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SOME RANDOM FIXED POINT THEOREMS FOR (θ, L)-WEAK CONTRACTIONS

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Abstract

In the present paper, stochastic generalizations of some fixed point theorems for operators satisfying a (θ, L)-weak contraction condition and some other contractive conditions have been proved.

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1. Introduction

Fixed point theory has the diverse applications in different branches of mathematics, statistics, engineering, and economics in dealing with the problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations, and others. Developments in the investigation on fixed points of nonexpansive mapings, contractive mappings in different spaces like metric spaces, Banach spaces, Fuzzy metric spaces, Cone metric spaces have almost been saturated. After the study of random fixed point theorems initiated by the Prague school of Probability in the 1950's, considerable attention has been given to the study of random fixed point theorems because of its importance in probabilistic functional analysis and probabilistic models with numerous applications. The introduction of randomness however leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions.

It is well known that random fixed point theorems are stochastic generalization of classical fixed point theorems what are known as deterministic results. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Špaček [34] and Hanš (see [13]-[14]). The survey article by Bharucha-Reid [9] in 1976 attracted the attention of several mathematicians and gave wings to this theory. Itoh [16] extended Špaček 's and Hanš's theorems to multivalued contraction mappings.

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A. Mukherjee [22] gave a random version of Schaduer's fixed point Theorem on an atomic probability measure space while Bharucha-Reid ([9]-[10]), generalized Mukherjee's result on a general probability measure space. Random fixed point theorems with an application to Random differential equations in Banach spaces are obtained by Itoh [16]. Sehgal and Waters [33] had obtained several random fixed point theorems including a random analogue of the classical results due to Rothe [27]. In some recent papers of Saha et al. ([31], [32]), some random fixed point theorems over separable Banach spaces and separable Hilbert spaces have been established.

On the other hand, the first fundamental fixed point theorem in deterministic form was due to S. Banach [4]. In a metric space setting this theorem runs as follows:

1.1. Theorem. (Banach contraction principle) Let (X, d) be a complete metric space, $c \in (0,1)$ and $T : X \to X$ a mapping such that for each $x, y \in X$,

$$d(Tx, Ty) \leq cd(x, y).$$

Then T has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} T^n x = a$. □

After this classical result, Kannan [19] gave a substantially new contractive mapping where the mapping T need not be continuous on X, (but continuous at their fixed points, see [26]). He considered the contractive condition as follows: there exists a constant $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$.

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed points for a class of contractive type conditions that do not require the continuity of T (see for examples [6, 30]). Another fixed point theorem due to Chatterjea [11] is based on a similar condition to Kannan: there exists constant a $c \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$.

Picking up these results obtained by Banach, Kannan and Chatterjea, Zamfirescu [37] successfully generalized these fixed point theorems in 1972. Another generalization was obtained by Ćirić [12] in 1974: there exists $0 < h < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

This mapping is commonly known as a Ćirić quasi contraction. It is obvious that the contractive conditions in each of Banach, Kannan and Chatterjea's results do imply Ćirić quasi contraction. The contractive condition considered in Zamfirescu's [37] fixed point theorem has been further extended to the class of weak contraction by V. Berinde [5] in 2004. In several papers (see for details [5, 7, 8]), weak contraction is commonly termed an *almost contraction*.

1.1. Weak contractions. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a weak contraction or (θ, L)-weak contraction if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X. \tag{1.1}$$

Due to the symmetry of the distance, the weak contraction condition (1.1) implicitly includes the following dual inequality:

$$d(Tx, Ty) \leq \theta d(x, y) + Ld(x, Ty) \text{ for all } x, y \in X. \tag{1.2}$$

which is obtained formally by interchanging x and y in (1.1). Therefore, in order to check the weak contractiveness of a given operator, it is necessary to check both conditions (1.1) and (1.2). It was shown in [5] that any strict contraction, the Kannan [19] and Zamfirescu [37] operators, as well as a large class of quasi-contractions [12], are all weak contractions.

A weak contraction has always at least one fixed point and there exist weak contractions that have infinitely many fixed points (see [5, Example 4]). Note also that the weak contraction condition (1.1) implies the so called Banach orbital condition

$$d(Tx, T^2x) \leq \theta d(x, Tx), \text{ for all } x \in X,$$

studied by various authors in the context of fixed point theorems, see for example Hicks and Rhoades [15], Ivanov [17], Rus ([28, 29, 30]) and Taskovic [35].

Some examples of weak contractions are cited here for ready references.

1.2. Example. (see [20, 24]) Let $[0,1]$ be the unit interval with the usual norm and let $T : [0,1] \to [0,1]$ be given by $Tx = \frac{2}{3}$ for all $x \in [0,1)$ and $T1 = 0$. Then T satisfies the inequality (1.1) with $1 > \theta \geq \frac{2}{3}$ and $L \geq \theta$. Note that T has a unique fixed point $x = \frac{2}{3}$ and also T is not continuous at that point.

1.3. Example. [5] Any quasi contraction, i.e. any operator for which there exists $0 < h < 1$ such that

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, is a weak contraction if $h < \frac{1}{2}$.

All these operators, namely the Chattrejea [11], Kannan [19], Zamfirescu [37], or Quasi-contraction, have a unique fixed point. Berinde also showed in the same paper [5], that a weak contraction may have infinitely many fixed points.

Berinde [5] introduced the notion of a (θ, L)-weak contraction and proved that a lot of well-known contractive conditions do imply (θ, L)-weak contraction. The concept of (θ, L)-weak contraction does not require $\theta + L$ to be less than 1 as happens in many kinds of fixed point theorems for contractive conditions that involve one or more of the displacements $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, $d(y, Tx)$. For more details, we refer to [20, 25] and allied references cited therein.

Our main aim of this paper is to define the random analogue of a (θ, L)-weak contraction and thereby prove the stochastic version of the deterministic fixed point theorem in a separable Banach space. Also some more random fixed point theorems have been established in separable Banach space to investigate this relatively new field of research extensively.

These results are stochastic generalizations of earlier results in the literature (see [4, 19, 11, 37, 5]) of deterministic fixed point theorems.

2. Some basic ideas and definitions

In order to prove our main results, we need to recall the following concepts and results.

Let (X, β_X) be a separable Banach space where β_X is a σ-algebra of Borel subsets of X, and let (Ω, β, μ) denote a complete probability measure space with measure μ, and let β be a σ-algebra of subsets of Ω. For more details one can see Joshi and Bose [18].

2.1. Definition. A mapping $x : \Omega \to X$ is said to be an *X-valued random variable*, if the inverse image under the mapping x of every Borel set B of X belongs to β; that is, $x^{-1}(B) \in \beta$ for all $B \in \beta_X$.

