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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY κ-STRICT PSEUDO-CONTRACTIONS IN A BANACH SPACE

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Abstract

In this paper, weak convergence theorems of a finite family of asymptotically k-strict pseudo-contractions are established in the framework of 2-uniformly smooth and uniformly convex Banach spaces.

Keywords: Asymptotically k-strict pseudo-contraction, Fixed point, Non-expansive mapping, Strictly pseudo-contractive mapping, Uniformly smooth Banach space.

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1. Introduction and Preliminaries

Let E be an arbitrary real Banach space and J_q $(q > 1)$ denotes the generalized duality mapping from E into 2^{E^*} give by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^q,\ \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping which is usually denoted by J. In this paper, we use j to denote the single-valued normalized duality mapping. It is well known (see, for example, [14]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$. If E is a Hilbert space, then $J = I$, where I denotes the identity mapping.

Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta, \quad \forall x, y \in U_E.$$

A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

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exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$. The norm of E is said to be Fréchet differentiable if, for all $x \in U_E$, the above limit is attained uniformly for all $y \in U_E$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1,\ \|y\| \leq \tau\right\},\ \forall\, \tau \geq 0.$$

The Banach space E is uniformly smooth if and only if $\lim_{\tau\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$. Let $q > 1$. The Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. It is shown in [14] that there is no Banach space which is q-uniformly smooth with $q > 2$. Hilbert spaces, L^p (or l^p) spaces and Sobolev space W_m^p, where $p \geq 2$ are 2-uniformly smooth. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

It is known that if E is 2-uniformly smooth with the best smooth constant K, then the following inequality holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2,\ \forall\, x, y \in E. \tag{1.1}$$

Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. Recall that the mapping T is said to be a κ-strict pseudo-contraction if there exist a constant $\kappa \in (0, \frac{1}{2}]$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \kappa\|(I-T)x - (I-T)y\|^2,\ \forall\, x, y \in C. \tag{1.2}$$

The class of strict pseudo-contractions was first introduced by Browder and Petryshyn in Hilbert spaces, see [2] for more details. If I denotes the identity mapping, then (1.2) can be rewritten as follows

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \qquad \geq \kappa\|(I-T)x - (I-T)y)\|^2,\ \forall\, x, y \in C. \tag{1.3}$$

If E is a Hilbert space, then (1.2) is reduced to

$$\langle Tx - Ty, x-y\rangle \leq \|x-y\|^2 - \kappa\|(I-T)x - (I-T)y\|^2,\ \forall\, x, y \in C,$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + (1-2\kappa)\|(I-T)x - (I-T)y\|^2,\ \forall x, y \in C. \tag{1.4}$$

If $\kappa = \frac{1}{2}$, then (1.4) is reduced to

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x, y \in C.$$

That is, T is non-expansive. Recall that the mapping T is said to be an asymptotically κ-strict pseudo-contraction if there exist a constant $\kappa \in (0, \frac{1}{2}]$, a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that

$$\begin{aligned}\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2 - \kappa\|(I-T^n)x - (I-T^n)y\|^2,\\ \forall\, x, y \in C,\ \forall\, n \geq 1.\end{aligned} \tag{1.5}$$

The class of asymptotically κ-strict pseudo-contraction was first introduced by Qihou in Hilbert spaces, see [11] for more details. If I denotes the identity mapping, then (1.5) can be rewritten as follows

$$\begin{aligned}&\langle (I-T^n)x - (I-T^n)y, j(x-y)\rangle\\ &\qquad \geq \kappa\|(I-T^n)x - (I-T^n)y)\|^2 - (k_n - 1)\|x-y\|^2,\ \forall\, x, y \in C,\ \forall\, n \geq 1.\end{aligned} \tag{1.6}$$

If E is a Hilbert space, then (1.5) is reduced to

$$\langle T^n x - T^n y, x - y\rangle \le k_n\|x-y\|^2 - \kappa\|(I-T^n)x - (I-T^n)y\|^2, \quad \forall\, x, y \in C,\ \forall\, n \ge 1, \tag{1.7}$$

which is equivalent to

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x-y\|^2 + (1-2\kappa)\|(I-T^n)x - (I-T^n)y\|^2, \quad \forall\, x, y \in C,\ \forall\, n \ge 1. \tag{1.8}$$

