseteq \Psi(V)$. Since $\sigma \subseteq \sigma_{\mathcal{G}}$, $\mathrm{Cl}(V) \supseteq \Psi(V)$ and $\Phi(V) = \mathrm{Cl}(V) = \Psi(V)$.

(3) $\Longrightarrow$ (1) This is obvious. □

4.7. Theorem. *A function* $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ *is weakly* Φ*-continuous if and only if for each open set* $V \subseteq Y$*,* $f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(\Psi(V)))$.

Proof. Necessity. Let V be any open set of Y and $x \in f^{-1}(V)$. Since f is weakly Φ-continuous, there exists an open set U such that $x \in U$ and $f(U) \subseteq \Psi(V)$. Hence $x \in U \subseteq f^{-1}(\Psi(V))$ and $x \in \mathrm{Int}(f^{-1}(\Psi(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(\Psi(V)))$.

Sufficiency. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(\Psi(V)))$. Let $U = \mathrm{Int}(f^{-1}(\Psi(V)))$. Then $x \in U$ and $f(U) = f(\mathrm{Int}(f^{-1}(\Psi(V)))) \subseteq f(f^{-1}(\Psi(V))) \subseteq \Psi(V)$. This shows that f is weakly Φ-continuous. □

4.8. Definition. A subset A of a topological space X is said to be *semi-open* [10] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$.

The following theorem gives characterizations of weakly Φ-continuous functions.

4.9. Theorem. *Let* $(Y, \sigma, \mathcal{G})$ *be an* $A\mathcal{G}$*-space. Then for a function* $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$*, the following properties are equivalent:*

(1) f *is weakly* Φ*-continuous;*
(2) *For every semi-open set* V *in* Y*, there exists an open set* U *in* Y *such that* $U \subseteq V$ *and* $f^{-1}(U) \subseteq \mathrm{Int}(f^{-1}(\Phi(V)))$*;*

(3) $f^{-1}(U) \subseteq \mathrm{Int}(f^{-1}(\Phi(U)))$ *for every open set* U *in* Y.

Proof. (1) $\Longrightarrow$ (2) Suppose f is weakly Φ-continuous and V is semi-open in (Y, σ). Since V is semi-open in (Y, σ), there exists an open set U in (Y, σ) such that $U \subseteq V \subseteq \mathrm{Cl}(U)$. By Theorem 4.6, $\Phi(U) = \mathrm{Cl}(U) = \Psi(U)$. Therefore $U \subseteq V \subseteq \Phi(U)$ so that $\Phi(U) = \Phi(V) = \Psi(U)$. By Theorem 4.7, $f^{-1}(U) \subseteq \mathrm{Int}(f^{-1}(\Psi(U))) = \mathrm{Int}(f^{-1}(\Phi(V)))$, which proves (2).

(2) $\Longrightarrow$ (3) Clear since every open set is semi-open.

(3) $\Longrightarrow$ (1) Since $\Phi(U) \subseteq \Psi(U)$, the proof follows from Theorem 4.7. □

4.10. Theorem. *Let* $(Y, \sigma, \mathcal{G})$ *be an* $A\mathcal{G}$*-space. Then for a function* $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$, *the following properties are equivalent:*

(1) f *is weakly* Φ*-continuous;*
(2) $\mathrm{Cl}(f^{-1}(U)) \subseteq f^{-1}(\Psi(U))$ *for each open set* $U \subseteq Y$*;*
(3) f *is weakly continuous.*

Proof. (1) $\Longrightarrow$ (2) Suppose $x \in \mathrm{Cl}(f^{-1}(U)) - f^{-1}(\Psi(U))$ for some open set U of Y. Then $x \in \mathrm{Cl}(f^{-1}(U))$ and $x \notin f^{-1}(\Psi(U))$. Now $x \notin f^{-1}(\Psi(U))$ implies that $f(x) \notin \Psi(U) = \mathrm{Cl}(U)$, by Theorem 4.6. Therefore, there exists an open set W containing $f(x)$ such that $W \cap U = \phi$; hence $\mathrm{Cl}(W) \cap U = \phi$ and so $\Psi(W) \cap U = \phi$. Since f is weakly Φ-continuous, there is an open set V containing x in X such that $f(V) \subseteq \Psi(W)$ and so $f(V) \cap U = \phi$. Now $x \in \mathrm{Cl}(f^{-1}(U))$ implies that $V \cap f^{-1}(U) \neq \phi$ and so $f(V) \cap U \neq \phi$ which is a contradiction. This completes the proof.