2.2. Definition. A mapping $x : \Omega \to X$ is said to be a *finitely valued random variable*, if it is constant on each of a finite number of disjoint sets $A_i \in \beta$ and is equal to 0 on $\Omega - \left(\bigcup_{i=1}^{n} A_i\right)$. X is called a *simple random variable* if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

2.3. Definition. A mapping $x : \Omega \to X$ is said to be a *strong random variable*, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega - A_0$.

2.4. Definition. A mapping $x : \Omega \to X$ is said to be a *weak random variable*, if the function $x^*(x(\omega))$ is a real valued random variables for each $x^* \in X^*$, the space X^* denoting the first normed dual space of X.

In a separable Banach space X, the notions of strong and weak random variables $x : \Omega \to X$ coincide (see Joshi and Bose [18, Corollary 1]) and in respect of such a space X, x is called as a random variable. We recall the following results.

2.5. Theorem. (Joshi and Bose [18, Theorem 6.1.2 (a)]) *Let x, $y : \Omega \to X$ be a strong random variables and α, β constants. Then the following statement holds:*

(a) $\alpha\, x(\omega) + \beta\, y(\omega)$ *is a strong random variable.*
(b) *If $f(\omega)$ is a real valued random variable and $x(\omega)$ is a strong random variable, then $f(\omega)\, x(\omega)$ is a strong random variable.*
(c) *If $\{x_n(\omega)\}$ is a sequence of strong random variables converging strongly to $x(\omega)$ almost surely, i.e., if there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n\to\infty} \|x_n(\omega) - x(\omega)\| = 0$ for every $\omega \notin A_0$, then $x(\omega)$ is a strong random variable.* □

2.6. Remark. If X is a separable Banach space, then the σ-algebra generated by the class of all spherical neighourhoods of X is equal to the σ-algebra of all Borel sets of X and hence every strong and also weak random variable is measurable in the sense of Definition 2.1.

Let Y be another Banach space. We also need the following definitions from Joshi and Bose [18].

2.7. Definition. A mapping $F : \Omega \times X \to Y$ is said to be a *random mapping* if $F(\omega, x) = Y(\omega)$ is a Y-valued random variable for every $x \in X$.

2.8. Definition. A mapping $F : \Omega \times X \to Y$ is said to be a *continuous random mapping* if the set of all $\omega \in \Omega$ for which $F(\omega, x)$ is a continuous function of x has measure one.

2.9. Definition. A random mapping $F : \Omega \times X \to Y$ is said to be *demi-continuous* at $x \in X$ if

$$\|x_n - x\| \to 0 \text{ implies } F(\omega, x_n) \xrightarrow{\text{weakly}} F(\omega, x) \text{ almost surely.}$$

2.10. Theorem. (Joshi and Bose [18, Theorem 6.2.2.]) *Let $F : \Omega \times X \to Y$ be a demi-continuous random mapping where the Banach space Y is separable. Then for any X-valued random variable x, the function $F(\omega, x(\omega))$ is a Y-valued random variable.* □

2.11. Remark. (see [31]) Since a continuous random mapping is a demi-continuous random mapping, Theorem 2.5 is also true for a continuous random mapping.

The following definitions are also given in the book of Joshi and Bose [18].

2.12. Definition. An equation of the type $F(\omega, x(\omega)) = x(\omega)$, where $F : \Omega \times X \to X$ is a random mapping, is called a *random fixed point equation*.

2.13. Definition. Any mapping $x : \Omega \to X$ which satisfies the random fixed point equation $F(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a *wide sense solution* of the fixed point equation.

2.14. Definition. Any X-valued random variable $x(\omega)$ which satisfies

$$\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$$

is said to be a *random solution* of the fixed point equation, or a *random fixed point* of F.

2.15. Remark. A random solution is a wide sense solution of the fixed point equation. But the converse is not necessarily true. This is evident from the following example as found under Joshi and Bose [18, Remark 1].

2.16. Example. Let X be the set of all real numbers and let E be a non measurable subset of X. Let $F : \Omega \times X \to Y$ be the random mapping defined as $F(\omega, x) = x^2 + x - 1$ for all $\omega \in \Omega$.

In this case, the real valued function $x(\omega)$, defined as $x(\omega) = 1$ for all $\omega \in \Omega$, is a random fixed point of F. However the real valued function $y(\omega)$ defined as

$$y(\omega) = \begin{cases} -1, & \omega \notin E, \\ 1, & \omega \in E, \end{cases}$$

is a wide sense solution of the fixed point equation $F(\omega, x(\omega)) = x(\omega)$, without being a random fixed point of F.

Now we define the random version of a (θ, L)-weak contraction map and then establish a random fixed point theorem for (θ, L)-weak contraction.

3. Random analogue of (θ, L)-weak contraction

3.1. Definition. Let X be a separable Banach space and (Ω, β, μ) a complete probability measure space. Then $T : \Omega \times X \to X$ is called a *random weak contraction* if there exist a real valued random variable $\theta(\omega) \in (0, 1)$ and a finitely valued real random variable $L(\omega) \geq 0$ almost surely, such that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq \theta(\omega) \|x_1 - x_2\| + L(\omega) \|x_2 - T(\omega, x_1)\| \tag{3.1}$$

for all $x_1, x_2 \in X$.

3.2. Theorem. *Let X be a separable Banach space and (Ω, β, μ) a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator satisfying* (3.1) *almost surely, where $0 < \theta(\omega) < 1$ is a real valued random variable and $L(\omega) \geq 0$ is a finitely valued real random variable almost surely. Then there exists a random fixed point of T.*

Proof. Let

$$A = \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\}$$

$$C_{x_1, x_2} = \big\{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \leq \theta(\omega) \|x_1 - x_2\| + L(\omega) \|x_2 - T(\omega, x_1)\| \big\}$$

and

$$B = \{\omega \in \Omega : 0 \leq \theta(\omega) < 1\} \cap \{\omega \in \Omega : L(\omega) \geq 0\}.$$

Let S be a countable dense subset of X. We next prove that

$$\bigcap_{x_1,x_2\in X}(C_{x_1,x_2}\cap A\cap B)=\bigcap_{s_1,s_2\in S}(C_{s_1,s_2}\cap A\cap B).$$

Let $\omega\in\bigcap_{s_1,s_2\in S}(C_{s_1,s_2}\cap A\cap B)$, then for all $s_1,s_2\in S$,

$$\|T(\omega,s_1)-T(\omega,s_2)\|\le\theta(\omega)\|s_1-s_2\|+L(\omega)\|s_2-T(\omega,x_1)\|. \tag{3.2}$$

Let $x_1,x_2\in X$. We have

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\|&\le\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_1-T(\omega,s_2)\|\\&\qquad+\|T(\omega,s_2)-T(\omega,x_2)\|\\&\le\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\&\qquad+\theta(\omega)\|s_1-s_2\|+L(\omega)\|s_2-T(\omega,s_1)\|\end{aligned}\tag{3.3}$$

Now

$$\|s_2-T(\omega,s_1)\|\le\|s_2-x_2\|+\|x_2-T(\omega,x_1)\|+\|T(\omega,x_1)-T(\omega,s_1)\|. \tag{3.4}$$