If $\kappa = \frac{1}{2}$, then (1.8) is reduced to

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x-y\|^2,\ \forall\, x, y \in C,\ \forall\, n \ge 1. \tag{1.9}$$

That is, T is an asymptotically non-expansive mapping which was introduced by Goebel and Kirk [3] as a generalization of the class of non-expansive mappings. It is clear that if $k_n = 1$ for each $n \ge 1$, then the class of asymptotically non-expansive mappings is reduced to the class of non-expansive mappings. If C is a nonempty bounded closed convex subset of a uniformly convex Banach space, then every asymptotically non-expansive mapping has a unique fixed point in C.

The normal Mann iterative process [5] generates a sequence $\{x_n\}$ in the following manner

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n,\ \forall\, n \ge 1, \tag{1.10}$$

where x_1 ia an initial value and $\{\alpha_n\}$ is a sequence in the interval $(0,1)$.

In 1979, Reich [12] obtained the following celebrated weak convergence theorem.

1.1. Theorem. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differential norm, $T : C \to C$ a non-expansive mapping with a fixed point, and $\{\alpha_n\}$ a real sequence such that $0 \le \alpha_n \le 1$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Let $\{x_n\}$ be a sequence generated in* (1.10)*. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Marino and Xu [6] extended the results of Reich [12] from the class of non-expansive mappings to the class of strict pseudo-contractions and obtained a weak convergence theorem based on the normal Mann iterative process in Hilbert spaces. More precisely, they proved the following results.

1.2. Theorem. *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strictly pseudo-contractive mapping defined in* (1.4) *and assume that T admits a fixed point in C. Let $\{x_n\}$ be the sequence generated in the normal Mann iterative process algorithm* (1.10)*. Assume that the control sequence $\{\alpha_n\}$ is chosen so that $1-2\kappa < \alpha_n < 1$ for all $n \ge 1$ and $\sum_{n=1}^{\infty}(\alpha_n - 1 + 2\kappa)(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Recently, Acedo and Xu [1], still in the framework of Hilbert spaces, introduced the following cyclic iterative algorithm.

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=0}^{N-1}$ be κ_i-strict pseudo-contractions on C such that $\bigcap_{i=0}^{N-1} F(T_i) \ne \emptyset$. Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 = \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\ x_2 = \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ \cdots \\ x_N = \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_0 x_N, \\ \cdots \end{cases}$$

In a compact form, x_{n+1} can be rewritten as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]} x_n, \tag{1.11}$$

where $T_{[n]} = T_i$ with $i = n \pmod N$ for $0 \le i \le N-1$.

They also established a weak convergence theorem based on the cyclic iterative algorithm (1.11) in the framework of Hilbert spaces. To be more precise, they proved the following results.

1.3. Theorem. *Let C be a closed convex subset of a Hilbert space H. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $T_i : C \to C$ be a κ_i-strict pseudo-contraction as defined in* (1.4). *Let $\kappa = \min\{\kappa_i : 1 \le i \le N\}$. Assume that the common fixed point set of $\{T_i\}_{i=0}^{N-1}$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated in the cyclic algorithm* (1.11). *Assume that the control sequence $\{\alpha_n\}$ is chosen so that $1 - 2\kappa + \epsilon \le \alpha_n \le 1 - \epsilon$ for all $n \ge 0$ and some $\epsilon \in (0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$.*

In this paper, we introduce the following iterative process for a family of asymptotically strict pseudo-contractions. Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $(0,1)$. The sequence $\{x_n\}$ generated in the following manner:

$$\begin{cases} x_1 = \alpha_0 x_0 + (1-\alpha_0)T_1 x_0, \\ x_2 = \alpha_1 x_1 + (1-\alpha_1)T_2 x_1, \\ \cdots \\ x_N = \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_N x_{N-1}, \\ x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_1^2 x_N, \\ \cdots \\ x_{2N} = \alpha_{2N-1} x_{2N-1} + (1-\alpha_{2N-1})T_N^2 x_{2N-1} \\ x_{2N+1} = \alpha_{2N} x_{2N} + (1-\alpha_{2N})T_1^3 x_{2N} \\ \cdots \end{cases}$$

is called the explicit iterative sequence of a finite family of asymptotically strict pseudo-contractions $\{T_1, T_2, \ldots, T_N\}$.