(2) $\Longrightarrow$ (3) Since $(Y, \sigma, \mathcal{G})$ is an $A\mathcal{G}$-space, by Theorem 4.6 and (2) we have $\mathrm{Cl}(f^{-1}(U)) \subseteq f^{-1}(\mathrm{Cl}(U))$ for every $U \in \sigma$. It follows from [13, Theorem 7] that f is weakly continuous.

(3) $\Longrightarrow$ (1) It is shown in [9, Theorem 1] that $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous if and only if $f^{-1}(V) \subseteq \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ for every $V \in \sigma$. Since $(Y, \sigma, \mathcal{G})$ is an $A\mathcal{G}$-space, by Theorem 4.6 $\mathrm{Cl}(V) = \Psi(V)$ for every $V \in \sigma$ and by Theorem 4.7, f is weakly Φ-continuous. □

4.11. Definition. A grill topological space $(X, \tau, \mathcal{G})$ is called an $R\mathcal{G}$*-space* if for each $x \in X$ and each open neighbourhood V of x, there exists an open neighbourhood U of x such that $x \in U \subseteq \Psi(U) \subseteq V$.

4.12. Remark. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$ the ideal on X. If $(X, \tau, \mathcal{G})$ is an $R\mathcal{G}$-space, then $(X, \tau, \mathcal{I})$ is the RI-space given in [1, Definition 2.2].

4.13. Theorem. *Let* $(Y, \sigma, \mathcal{G})$ *be an* $R\mathcal{G}$*-space. Then* $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ *is weakly* Φ*-continuous if and only if* f *is continuous.*

Proof. The sufficiency is clear.

Necessity. Let $x \in X$ and V be an open set of Y containing $f(x)$. Since Y is an $R\mathcal{G}$-space, there exists an open set W of Y such that $f(x) \in W \subseteq \Psi(W) \subseteq V$. Since f is weakly Φ-continuous, there exists an open set U such that $x \in U$ and $f(U) \subseteq \Psi(W)$. Hence we obtain that $f(U) \subseteq \Psi(W) \subseteq V$. Thus, f is continuous. □

We now introduce a complementary form of weak Φ-continuity.

4.14. Definition. Let A be a subset of a grill topological space $(Y, \sigma, \mathcal{G})$. The $\mathcal{G}$*-frontier of* A is defined as $\Phi(A) - \mathrm{Int}(A)$ and is denoted by $fr_{\mathcal{G}}(A)$.

4.15. Definition. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is said to be *weakly** Φ*-continuous* if for each open V in Y, $f^{-1}(fr_G(V))$ is closed in X.

4.16. Definition. [1] A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is said to be *weakly* $\mathcal{I}$-continuous* if for each open V in Y, $f^{-1}(fr^*(V))$ is closed in X, where $fr^*(V)$ is the $*$-frontier of V defined by $V^* - int(V)$.

4.17. Remark. Let (X, τ), (Y, σ) be two topological spaces and $\mathcal{G}$ a grill on Y. If $\mathcal{I} = \mathcal{P}(Y) - \mathcal{G}$, then by Lemma 2.3, a weakly* Φ-continuous function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ coincides with the weakly* $\mathcal{I}$-continuous function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$.

4.18. Theorem. *A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is continuous if and only if it is both weakly Φ-continuous and weakly* Φ-continuous.*

Proof. Necessity. The weak Φ-continuity is clear. By Proposition 3.7, $\Phi(A)$ is closed in $(Y, \sigma, \mathcal{G})$ for every subset A of Y and $fr_{\mathcal{G}}(A)$ is closed in Y. Therefore, f is weakly* Φ-continuous.