Again

$$\|s_1-s_2\|\le\|s_1-x_1\|+\|x_1-x_2\|+\|x_2-s_2\|. \tag{3.5}$$

Using (3.2), (3.3), (3.4) and (3.5) we get

$$\begin{aligned}\|T(\omega,x_1)-T(\omega,x_2)\|&\le\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\&\qquad+\theta(\omega)\left[\|s_1-x_1\|+\|x_1-x_2\|+\|x_2-s_2\|\right]\\&\qquad\quad+L(\omega)\left[\|s_2-x_2\|+\|x_2-T(\omega,x_1)\|\right.\\&\qquad\qquad\left.+\|T(\omega,x_1)-T(\omega,s_1)\|\right]\\&=(1+L(\omega))\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\&\qquad+\theta(\omega)\|s_1-x_1\|+\theta(\omega)\|x_1-x_2\|\\&\qquad\quad+(\theta(\omega)+L(\omega))\|x_2-s_2\|+L(\omega)\|x_2-T(\omega,x_1)\|\\&<(1+L(\omega))\|T(\omega,x_1)-T(\omega,s_1)\|+\|T(\omega,s_2)-T(\omega,x_2)\|\\&\qquad+\|s_1-x_1\|+(1+L(\omega))\|x_2-s_2\|\\&\qquad\quad+\left[\theta(\omega)\|x_1-x_2\|+L(\omega)\|x_2-T(\omega,x_1)\|\right]\end{aligned}\tag{3.6}$$

Since for a particular $\omega\in\Omega$, $T(\omega,x)$ is a continuous function of x, so for any $\epsilon>0$, there exists $\delta_i(x_i)>0$; $(i=1,2)$ such that

$$\|T(\omega,x_1)-T(\omega,s_1)\|<\frac{\epsilon}{4(1+L(\omega))},\ \text{whenever}\ \|x_1-s_1\|<\delta_1(x_1)$$

and

$$\|T(\omega,x_2)-T(\omega,s_2)\|<\frac{\epsilon}{4},\ \text{whenever}\ \|x_2-s_2\|<\delta_2'(x_2),$$
$$\text{where}\ \delta_2'(x_2)=\frac{\delta_2(x_2)}{(1+L(\omega))}.$$

Now choose

$$\rho_1=\min\left(\delta_1(x_1),\frac{\epsilon}{4}\right)$$

and

$$\rho_2=\min\left(\delta_2'(x_2),\frac{\epsilon}{4}\right).$$

For such a choice of ρ_1, ρ_2, from (3.6) we have

$$\|T(\omega,x_1)-T(\omega,x_2)\| \leq \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4} + [\theta(\omega)\,\|x_1-x_2\|+L(\omega)\,\|x_2-T(\omega,x_1)\|].$$

Since ϵ is arbitrary, so we have

$$\|T(\omega,x_1)-T(\omega,x_2)\| \leq \theta(\omega)\,\|x_1-x_2\|+L(\omega)\,\|x_2-T(\omega,x_1)\|\,.$$

Thus,

$$\omega \in \bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right),$$

which implies that

$$\bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right)\subset \bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right).$$

Also it is obvious that

$$\bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right)\subset \bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right),$$

and so

$$\bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right)= \bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right).$$

Let

$$N=\bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right).$$

Then $\mu\,(N)=1$ and for each $\omega\in N$, $T\,(\omega,x)$ are deterministic continuous operators satisfying the mapping referred to in [5, Theorem 1] and hence, these have a wide sense solution $x\,(\omega)$. To prove the randomness and measurability of $x\,(\omega)$, we generate an approximating sequence of random variables $x_n\,(\omega)$ as follows. Let $x_0\,(\omega)$ be an arbitrary random variable. Let $x_1\,(\omega)=T\,(\omega,x_0\,(\omega))$. Then it follows that $x_1\,(\omega)$ is a random variable. We then consider

$$x_{n+1}\,(\omega)=T\,(\omega,x_n\,(\omega))\,.$$

By repeated application, it gives that $\{x_n\,(\omega)\}_{n=1,2,\ldots}$ is a sequence of random variables converging to $x\,(\omega)$. Thus, it follows that $x\,(\omega)$ is a random variable and hence $x\,(\omega)$ is measurable. Hence $x\,(\omega)$ is a random fixed point of T. □

3.3. Example. Let X be the set of all real numbers and let E be a non measurable subset of X. Let $T:\Omega\times X\to X$ be the random mapping defined as $T\,(\omega,x)=\frac{1}{5}$ for all $\omega\in\Omega$. Clearly T satisfies (3.1) and also the real valued function $x\,(\omega)$ defined as $x\,(\omega)=\frac{1}{5}$ for all $\omega\in\Omega$, is a unique random fixed point of T.

4. Some other random fixed point theorems

In this section, we prove the stochastic version of deterministic fixed point theorems due to [1] and some other related results.

4.1. Theorem. *Let X be a separable Banach space and (Ω,β,μ) a complete probability measure space. Let $R,T:\Omega\times X\to X$ be continuous random operators satisfying:*

$$\alpha(\omega)\,\|R(\omega,x_1)-T(\omega,x_2)\|+\beta(\omega)\,\|x_1-R(\omega,x_1)\|+\gamma(\omega)\,\|x_2-T(\omega,x_2)\| \leq \delta(\omega)\,\|x_1-x_2\| \tag{4.1}$$

almost surely for all $x_1, x_2 \in X$, where $\beta(\omega)$, $\gamma(\omega)$, $\delta(\omega)$ are nonnegative real random variables and $\alpha(\omega)$ is a nonnegative finitely valued real random variable such that $\beta(\omega), \gamma(\omega) < \delta(\omega)$ and $\delta(\omega) < \alpha(\omega)$ almost surely. Then there exists a unique common random fixed point of R and T.

Proof. Let

$$\begin{aligned} A &= \{\omega \in \Omega : R(\omega, x) \text{ is a continuous function of } x\} \\ &\quad \cap \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\} \\ C_{x_1,x_2} &= \{\omega \in \Omega : \alpha(\omega)\,\|R(\omega,x_1) - T(\omega,x_2)\| + \beta(\omega)\,\|x_1 - R(\omega,x_1)\| \\ &\qquad + \gamma(\omega)\,\|x_2 - T(\omega,x_2)\| \leq \delta(\omega)\,\|x_1 - x_2\|\} \end{aligned}$$

and

$$B = \{\omega \in \Omega : \beta(\omega), \gamma(\omega) < \delta(\omega) \text{ and } \delta(\omega) < \alpha(\omega)\}.$$

Let S be a countable dense subset of X. We next prove that

$$\bigcap_{x_1,x_2 \in X} (C_{x_1,x_2} \cap A \cap B) = \bigcap_{s_1,s_2 \in S} (C_{s_1,s_2} \cap A \cap B).$$