Since, for each $n \ge 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \cdots, N\}$, $h = h(n) \ge 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Hence we can rewrite the above table in the following compact form:

$$x_n = \alpha_{n-1} x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}, \ \forall\, n \ge 1. \tag{1.12}$$

In this paper, motivated by the research announced in [1,7-10], we consider the weak convergence of the iteration process (1.12) for a finite family of asymptotically strict pseudo-contraction in a 2-uniformly smooth and uniformly convex Banach space. The results presented in this paper improve and extend the corresponding results in Acedo and Xu [1] and Marino and Xu [6].

In order to prove our main results, we need the following lemmas.

1.4. Lemma (Krüppel [4]). *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space E and $T : C \to E$ a non-expansive mapping. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to some point x. Then there exists an increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ depending on the diameter of C such that $h(\|x - Tx\|) \le \liminf_{n\to\infty} \|x_n - Tx_n\|$.*

1.5. Lemma. *Let C be a nonempty subset of a Banach space E and $T : C \to C$ an asymptotically κ-strict pseudo-contraction. Then T is uniformly L-Lipschitz.*

Proof. Note that (1.5) is equivalent to

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x - y\|^2 + (1 - 2\kappa)\|x - y - (T^n x - T^n y)\|^2, \quad \forall\, x, y \in C,\ \forall\, n \ge 1. \tag{1.13}$$

It follows that

$$\begin{aligned}\|T^n x - T^n y\|^2 &\le (2k_n - 1)\|x - y\|^2 + (1 - 2\kappa)\big(\|x - y\|^2 \\ &\qquad - 2\langle T^n x - T^n y, j(x - y)\rangle + \|T^n x - T^n y\|^2\big) \\ &\le 2(k_n - \kappa)\|x - y\|^2 + 2(1 - 2\kappa)\|T^n x - T^n y\|\|x - y\| \\ &\qquad + (1 - 2\kappa)\|T^n x - T^n y\|^2,\ \forall\, x, y \in C,\ \forall\, n \ge 1.\end{aligned}$$

This implies that

$$2\kappa\|T^n x - T^n y\|^2 - 2(1 - 2\kappa)\|x - y\|\|T^n x - T^n y\| - 2(k_n - \kappa)\|x - y\|^2 \le 0, \quad \forall x, y \in C, \forall n \ge 1. \tag{1.14}$$

Solving the quadratic inequality, we see that

$$\|T^n x - T^n y\| \le \frac{1 - 2\kappa + \sqrt{1 + 4\kappa(k_n - 1)}}{2\kappa}\|x - y\|, \quad \forall\, x, y \in C,\ \forall\, n \ge 1. \tag{1.15}$$

Putting $L = \sup_{n\ge}\{\frac{1-2\kappa+\sqrt{1+4\kappa(k_n-1)}}{2\kappa}\}$, we arrive at

$$\|T^n x - T^n y\| \le L\|x - y\|,\ \forall\, x, y \in C,\ \forall\, n \ge 1.$$

This shows that T is uniformly Lipschitz continuous. □

1.6. Lemma (Tan and Xu [13]). *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \le (1 + s_n)r_n + t_n,\ \forall\, n \ge 0.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists. □

2. Main Results

Now we are ready to give our main results in this paper.

2.1. Theorem. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K which also satisfies Opial's condition and C a nonempty closed convex subset of E. Let $N \ge 1$ be an integer and, for each $1 \le i \le N$, $T_i : C \to C$ an asymptotically κ_i-strict pseudo-contraction defined in (1.5) with the sequence $k_{n,i}$ in $[1, \infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i} - 1) < \infty$. Let $\kappa = \min\{\kappa_i : 1 \le i \le N\}$ and $k_n = \max\{k_{n,i} : 1 \le i \le N\}$. Assume that the set $\bigcap_{i=1}^{N} F(T_i)$ of common fixed points of $\{T_i\}_{i=1}^{N}$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated in the cyclic iterative algorithm (1.12). Assume that the control sequence $\{\alpha_n\}$ is chosen such that*

$1-\frac{\kappa}{K^2} < a \le \alpha_n \le b < 1$ for all $n \ge 1$ and for some $a, b \in (0,1)$. Then $\{x_n\}$ converges weakly to some common fixed point of $\{T_i\}_{i=1}^N$.