Sufficiency. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since f is weakly Φ-continuous, there exists an open set U containing x such that $f(U) \subseteq \Psi(V)$. Now $fr_G(V) = \Phi(V) - \mathrm{Int}(V)$ and thus $f(x) \notin fr_G(V)$. Hence $x \notin f^{-1}(fr_G(V))$ and $U - f^{-1}(fr_G(V))$ is an open set containing x since f is weakly* Φ-continuous. The proof will be complete when we show $f(U - f^{-1}(fr_G(V))) \subseteq V$. To this end let $z \in U - f^{-1}(fr_G(V))$. Then $z \in U$ and hence $f(z) \in \Psi(V)$. But $z \notin f^{-1}(fr_G(V))$ and thus $f(z) \notin fr_G(V) = \Phi(V) - V = \Psi(V) - V$. This implies that $f(z) \in V$. □

The following two examples show that weak Φ-continuity and weak* Φ-continuity are independent of each other.

4.19. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, d\}\}$ and $\sigma = \{\phi, X, \{b\}, \{b, d\}\}$. Let $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ be a grill on X. The identity function $f : (X, \tau) \to (X, \sigma, \mathcal{G})$ is weakly* Φ-continuous but it is not weakly Φ-continuous.

(i) Let $V = X \in \sigma$, then $\Phi(V) = \{a, c\}$ and $fr_{\mathcal{G}}(V) = \Phi(V) - \mathrm{Int}(V) = \phi$ and hence $f^{-1}(fr_{\mathcal{G}}(V))$ is closed.
(ii) Let $V = \{b\} \in \sigma$, then $\Phi(V) = \phi$ and $fr_{\mathcal{G}}(V) = \Phi(V) - \mathrm{Int}(V) = \phi$ and hence $f^{-1}(fr_{\mathcal{G}}(V))$ is closed.
(iii) Let $V = \{b, d\} \in \sigma$, then $\Phi(V) = \phi$ and $fr_{\mathcal{G}}(V) = \Phi(V) - \mathrm{Int}(V) = \phi$ and hence $f^{-1}(fr_{\mathcal{G}}(V))$ is closed.

By (i), (ii) and (iii), f is weakly* Φ-continuous. On the other hand, for $V = \{b\} \in \sigma$, $\Phi(V) = \phi$ and $\Psi(V) = \{b\}$. Then, there exists only one open set $U = X \in \tau$ such that $b \in U$. Since $f(U) = X \nsubseteq \Psi(V)$, f is not weakly Φ-continuous.

4.20. Example. Let $X = \{a, b\}$ and $\tau = \{\phi, X, \{a\}\}$. Let $\mathcal{G} = \{\{a\}, X\}$ be a grill on X. Define a function $f : (X, \tau) \to (X, \tau, \mathcal{G})$ as $f(a) = b$ and $f(b) = a$. Then f is weakly Φ-continuous but it is not weakly* Φ-continuous.

(i) Let $a \in X$ and $V \in \tau$ such that $f(a) = b \in V = X$, then there exists an open set $U = \{a\}$ such that $a \in U$ and $f(U) = \{b\} \subseteq \Psi(V) = X$.
(ii) Let $b \in X$ and $V \in \tau$ such that $f(b) = a \in V$, then $V = \{a\}$ or $V = \{X\}$, then there exists an open set $U = X$ such that $b \in U$ and $f(U) = X \subseteq \Psi(V) = X$.

By (i) and (ii), f is weakly Φ-continuous. On the other hand, for $V = \{a\} \in \tau$, then $\Phi(V) = \{a, b\}$ and $fr_{\mathcal{G}}(V) = \Phi(V) - \mathrm{Int}(V) = \{b\}$, hence $f^{-1}(fr_{\mathcal{G}}(V)) = \{a\}$ which is not closed. Hence f is not weakly* Φ-continuous.

4.21. Proposition. *If $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is precontinuous and $\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(\Psi(V))$ for each open set $V \subseteq Y$, then f is weakly Φ-continuous.*

Proof. For any point $x \in X$ and any open set $V \subseteq Y$ containing $f(x)$, by the hypothesis, we have $\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(\Psi(V))$. Since f is precontinuous, $x \in f^{-1}(V) \subseteq \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ and hence there exists an open set $U \subseteq X$ such that $x \in U \subseteq \mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(\Psi(V))$. Thus $f(U) \subseteq \Psi(V)$. This implies that f is weakly Φ-continuous. □

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *θ-continuous at x_0* [5] if for each open set V of $f(x_0)$, there exists an open set U containing x_0 such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. The function f is said to be *θ-continuous* if it is θ-continuous at each point in X.