Let

$$\omega \in \bigcap_{s_1,s_2 \in S} (C_{s_1,s_2} \cap A \cap B),$$

then for all $s_1, s_2 \in S$,

$$\begin{aligned} &\alpha(\omega)\,\|R(\omega,s_1) - T(\omega,s_2)\| + \beta(\omega)\,\|s_1 - R(\omega,s_1)\| + \gamma(\omega)\,\|s_2 - T(\omega,s_2)\| \\ &\qquad \leq \delta(\omega)\,\|s_1 - s_2\| \end{aligned} \tag{4.2}$$

Let $x_1, x_2 \in X$, we have

$$\begin{aligned} &\alpha(\omega)\,\|R(\omega,x_1) - T(\omega,x_2)\| + \beta(\omega)\,\|x_1 - R(\omega,x_1)\| + \gamma(\omega)\,\|x_2 - T(\omega,x_2)\| \\ &\quad \leq \alpha(\omega)\,[\|R(\omega,x_1) - R(\omega,s_1)\| + \|R(\omega,s_1) - T(\omega,s_2)\| + \|T(\omega,s_2) - T(\omega,x_2)\|] \\ &\qquad + \beta(\omega)\,[\|x_1 - s_1\| + \|s_1 - R(\omega,s_1)\| + \|R(\omega,s_1) - R(\omega,x_1)\|] \\ &\qquad\quad + \gamma(\omega)\,[\|x_2 - s_2\| + \|s_2 - T(\omega,s_2)\| + \|T(\omega,s_2) - T(\omega,x_2)\|] \\ &\quad = \{\alpha(\omega) + \beta(\omega)\}\,\|R(\omega,x_1) - R(\omega,s_1)\| + \{\alpha(\omega) + \gamma(\omega)\}\,\|T(\omega,x_2) - T(\omega,s_2)\| \\ &\qquad + [\alpha(\omega)\,\|R(\omega,s_1) - T(\omega,s_2)\| + \beta(\omega)\,\|s_1 - R(\omega,s_1)\| + \gamma(\omega)\,\|s_2 - T(\omega,s_2)\|] \\ &\qquad\quad + \beta(\omega)\,\|x_1 - s_1\| + \gamma(\omega)\,\|x_2 - s_2\| \\ &\quad \leq \{\alpha(\omega) + \beta(\omega)\}\,\|R(\omega,x_1) - R(\omega,s_1)\| + \{\alpha(\omega) + \gamma(\omega)\}\,\|T(\omega,x_2) - T(\omega,s_2)\| \\ &\qquad + \delta(\omega)\,\|s_1 - s_2\| + \beta(\omega)\,\|x_1 - s_1\| + \gamma(\omega)\,\|x_2 - s_2\| \end{aligned} \tag{4.3}$$

Again

$$\|s_1 - s_2\| \leq \|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\| \tag{4.4}$$

Using (4.3) and (4.4) we get

$$\begin{aligned} &\alpha(\omega)\,\|R(\omega,x_1) - T(\omega,x_2)\| + \beta(\omega)\,\|x_1 - R(\omega,x_1)\| + \gamma(\omega)\,\|x_2 - T(\omega,x_2)\| \\ &\quad \leq \{\alpha(\omega) + \beta(\omega)\}\,\|R(\omega,x_1) - R(\omega,s_1)\| + \{\alpha(\omega) + \gamma(\omega)\}\,\|T(\omega,x_2) - T(\omega,s_2)\| \\ &\qquad\quad + \delta(\omega)\,\|x_1 - x_2\| + \{\beta(\omega) + \delta(\omega)\}\,\|x_1 - s_1\| + \{\gamma(\omega) + \delta(\omega)\}\,\|x_2 - s_2\| \\ &\quad \leq \{\alpha(\omega) + \delta(\omega)\}\,\|R(\omega,x_1) - R(\omega,s_1)\| + \{\alpha(\omega) + \delta(\omega)\}\,\|T(\omega,x_2) - T(\omega,s_2)\| \\ &\qquad\quad + \delta(\omega)\,\|x_1 - x_2\| + 2\delta(\omega)\,\|x_1 - s_1\| + 2\delta(\omega)\,\|x_2 - s_2\| \\ &\quad \leq 2\alpha(\omega)\,\|R(\omega,x_1) - R(\omega,s_1)\| + 2\alpha(\omega)\,\|T(\omega,x_2) - T(\omega,s_2)\| \\ &\qquad\quad + \delta(\omega)\,\|x_1 - x_2\| + 2\alpha(\omega)\,\|x_1 - s_1\| + 2\alpha(\omega)\,\|x_2 - s_2\|. \end{aligned} \tag{4.5}$$

Since for a particular $\omega \in \Omega$, $R(\omega, x), T(\omega, x)$ are continuous functions of x, so for any $\epsilon > 0$, there exists $\theta_i(x_i) > 0$; $(i = 1, 2)$ such that

$$\|R(\omega, x_1) - R(\omega, s_1)\| < \frac{\epsilon}{8\alpha(\omega)}, \text{ whenever } \|x_1 - s_1\| < \frac{\theta_1(x_1)}{2\alpha(\omega)} = \theta_1'(x_1)$$

and

$$\|T(\omega, x_2) - T(\omega, s_2)\| < \frac{\epsilon}{8\alpha(\omega)}, \text{ whenever } \|x_2 - s_2\| < \frac{\theta_2(x_2)}{2\alpha(\omega)} = \theta_2'(x_2).$$

Now choose

$$\rho_1 = \min\left(\theta_1'(x_1), \frac{\epsilon}{4}\right)$$

and

$$\rho_2 = \min\left(\theta_2'(x_2), \frac{\epsilon}{4}\right)$$

For such a choice of ρ_1, ρ_2, from (4.5) we have

$$\begin{aligned}\alpha(\omega)\|R(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\|x_1 - R(\omega, x_1)\| + \gamma(\omega)\|x_2 - T(\omega, x_2)\| \\ \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \delta(\omega)\|x_1 - x_2\|.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have

$$\begin{aligned}\alpha(\omega)\|R(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\|x_1 - R(\omega, x_1)\| + \gamma(\omega)\|x_2 - T(\omega, x_2)\| \\ \leq \delta(\omega)\|x_1 - x_2\|\end{aligned} \tag{4.6}$$

Thus,

$$\omega \in \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B),$$

which implies that

$$\bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B) \subset \bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B).$$

Also it is obvious that

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) \subset \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)$$

and so

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) = \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)$$