Proof. Let $p \in \bigcap_{i=1}^N F(T_i)$. It follows from (1.1) and (1.6) that

$$\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}x_{n-1} - p\|^2 \\
&= \|x_{n-1} - p - (1-\alpha_{n-1})(x_{n-1} - T_{i(n)}^{h(n)}x_{n-1})\|^2 \\
&\le \|x_{n-1} - p\|^2 - 2(1-\alpha_{n-1})\langle x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}, j(x_{n-1}-p)\rangle \\
&\quad + 2K^2(1-\alpha_{n-1})^2\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\
&\le \|x_{n-1} - p\|^2 - 2(1-\alpha_{n-1})\big(\kappa\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\
&\quad - (k_{h(n)} - 1)\|x_{n-1} - p\|^2\big) + 2K^2(1-\alpha_{n-1})^2\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\
&= \big(1 + 2(1-\alpha_{n-1})(k_{h(n)} - 1)\big)\|x_{n-1} - p\|^2 \\
&\quad - 2(1-\alpha_{n-1})\big(\kappa - K^2(1-\alpha_{n-1})\big)\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\
&\le \big(1 + 2(1-\alpha_{n-1})(k_{h(n)} - 1)\big)\|x_{n-1} - p\|^2.
\end{aligned} \tag{2.1}$$

Since $\sum_{n=1}^\infty (k_{n,i} - 1) < \infty$ and $k_n = \max\{k_{n,i} : 1 \le i \le N\}$, we have $\sum_{n=1}^\infty (k_{h(n)} - 1) < \infty$. It follows from Lemma 1.6 that $\lim_{n\to\infty}\|x_n - p\|^2$ exists. It follows from (2.1) that

$$\begin{aligned}
\|x_n - p\|^2 &\le \|x_{n-1} - p\|^2 + 2(1-\alpha_{n-1})(k_{h(n)} - 1)M_1 \\
&\quad - 2(1-\alpha_{n-1})\big(\kappa - K^2(1-\alpha_{n-1})\big)\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2,
\end{aligned} \tag{2.2}$$

where M_1 is an constant such that $M_1 \ge \sup_{n\ge 0}\{\|x_n - p\|^2\}$. It follows from (2.2) and the assumptions that

$$\begin{aligned}
&2(1-b)\big(\kappa - K^2(1-a)\big)\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\
&\qquad \le \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + 2(1-\alpha_{n-1})(k_{h(n)} - 1)M_1.
\end{aligned} \tag{2.3}$$

Since $\lim_{n\to\infty}\|x_n - p\|^2$ exists, we obtain that

$$\lim_{n\to\infty}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\| = 0. \tag{2.4}$$

Notice that

$$\|x_n - x_{n-1}\| = (1-\alpha_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|.$$

It follows that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{2.5}$$

On the other hand, we have

$$\|x_{n-1} - T_{i(n)}^{h(n)}x_n\| \le \|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\| + \|T_{i(n)}^{h(n)}x_{n-1} - T_{i(n)}^{h(n)}x_n\|.$$

Note that T_l is uniformly Lipschitz for each $l \in \{1, 2, \ldots, N\}$. Combining (2.4) with (2.5) yields that

$$\lim_{n\to\infty}\|x_{n-1} - T_{i(n)}^{h(n)}x_n\| = 0. \tag{2.6}$$

In view of (2.5), we see that

$$\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0,\ \forall\, j \in \{1, 2, \cdots, N\}. \tag{2.7}$$