4.22. Theorem. *Let $(Y, \sigma, \mathcal{G})$ be an $A\mathcal{G}$-space. For a function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$, the implications (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) hold. If (X, τ) is regular, they are all equivalent.*

(1) *f is θ-continuous;*
(2) *f is weakly Φ-continuous;*
(3) *f is weakly continuous.*

Proof. (1) $\Longrightarrow$ (2) Let f be θ-continuous, $x \in X$ and V any open set of Y containing $f(x)$. Since f is θ-continuous, there exists an open set U containing x such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. Then since $(Y, \sigma, \mathcal{G})$ is an $A\mathcal{G}$-space, by Theorem 4.6 $f(U) \subseteq f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V) = \Psi(V)$. Thus f is weakly Φ-continuous.

(2) $\Longleftrightarrow$ (3) This follows from Theorem 4.10.

(3) $\Longrightarrow$ (1) Suppose that (X, τ) is regular. Let f be weakly continuous, $x \in X$ and V any open set of Y containing $f(x)$. Then, there exists an open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}(V)$. Since (X, τ) is a regular space, there exists an open set H of x such that $x \in H \subseteq \mathrm{Cl}(H) \subseteq U$. Then $f(\mathrm{Cl}(H)) \subseteq \mathrm{Cl}(V)$. Thus f is θ-continuous. □

4.23. Theorem. *Let $(Y, \sigma, \mathcal{G})$ be a grill topological space such that $Y - V \subseteq \Phi(V)$ for every $V \in \sigma$. Then*

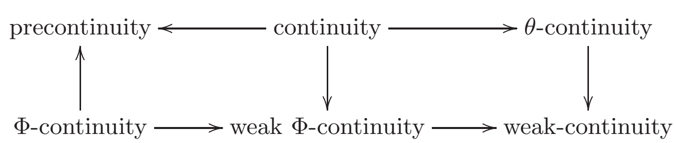
(1) *Every function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is θ-continuous and weakly Φ-continuous.*
(2) *A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is continuous if and only if it is weakly* Φ-continuous.*

Proof. (1) By hypothesis $\Psi(V) = Y$ for every $V \in \sigma$ and every function f is weakly Φ-continuous. Furthermore $\mathrm{Cl}(V) = Y$ for every $V \in \sigma$ since $\Psi(V) \subseteq \mathrm{Cl}(V)$. Thus every function f is θ-continuous.

(2) By (1), every function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is weakly Φ-continuous and by Theorem 4.18, f is continuous. □

5. Examples

It is well known that continuity implies both θ-continuity and precontinuity and also θ-continuity implies weak continuity. Therefore, we have the following diagram.



5.1. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, d\}\}$ and $\sigma = \{\phi, X, \{b\}, \{b, d\}\}$. Let $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ be a grill on X. The identity function $f : (X, \tau) \to (X, \sigma, \mathcal{G})$ is θ-continuous but it is not weakly Φ-continuous (for more details see [1, Example 2.1] and Example 4.19).

5.2. Example. Let $X = \{a, b\}$ and $\tau = \{\phi, X, \{a\}\}$. Let $\mathcal{G} = \{\{a\}, X\}$ be a grill on X. Define a function $f : (X, \tau) \to (X, \tau, \mathcal{G})$ as follows: $f(a) = b$ and $f(b) = a$. Then f is weakly Φ-continuous but it is not continuous (for more details see Example 4.20).

5.3. Example. Let $X = \{a, b\}$ and $\tau = \{\phi, X, \{a\}\}$. Let $\mathcal{G} = \{\{a\}, X\}$ be a grill on X. Define a function $f : (X, \tau) \to (X, \tau, \mathcal{G})$ as follows: $f(a) = b$ and $f(b) = a$. Then f is weakly Φ-continuous but it is not Φ-continuous, since $f^{-1}(\{a\}) = \{b\}$ is not Φ-open (for more details see Example 4.20).

5.4. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, d\}, \{a, b, d\}\}$, $Y = \{a, b\}$ and $\sigma = \{\phi, Y, \{a\}\}$. Let $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ be a grill on X. Define a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ as follows: $f(a) = f(b) = f(d) = a$ and $f(c) = b$. Then f is precontinuous since $f^{-1}(\{a\}) = \{a, b, d\}$ is preopen. But it is not Φ-continuous, since $f^{-1}(\{a\}) = \{a, b, d\}$ is not Φ-open.

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