Let

$$N = \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B).$$

Then $\mu(N) = 1$ and for each $\omega \in N$, $T(\omega, x)$, is a deterministic continuous operator satisfying [1, Theorem 2.1], and hence it has a unique wide sense common solution $x(\omega)$. The uniqueness of $x(\omega)$ follows from the deterministic case. To prove the randomness and measurability of $x(\omega)$, we generate an approximating sequence of random variables $x_n(\omega)$ as follows. Let $x_0(\omega)$ be an arbitrary random variable. Let $x_1(\omega) = R(\omega, x_0(\omega))$, $x_2(\omega) = T(\omega, x_1(\omega))$. Then it follows that $x_1(\omega)$ and $x_2(\omega)$ are random variables. We then consider

$$\begin{aligned}x_{2n+1}(\omega) &= R(\omega, x_{2n}(\omega)), \\ x_{2n+2}(\omega) &= T(\omega, x_{2n+1}(\omega)).\end{aligned}$$

By repeated application, this gives that $\{x_n(\omega)\}_{n=1,2,\ldots}$ is a sequence of random variables converging to $x(\omega)$. Thus, it follows that $x(\omega)$ is a random variable and hence $x(\omega)$ is measurable. Hence $x(\omega)$ is the unique common random fixed point of R and T. □

4.2. Corollary. *Let X be a separable Banach space and (Ω, β, μ) a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator satisfying:*

$$\alpha(\omega)\|T(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\|x_1 - T(\omega, x_1)\| + \gamma(\omega)\|x_2 - T(\omega, x_2)\| \\ \leq \delta(\omega)\|x_1 - x_2\|$$

almost surely for all $x_1, x_2 \in X$, where $\beta(\omega)$, $\gamma(\omega)$, $\delta(\omega)$ are nonnegative real random variables and $\alpha(\omega)$ is a nonnegative finitely valued real random variable such that $\beta(\omega), \gamma(\omega) < \delta(\omega)$ and $\delta(\omega) < \alpha(\omega)$ almost surely. Then there exists a unique random fixed point of T.

Proof. Let

$$A = \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\},$$
$$C_{x_1,x_2} = \{\omega \in \Omega : \alpha(\omega)\|T(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\|x_1 - T(\omega, x_1)\| \\ +\gamma(\omega)\|x_2 - T(\omega, x_2)\| \leq \delta(\omega)\|x_1 - x_2\|\},$$

and

$$B = \{\omega \in \Omega : \beta(\omega), \gamma(\omega) < \delta(\omega) \text{ and } \delta(\omega) < \alpha(\omega)\}.$$

Then in a similar fashion to that in Theorem 4.1, we can prove that T has a unique wide sense solution $x(\omega)$. The uniqueness of $x(\omega)$ is also clear. To prove the randomness and measurability of $x(\omega)$, we generate an approximating sequence of random variables $x_n(\omega)$ as follows. Let $x_0(\omega)$ be an arbitrary random variable. Let $x_1(\omega) = T(\omega, x_0(\omega))$. Then it follows that $x_1(\omega)$ is a random variable. We then consider

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)) \text{ for } n = 2, 3, \ldots.$$

By repeated application, this gives that $\{x_n(\omega)\}_{n=1,2,\ldots}$ is a sequence of random variables converging to $x(\omega)$. Thus, it follows that $x(\omega)$ is a random variable and hence $x(\omega)$ is measurable. Hence $x(\omega)$ is a unique random fixed point of T. □

4.3. Note. It is to be noted that a (θ, L)-weak contraction satisfying (1.1) is not the same as the following contractive type mapping in a Banach or metric space. Example 4.5 clearly supports our contention.

Before going into our next stochastic result, we first prove a deterministic fixed point theorem in a Banach space.

4.4. Theorem. *Let R, T be two continuous self mapping of a Banach space $(X, \|\cdot\|)$ satisfying the following condition:*

$$\alpha\|Rx - Ty\| + \beta\|x - Ty\| + \gamma\|y - Rx\| \leq \delta\|x - y\| \tag{4.7}$$

for all $x, y \in X$, $\alpha, \beta, \gamma, \delta \geq 0$, $\gamma, \delta < \alpha$ and $\alpha < \beta$. Then R and T have a unique common fixed point $z \in X$.

Proof. Let $x_0 \in X$ and define $\{x_n\}$ by $x_{2n+1} = Rx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$. Assume $x_n \neq x_{n+1}$ for each n. Then from (4.7) we have by letting $x = x_{2n}$ and $y = x_{2n+1}$,

$$\alpha\|Rx_{2n} - Tx_{2n+1}\| + \beta\|x_{2n} - Tx_{2n+1}\| + \gamma\|x_{2n+1} - Rx_{2n}\| \leq \delta\|x_{2n} - x_{2n+1}\|,$$

which implies that

$$\alpha \|x_{2n+1} - x_{2n+2}\| + \beta \|x_{2n} - x_{2n+2}\| \leq \delta \|x_{2n} - x_{2n+1}\|, \text{ or}$$
$$\alpha \|x_{2n+1} - x_{2n+2}\| + \beta \|x_{2n} - x_{2n+2}\| \leq \delta \|x_{2n} - x_{2n+2}\| + \delta \|x_{2n+1} - x_{2n+2}\|, \text{ or}$$
$$(\alpha - \delta) \|x_{2n+1} - x_{2n+2}\| \leq (\delta - \beta) \|x_{2n} - x_{2n+2}\|, \text{ or}$$
$$(\alpha - \delta) \|x_{2n+1} - x_{2n+2}\| \leq (\delta - \beta) \|x_{2n} - x_{2n+1}\| + (\delta - \beta) \|x_{2n+1} - x_{2n+2}\|,$$

i.e.,

$$\|x_{2n+1} - x_{2n+2}\| \leq k \|x_{2n} - x_{2n+1}\|, \text{ where } k = \frac{\delta - \beta}{\alpha + \beta - 2\delta} < 1,$$
$$\text{since } \gamma, \delta < \alpha \text{ and } \alpha < \beta.$$

Similarly one can obtain

$$\|x_{2n} - x_{2n+1}\| \leq k \|x_{2n-1} - x_{2n}\|,$$

so that

$$\|x_{2n+1} - x_{2n+2}\| \leq k^2 \|x_{2n-1} - x_{2n}\| \leq \cdots \leq k^{2n} \|x_1 - x_2\|$$

and

$$\|x_{2n} - x_{2n+1}\| \leq k^{2n} \|x_0 - x_1\|.$$

Now let $r(x_0) = \max\{\|x_0 - x_1\|, \|x_1 - x_2\|\}$. Then for any $m > n$,

$$\begin{aligned}\|x_m - x_n\| &\leq \sum_{i=0}^{m-n-1} \|x_{n+i} - x_{n+i+1}\| \\ &\leq \sum_{i=0}^{m-n-1} k^{2(n+i)} r(x_0) \\ &\leq \frac{k^{2n}}{1-k^2} r(x_0) \to 0, \text{ as } n \to \infty.\end{aligned}$$

So $\{x_n\}$ is a Cauchy sequence and hence convergent. Call the limit z. Also by the continuity of R and T, we get $Rz = z = Tz$.