Any positive integer $n > N$ can be written as $n = (k(n)-1)N + i(n)$, where $i(n) \in \{1, 2, \cdots, N\}$. Observe that

$$\begin{aligned} &\|x_{n-1} - T_n x_{n-1}\| \\ &\quad \le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + \|T_{i(n)}^{h(n)} x_{n-1} - T_n x_{n-1}\| \\ &\quad = \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + \|T_{i(n)}^{h(n)} x_{n-1} - T_{i(n)} x_{n-1}\| \\ &\quad \le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + L\|T_{i(n)}^{h(n)-1} x_{n-1} - x_{n-1}\| \\ &\quad \le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| + L\big(\|T_{i(n)}^{h(n)-1} x_{n-1} - T_{i(n-N)}^{h(n)-1} x_{n-N}\| \\ &\qquad + \|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\|\big). \end{aligned} \tag{2.8}$$

Since, for each $n > N$, $n = (n-N) \pmod N$, we notice that $n = (k(n)-1)N + i(n)$. Therefore, we have

$$n - N = (k(n)-1)N + i(n) - N = (k(n-N)-1)N + i(n-N),$$

that is,

$$h(n-N) = h(n) - 1, \quad i(n-N) = i(n).$$

Observe that

$$\begin{aligned} \|T_{i(n)}^{h(n)-1} x_{n-1} - T_{i(n-N)}^{h(n)-1} x_{n-N}\| &= \|T_{i(n)}^{h(n)-1} x_{n-1} - T_{i(n)}^{h(n)-1} x_{n-N}\| \\ &\le L\|x_{n-1} - x_{n-N}\| \end{aligned} \tag{2.9}$$

and

$$\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\| = \|T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\|. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), we arrive at

$$\begin{aligned} \|x_{n-1} - T_n x_{n-1}\| \le \|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| &+ L\big(L\|x_n - x_{n-N}\| \\ &+ \|T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\|\big). \end{aligned} \tag{2.11}$$

It follows from (2.4), (2.6) and (2.7) that

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_{n-1}\| = 0. \tag{2.12}$$

Note that

$$\begin{aligned} \|x_n - T_n x_n\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_{n-1}\| + \|T_n x_{n-1} - T_n x_n\| \\ &\le (1+L)\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_{n-1}\|. \end{aligned} \tag{2.13}$$

From (2.5) and (2.12), we obtain that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{2.14}$$

On the other hand, we have

$$\begin{aligned} \|x_n - T_{n+j} x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\ &\le (1+L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\|, \\ &\forall\, j \in \{1, 2, \ldots, N\}. \end{aligned} \tag{2.15}$$

In view of (2.7) and (2.14), we see that

$$\lim_{n\to\infty} \|x_n - T_{n+j} x_n\| = 0, \ \forall\, j \in \{1, 2, \ldots, N\},$$

which gives that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \ \forall\, l \in \{1, 2, \ldots, N\}. \tag{2.16}$$

Since E is a q-uniformly smooth Banach space and $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $p_1 \in C$.

Next, we show that $p_1 \in \bigcap_{l=1}^N F(T_l)$. It follows from (1.1) and (1.6) that

$$\begin{aligned}
&\|(aI + (1-a)T_l^n)x - (aI + (1-a)T_l^n)y\|^2 \\
&\quad = \|x - y - (1-a)\big((I - T_l^n)x - (I - T_l^n)y\big)\|^2 \\
&\quad \le \|x-y\|^2 - 2(1-a)\langle (I - T_l^n)x - (I - T_l^n)y), j(x-y)\rangle \\
&\qquad + 2K^2(1-a)^2\|(I - T_l^n)x - (I - T_l^n)y)\|^2 \\
&\quad \le \|x-y\|^2 - 2(1-a)\big(\kappa\|(I - T_l^n)x - (I - T_l^n)y\|^2 \\
&\qquad - (k_n - 1)\|x-y\|^2\big) + 2K^2(1-a)^2\|(I - T_l^n)x - (I - T_l^n)y)\|^2 \\
&\quad = \big(1 + 2(1-a)(k_n - 1)\big)\|x-y\|^2 \\
&\qquad - 2(1-a)\big(\kappa - K^2(1-a)\big)\|(I - T_l^n)x - (I - T_l^n)y)\|^2.
\end{aligned}$$