Next suppose that $(v \neq z)$ is another common fixed point of R and T. Then from (4.7) we have

$$\alpha \|Rz - Tv\| + \beta \|z - Tv\| + \gamma \|v - Rz\| \leq \delta \|z - v\|$$
$$\text{implies } (\alpha + \beta + \gamma - \delta) \|z - v\| \leq 0,$$

which shows that $z = v$ and so the fixed point is unique. □

4.5. Example. Let $[0,1]$ be the unit interval with its usual norm and let $T : [0,1] \to [0,1]$ be given by

$$Tx = \tfrac{1}{2} \text{ for } x \in [0,1) = 0 \text{ for } x = 1.$$

Then the following results hold:

1) T is a (θ, L)-weak contraction,
2) T has a unique fixed point ($x = \frac{1}{2}$), but
3) T does not satisfy the contractive condition of type (4.7) as we can check by taking $x = \frac{1}{4}$ and $y = \frac{1}{8}$.

Now we give the random analogue of the above theorem.

4.6. Theorem. *Let X be a separable Banach space and (Ω, β, μ) a complete probability measure space. Let $R, T : \Omega \times X \to X$ be continuous random operators satisfying:*

$$\begin{aligned} \alpha(\omega)\,\|R(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\,\|x_1 - T(\omega, x_2)\| + \gamma(\omega)\,\|x_2 - R(\omega, x_1)\| \\ \leq \delta(\omega)\,\|x_1 - x_2\| \end{aligned}$$

almost surely for all $x_1, x_2 \in X$, where $\alpha(\omega)$, $\gamma(\omega)$, $\delta(\omega)$ are nonnegative real random variables and $\beta(\omega)$ is a nonnegative finitely valued real random variable such that $\gamma(\omega), \delta(\omega) < \alpha(\omega)$ and $\alpha(\omega) < \beta(\omega)$ almost surely. Then there exists a unique common random fixed point of R and T.

Proof. Let

$$\begin{aligned} A &= \{\omega \in \Omega : R(\omega, x) \text{ is a continuous function of } x\} \\ &\qquad \cap \{\omega \in \Omega : T(\omega, x) \text{ is a continuous function of } x\} \\ C_{x_1, x_2} &= \{\omega \in \Omega : \alpha(\omega)\,\|R(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\,\|x_1 - T(\omega, x_2)\| \\ &\qquad + \gamma(\omega)\,\|x_2 - R(\omega, x_1)\| \leq \delta(\omega)\,\|x_1 - x_2\|\} \end{aligned}$$

and

$$B = \{\omega \in \Omega : \gamma(\omega),\ \delta(\omega) < \alpha(\omega) \text{ and } \alpha(\omega) < \beta(\omega)\}\,.$$

Let S be a countable dense subset of X. We next prove that

$$\bigcap_{x_1, x_2 \in X} (C_{x_1, x_2} \cap A \cap B) = \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)\,.$$

Let

$$\omega \in \bigcap_{s_1, s_2 \in S} (C_{s_1, s_2} \cap A \cap B)\,,$$

then for all $s_1, s_2 \in S$,

$$\begin{aligned} \alpha(\omega)\,\|R(\omega, s_1) - T(\omega, s_2)\| + \beta(\omega)\,\|s_1 - T(\omega, s_2)\| + \gamma(\omega)\,\|s_2 - R(\omega, s_1)\| \\ \leq \delta(\omega)\,\|s_1 - s_2\|\,. \end{aligned} \tag{4.8}$$

Let $x_1, x_2 \in X$, we have

$$\begin{aligned} &\alpha(\omega)\,\|R(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\,\|x_1 - T(\omega, x_2)\| + \gamma(\omega)\,\|x_2 - R(\omega, x_1)\| \\ &\leq \alpha(\omega)\,[\|R(\omega, x_1) - R(\omega, s_1)\| + \|R(\omega, s_1) - T(\omega, s_2)\| + \|T(\omega, s_2) - T(\omega, x_2)\|] \\ &\quad + \beta(\omega)\,[\|x_1 - s_1\| + \|s_1 - T(\omega, s_2)\| + \|T(\omega, s_2) - T(\omega, x_2)\|] \\ &\quad + \gamma(\omega)\,[\|x_2 - s_2\| + \|s_2 - R(\omega, s_1)\| + \|R(\omega, s_1) - R(\omega, x_1)\|] \\ &= \{\alpha(\omega) + \gamma(\omega)\}\,\|R(\omega, x_1) - R(\omega, s_1)\| + \{\alpha(\omega) + \beta(\omega)\}\,\|T(\omega, x_2) - T(\omega, s_2)\| \\ &\quad + [\alpha(\omega)\,\|R(\omega, s_1) - T(\omega, s_2)\| + \beta(\omega)\,\|s_1 - T(\omega, s_2)\| + \gamma(\omega)\,\|s_2 - R(\omega, s_1)\|] \\ &\quad + \beta(\omega)\,\|x_1 - s_1\| + \gamma(\omega)\,\|x_2 - s_2\| \\ &\leq 2\beta(\omega)\,\|R(\omega, x_1) - R(\omega, s_1)\| + 2\beta(\omega)\,\|T(\omega, x_2) - T(\omega, s_2)\| \\ &\quad + \delta(\omega)\,\|s_1 - s_2\| + \beta(\omega)\,\|x_1 - s_1\| + \gamma(\omega)\,\|x_2 - s_2\| \end{aligned} \tag{4.9}$$

Again

$$\|s_1 - s_2\| \leq \|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\|\,. \tag{4.10}$$

Using (4.8), (4.9) and (4.10) we get

$$\begin{aligned}
&\alpha(\omega)\left\|R(\omega,x_1)-T(\omega,x_2)\right\|+\beta(\omega)\left\|x_1-T(\omega,x_2)\right\|+\gamma(\omega)\left\|x_2-R(\omega,x_1)\right\| \\
&\quad\le 2\beta(\omega)\left\|R(\omega,x_1)-R(\omega,s_1)\right\|+2\beta(\omega)\left\|T(\omega,x_2)-T(\omega,s_2)\right\| \\
&\qquad+\delta(\omega)\left\|x_1-x_2\right\|+\{\beta(\omega)+\delta(\omega)\}\left\|x_1-s_1\right\|+\{\gamma(\omega)+\delta(\omega)\}\left\|x_2-s_2\right\| \qquad (4.11)\\
&\quad\le 2\beta(\omega)\left\|R(\omega,x_1)-R(\omega,s_1)\right\|+2\beta(\omega)\left\|T(\omega,x_2)-T(\omega,s_2)\right\| \\
&\qquad+\delta(\omega)\left\|x_1-x_2\right\|+2\beta(\omega)\left\|x_1-s_1\right\|+2\beta(\omega)\left\|x_2-s_2\right\|
\end{aligned}$$

Since for a particular $\omega\in\Omega$, $R(\omega,x),T(\omega,x)$ are continuous functions of x, so for any $\epsilon>0$ there exists $\theta_i(x_i)>0$; $(i=1,2)$ such that

$$\left\|R(\omega,x_1)-R(\omega,s_1)\right\|<\frac{\epsilon}{8\beta(\omega)},\ \text{whenever}\ \left\|x_1-s_1\right\|<\frac{\theta_1(x_1)}{2\beta(\omega)}=\theta_1'(x_1)$$

and

$$\begin{aligned}
&\left\|T(\omega,x_2)-T(\omega,s_2)\right\| \\
&<\frac{\epsilon}{8\beta(\omega)},\ \text{whenever}\ \left\|x_2-s_2\right\|<\frac{\theta_2(x_2)}{2\beta(\omega)}=\theta_2'(x_2).
\end{aligned}$$

Now choose

$$\rho_1=\min\left(\theta_1'\left(x_1\right),\frac{\epsilon}{4}\right)$$

and

$$\rho_2=\min\left(\theta_2'\left(x_2\right),\frac{\epsilon}{4}\right).$$

For such a choice of ρ_1,ρ_2, from (4.11) we have

$$\begin{aligned}
&\alpha(\omega)\left\|R(\omega,x_1)-T(\omega,x_2)\right\|+\beta(\omega)\left\|x_1-R(\omega,x_1)\right\|+\gamma(\omega)\left\|x_2-T(\omega,x_2)\right\| \\
&\qquad\le\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\delta(\omega)\left\|x_1-x_2\right\|.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, so we have

$$\begin{aligned}
&\alpha(\omega)\left\|R(\omega,x_1)-T(\omega,x_2)\right\|+\beta(\omega)\left\|x_1-R(\omega,x_1)\right\|+\gamma(\omega)\left\|x_2-T(\omega,x_2)\right\| \\
&\qquad\le\delta(\omega)\left\|x_1-x_2\right\|.
\end{aligned}$$

Thus,

$$\omega\in\bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right)$$

which implies that

$$\bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right)\subset\bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right)$$

Also it is obvious that

$$\bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right)\subset\bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right)$$

and so

$$\bigcap_{x_1,x_2\in X}\left(C_{x_1,x_2}\cap A\cap B\right)=\bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right).$$

Let

$$N=\bigcap_{s_1,s_2\in S}\left(C_{s_1,s_2}\cap A\cap B\right).$$

Then $\mu(N) = 1$ and for each $\omega \in N$, $T(\omega, x)$, is a deterministic continuous operator satisfying the above Theorem 4.4 and hence, it has a unique wide sense common solution $x(\omega)$. The uniqueness of $x(\omega)$ follows from the deterministic case. To prove the randomness and measurability of $x(\omega)$, we generate an approximating sequence of random variables $x_n(\omega)$ as follows. Let $x_0(\omega)$ be an arbitrary random variable. Let $x_1(\omega) = R(\omega, x_0(\omega))$, $x_2(\omega) = T(\omega, x_1(\omega))$. Then it follows that $x_1(\omega)$ and $x_2(\omega)$ are random variables. We then consider

$$x_{2n+1}(\omega) = R(\omega, x_{2n}(\omega))$$
$$x_{2n+2}(\omega) = T(\omega, x_{2n+1}(\omega)).$$

By repeated application, this gives that $\{x_n(\omega)\}_{n=1,2,\ldots}$ is a sequence of random variables converging to $x(\omega)$. Thus, it follows that $x(\omega)$ is a random variable and hence $x(\omega)$ is measurable. Hence $x(\omega)$ is the unique common random fixed point of R and T. □

4.7. Corollary. *Let X be a separable Banach space and (Ω, β, μ) a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator satisfying:*

$$\alpha(\omega)\|T(\omega, x_1) - T(\omega, x_2)\| + \beta(\omega)\|x_1 - T(\omega, x_2)\| + \gamma(\omega)\|x_2 - T(\omega, x_1)\| \leq \delta(\omega)\|x_1 - x_2\|$$

almost surely for all $x_1, x_2 \in X$, where $\alpha(\omega)$, $\gamma(\omega)$, $\delta(\omega)$ are nonnegative real random variables and $\beta(\omega)$ is a nonnegative finitely valued real random variable such that $\gamma(\omega), \delta(\omega) < \alpha(\omega)$ and $\alpha(\omega) < \beta(\omega)$ almost surely. Then there exists a unique random fixed point of T. □

5. Application to a random nonlinear integral equation

In this section, we apply Theorem 3.2 to prove the existence of a solution in a Banach space of a random nonlinear integral equation of the form:

$$x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega) f(s, x(s; \omega))\, d\mu_0(s), \tag{5.1}$$

where

(i) S is a locally compact metric space with metric d on $S \times S$, μ_0 is a complete σ-finite measure defined on the collection of Borel subsets of S;
(ii) $\omega \in \Omega$, where ω is a supporting set of the probability measure space (Ω, β, μ);
(iii) $x(t; \omega)$ is an unknown vector-valued random variable for each $t \in S$.
(iv) $h(t; \omega)$ is the stochastic free term defined for $t \in S$;
(v) $k(t, s; \omega)$ is the stochastic kernel defined for t and s in S, and
(vi) $f(t, x)$ is vector-valued function of $t \in S$ and x,

and the integral in equation (5.1) is a Bochner integral.

We will further assume that S is the union of a countable family of compact sets $\{C_n\}$ having the properties that $C_1 \subset C_2 \subset \cdots$ and that for any other compact set S there is a C_i which contains it (see [3]).

5.1. Definition. We define the space $C(S, L_2(\Omega, \beta, \mu))$ to be the space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$ with the topology of uniform convergence on compacta i.e. for each fixed $t \in S$, $x(t; \omega)$ is a vector valued random variable such that

$$\|x(t; \omega)\|^2_{L_2(\Omega,\beta,\mu)} = \int_\Omega |x(t; \omega)|^2\, d\mu(\omega) < \infty.$$

It may be noted that $C(S, L_2(\Omega, \beta, \mu))$ is locally convex space (see [36]) whose topology is defined by a countable family of seminorms given by

$$\|x(t;\omega)\|_n = \sup_{t \in C_n} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}, \; n = 1, 2, \ldots$$

Moreover $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to this topology since $L_2(\Omega, \beta, \mu)$ is complete.

We further define $BC = BC(S, L_2(\Omega, \beta, \mu))$ to be the Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ with norm

$$\|x(t;\omega)\|_{BC} = \sup_{t \in S} \|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)}.$$

The space $BC \subset C$ is the space of all second order vector-valued stochastic process defined on S which are bounded and continuous in mean square.