for all $x, y \in C$ and for all $l \in \{1, 2, \ldots, N\}$. In view of $1 - \frac{k}{K^2} < a$, we see that

$$\|(aI + (1-a)T_l^n)x - (\alpha_n I + (1-a)T_l^n)y\|^2 \le \big(1 + 2(1-a)(k_n - 1)\big)\|x-y\|^2,$$

that is,

$$\|(aI + (1-a)T_l^n)x - (aI + (1-a)T_l^n)y\| \le \gamma_n\|x-y\|,$$

where $\gamma_n = [1 + 2(1-a)(k_n - 1)]$, for all $x, y \in C$ and for all $l \in \{1, 2, \ldots, N\}$. It follows that the mapping $\frac{1}{\gamma_n}(aI + (1-a)T_l^n)$ is non-expansive for all $n \ge 1$ and $l \in \{1, 2, \ldots, N\}$. Since $\{x_n\}$ is bounded, we know that there exists $R > 0$ such that $\|x_n - p\| \le R$ for all $n \ge 1$. Let

$$B_R = \{x \in E : \|x - p\| \le R\}, \quad K = C \bigcap B_R.$$

Then K is nonempty closed convex and bounded and $\{x_n\}$ is a sequence in K. In view of Lemma 1.4, we see that there exists an increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ depending on the diameter of K such that

$$h(\|p_1 - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)p_1\|) \le \liminf_{n\to\infty} \|x_n - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)x_n\| \tag{2.17}$$

for each $m \ge 1$ and $l \in \{1, 2, \ldots, N\}$. On the other hand, we have

$$\begin{aligned}
&\|x_n - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)x_n\| \\
&\quad \le \|x_n - (aI + (1-a)T_l^m)x_n\| + (1 - \frac{1}{\gamma_m})\|(aI + (1-a)T_l^m)x_n - p + p\| \\
&\quad \le \|x_n - (aI + (1-a)T_l^m)x_n\| + (1 - \frac{1}{\gamma_m})\big(\gamma_m\|x_n - p\| + \|p\|\big) \\
&\quad \le \|x_n - (aI + (1-a)T_l^m)x_n\| + (1 - \frac{1}{\gamma_m})\big(\gamma_m R + \|p\|\big)
\end{aligned} \tag{2.18}$$

for each $m \ge 1$ and $l \in \{1, 2, \ldots, N\}$. Note that

$$\begin{aligned}
\|x_n - (aI + (1-a)T_l^m)x_n\| &\le \|x_n - T_l^m x_n\| \\
&\le \sum_{j=1}^m \|T^{j-1}x_n - T_l^j x_n\| \\
&\le Lm\|x_n - T_l x_n\|
\end{aligned}$$

for each $m \geq 1$ and $l \in \{1, 2, \ldots, N\}$. It follows from (2.16) that

$$\lim_{n\to\infty} \|x_n - (aI + (1-a)T_l^m)x_n\| = 0 \tag{2.19}$$

for each $m \geq 1$ and $l \in \{1, 2, \ldots, N\}$. It follows from (2.18) and (2.19) that

$$\limsup_{n\to\infty} \|x_n - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)x_n\| \leq (1 - \frac{1}{\gamma_m})(\gamma_m R + \|p\|)$$

for each $m \geq 1$ and $l \in \{1, 2, \ldots, N\}$. In view of (2.17), we see that

$$h(\|p_1 - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)p_1\|) \leq (1 - \frac{1}{\gamma_m})(\gamma_m R + \|p\|)$$

for each $m \geq 1$ and $l \in \{1, 2, \ldots, N\}$. Notice that $\lim_{m\to\infty} \gamma_m = 1$. It follows that

$$\lim_{m\to\infty} \|p_1 - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)p_1\| = 0 \tag{2.20}$$

for each $l \in \{1, 2, \ldots, N\}$. On the other hand, we have

$$\begin{aligned}
&\|p_1 - (aI + (1-a)T_l^m)p_1\| \\
&\quad\leq \|p_1 - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)p_1\| + (1 - \frac{1}{\gamma_m})\|(aI + (1-a)T_l^m)p_1\| \\
&\quad\leq \|p_1 - \frac{1}{\gamma_m}(aI + (1-a)T_l^m)p_1\| + (1 - \frac{1}{\gamma_m})M_2,
\end{aligned}$$