We will consider the function $h(t;\omega)$ and $f(t, x(t;\omega))$ to be in the space $C(S, L_2(\Omega, \beta, \mu))$ with respect to the stochastic kernel. We assume that for each pair (t, s), $k(t, s; \omega) \in L_\infty(\Omega, \beta, \mu)$, and denote the norm by

$$\|k(t, s; \omega)\| = \|k(t, s; \omega)\|_{L_\infty(\Omega,\beta,\mu)} = \mu - ess \sup_{\omega \in \Omega} |k(t, s; \omega)|.$$

Also we will suppose that $k(t, s; \omega)$ is such that $|||k(t, s; \omega)||| \cdot \|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable with respect to s for each $t \in S$ and $x(s;\omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ and there exists a real valued function G defined μ_0-a.e. on S, so that $G(S) \|x(s;\omega)\|_{L_2(\Omega,\beta,\mu)}$ is μ_0-integrable and for each pair $(t, s) \in S \times S$,

$$|||k(t, u; \omega) - k(s, u; \omega)|||\,.\,\|x(u, \omega)\|_{L_2(\Omega,\beta,\mu)} \leq G(u) \|x(u, \omega)\|_{L_2(\Omega,\beta,\mu)}$$

μ_0-a.e. Further, for almost all $s \in S$, $k(t, s; \omega)$ will be continuous in t from S into $L_\infty(\Omega, \beta, \mu)$.

We now define the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(5.2) \qquad (Tx)(t;\omega) = \int_S k(t, s; \omega) x(s;\omega)\, d\mu_0(s),$$

where the integral is a Bochner integral. Moreover, we have that for each $t \in S$, $(Tx)(t;\omega) \in L_2(\Omega, \beta, \mu)$ and that $(Tx)(t;\omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem. So $(Tx)(t;\omega) \in C(S, L_2(\Omega, \beta, \mu))$.

5.2. Definition. (see [2, 21]) Let B and D be Banach spaces. The pair (B, D) is said to be *admissible with respect to a random operator* $T(\omega)$ if $T(\omega)(B) \subset D$.

5.3. Lemma. (see [23]) *The linear operator T defined by* (5.2) *is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.* □

5.4. Lemma. (see [23, 21]) *If T is a continuous linear operator from $C(S, L_2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ such that (B, D) is admissible with respect to T, then T is continuous from B into D.* □

5.5. Remark. (see [23]) The operator T defined by (5.3) is a bounded linear operator from B into D.

It is to be noted that by a random solution of the equation (5.1) we will mean a function $x(t;\omega)$ in $C(S, L_2(\Omega, \beta, \mu))$ which satisfies the equation (5.1) μ-a.e.

We are now in a state to prove the following theorem.

5.6. Theorem. *We consider the stochastic integral equation* (5.1) *subject to the following conditions:*

(a) *B and D are Banach spaces stronger than* $C(\beta, L_2(\Omega,\beta,\mu))$ *such that* (B, D) *is admissible with respect to the integral operator defined by* (5.2)*;*

(b) $x(t;\omega) \to f(t, x(t;\omega))$ *is an operator from the set*

$$Q(\rho) = \left\{x(t;\omega) : x(t;\omega) \in D, \|x(t;\omega)\|_D \le \rho\right\}$$

into the space B satisfying

$$\begin{aligned}\|f(t, x_1(t;\omega)) - f(t, x_2(t;\omega))\|_B \le \theta(\omega)\, &\|x_1(t;\omega) - x_2(t;\omega)\|_D \\ &+ L(\omega)\, \|x_2(t;\omega) - f(t, x_1(t;\omega))\|_D\end{aligned} \tag{5.3}$$

for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$*, where* $0 < \theta(\omega) < 1$ *is a real valued random variable and* $L(\omega) \ge 0$ *is a finitely valued real random variable almost surely,*

(c) $h(t;\omega) \in D$.

Then there exists a unique random solution of (5.1) *in* $Q(\rho)$*, provided* $\frac{c(\omega)}{1-L(\omega)} < 1$ *and*

$$\|h(t;\omega)\|_D + \frac{c(\omega)}{1-L(\omega)} \|f(t;0)\|_B \le \rho\left(1 - \frac{c(\omega)\theta(\omega)}{1-L(\omega)}\right),$$

where $c(\omega)$ *is the norm of* $T(\omega)$*.*

Proof. Define the operator $U(\omega)$ from $Q(\rho)$ into D by

$$(Ux)(t;\omega) = h(t;\omega) + \int_S k(t,s;\omega) f(s, x(s;\omega)) d\mu_0(s).$$

Now

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &\le \|h(t;\omega)\|_D + c(\omega)\, \|f(t, x(t;\omega))\|_B \\ &\le \|h(t;\omega)\|_D + c(\omega)\, \|f(t;0)\|_B + c(\omega)\, \|f(t, x(t;\omega)) - f(t;0)\|_B\,.\end{aligned}$$

Then from condition (5.3) of this theorem

$$\begin{aligned}\|f(t, x(t;\omega)) - f(t;0)\|_B &\le [\theta(\omega)\, \|x(t;\omega)\|_D + L(\omega)\, \|f(t, x(t;\omega))\|_D] \\ &\le \theta(\omega)\rho + L(\omega)\, \|f(t, x(t;\omega)) - f(t;0)\|_D + L(\omega)\, \|f(t;0)\|_D\end{aligned}$$

implies

$$\|f(t, x(t;\omega)) - f(t;0)\|_B \le \frac{\theta(\omega)}{1-L(\omega)}\rho + \frac{L(\omega)}{1-L(\omega)} \|f(t;0)\|_B\,. \tag{5.4}$$

Therefore, by (5.4),

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &\le \|h(t;\omega)\|_D + c(\omega)\, \|f(t;0)\|_B + \frac{c(\omega)\theta(\omega)}{1-L(\omega)}\rho + \frac{c(\omega)L(\omega)}{1-L(\omega)} \|f(t;0)\|_B \\ &= \|h(t;\omega)\|_D + \frac{c(\omega)\theta(\omega)}{1-L(\omega)}\rho + \frac{c(\omega)}{1-L(\omega)} \|f(t;0)\|_B < \rho.\end{aligned}$$

Hence $(Ux)(t;\omega) \in Q(\rho)$. Then for $x_1(t;\omega), x_2(t;\omega) \in Q(\rho)$, we have by condition (b)

$$\begin{aligned}&\|(Ux_1)(t;\omega) - (Ux_2)(t;\omega)\|_D \\ &\quad = \left\|\int_S k(t,s;\omega)[f(s, x_1(s;\omega)) - f(s, x_2(s;\omega))] d\mu_0(s)\right\|_D \\ &\quad \le c(\omega)\, \|f(t, x_1(t;\omega)) - f(t, x_2(t;\omega))\|_B \\ &\quad \le \theta(\omega)\, \|x_1(t;\omega) - x_2(t;\omega)\|_D + L(\omega)\, \|x_2(t;\omega) - (Ux_1)(t;\omega)\|_D\,,\end{aligned}$$

since $\frac{c(\omega)}{1-L(\omega)} < 1$. Thus $U(\omega)$ is a random nonlinear (θ, L)-weak contraction operator on $Q(\rho)$. Hence, by Theorem 3.2 there exists a random fixed point of $U(\omega)$, which is the random solution of the Equation (5.1). □

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