where M_2 is an appropriate constant, for each $m \geq 1$ and $l \in \{1, 2, \ldots, N\}$. It follows from (2.20) and $\lim_{m\to\infty} \gamma_m = 1$ that

$$\lim_{m\to\infty} \|p_1 - T_l^m p_1\| = 0$$

for each $m \geq 1$ and $l \in \{1, 2, \ldots, N\}$. In view of Lemma 1.5, we have $p_1 = T_l p_1$ for each $l \in \{1, 2, \ldots, N\}$. This shows that $p_1 \in \bigcap_{l=1}^N F(T_l)$.

Next, we show $\{x_n\}$ converges weakly to p_1. Suppose the contrary. If $\{x_n\}$ has another subsequence $\{n_j\}$ which converges weakly to p_2 such that $p_2 \neq p_1$, then we also have $p_2 \in \bigcap_{l=1}^N F(T_l)$. Note that E satisfies Opial's condition. It follows that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - p_1\| < \lim_{n_i\to\infty} \|x_{n_i} - p_2\| \\
&= \lim_{n\to\infty} \|x_n - p_2\| = \lim_{n_j\to\infty} \|x_{n_j} - p_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - p_1\| = \lim_{n\to\infty} \|x_n - p_1\|.
\end{aligned}$$

This is a contradiction. This implies that $p_1 = p_2$. This shows that the sequence $\{x_n\}$ converges weakly to $p_1 \in \bigcap_{l=1}^N F(T_l)$. This completes the proof. □

In Hilbert spaces, we know that $K = \frac{\sqrt{2}}{2}$. The following results are not hard to derive from Theorem 2.1.

2.2. Corollary. *Let H be a Hilbert space and C a nonempty closed convex subset of H. Let $N \geq 1$ be an integer and, for each $1 \leq i \leq N$, $T_i : C \to C$ an asymptotically κ_i-strict pseudo-contraction as defined in* (1.7) *with the sequence $k_{n,i}$ in $[1, \infty)$ such that $\sum_{n=1}^\infty (k_{n,i} - 1) < \infty$. Let $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$ and $k_n = \max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that the set $\bigcap_{i=1}^N F(T_i)$ of common fixed points of $\{T_i\}_{i=1}^N$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated in the cyclic iterative algorithm* (1.12)*. Assume that the control sequence $\{\alpha_n\}$ is chosen such that $1 - 2\kappa < a \leq \alpha_n \leq b < 1$ for all $n \geq 1$ and for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to some common fixed point of $\{T_i\}_{i=1}^N$.* □

As corollaries of Theorem 2.1, we also have the following.

2.3. Corollary. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K which also satisfies Opial's condition and C a nonempty closed convex subset of E. Let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction defined in* (1.5) *with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*

$$x_0 \in C, \quad x_n = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T^n x_{n-1}, \ \forall\, n \geq 1.$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $1 - \frac{\kappa}{K^2} < a \leq \alpha_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to some fixed point of T. □

2.4. Corollary. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K which also satisfies Opial's condition and C a nonempty closed convex subset of E. Let $N \geq 1$ be an integer and, for each $1 \leq i \leq N$, $T_i : C \to C$ a κ_i-strict pseudo-contraction as defined in* (1.2). *Let $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$ and assume that the set $\bigcap_{i=1}^{N} F(T_i)$ of common fixed points of $\{T_i\}_{i=1}^{N}$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated in the following manner:*

$$x_0 \in C, \quad x_n = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_n x_{n-1}, \ \forall\, n \geq 1,$$

where $T_n = T_{n \bmod N}$. Assume that the control sequence $\{\alpha_n\}$ is chosen such that $1 - \frac{\kappa}{K^2}) < a \leq \alpha_n \leq b < 1$ for all $n \geq 1$ and for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to some fixed point of T. □

2.5. Remark. Corollary 2.4 is a version of Theorem 1.3 in the framework of Banach spaces.

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