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A NOTE ON CERTAIN CENTRAL DIFFERENTIAL IDENTITIES WITH GENERALIZED DERIVATIONS

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Abstract

Let R be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R. Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C, $m \geq 1$ a fixed integer, a a fixed element of R, G a non-zero generalized derivation of R. If $aG(f(r_1, \ldots, r_n))^m \in Z(R)$ for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(1) $aI = aG(I) = (0)$;
(2) $G(x) = qx$, for some $q \in U$ and $aqI = 0$;
(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;
(4) $G(x) = cx + [q, x]$ for all $x \in R$, where $c, q \in U$ such that $cI = 0$ and $[q, I]I = 0$;
(5) $\dim_C(RC) \leq 4$;
(6) $G(x) = \alpha x$, for some $\alpha \in C$; moreover $a \in C$ and $f(x_1, \ldots, x_n)^m$ is central valued on R.

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1. Introduction and Preliminaries

Throughout this paper unless specially stated, R always denotes a prime ring with center $Z(R)$, U its right Utumi quotient ring and C its extended centroid (which is the center of U). The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [1]. In any case, when R is a prime ring, all that we need about U is that:

(1) $R \subseteq U$;
(2) U is a prime ring with identity;
(3) The center of U, denoted by C, is a field which is called the extended centroid of R.

By a *derivation* of R we mean that an additive map d from R into itself satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. For $b \in U$, we use $\mathrm{ad}(b)$ to denote the *inner derivation* induced by b; that is , $\mathrm{ad}(b)(x) = [b, x]$ for $x \in R$. An additive mapping $g : R \to R$ is called a *generalized derivation* of R if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$ [8]. Obviously any derivation is a generalized derivation. Moreover, other basic examples of generalized derivations are the following: (i) $g(x) = ax + xb$, for $a, b \in R$; (ii) $g(x) = ax$, for some $a \in R$. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [8, 11, 16]).

In [2] M. Bresar proved that if R is a semiprime ring, d a nonzero derivation of R and $a \in R$ such that $\mathrm{ad}(x)^m = 0$, for all $x \in R$, where m is a fixed integer, then $\mathrm{ad}(R) = 0$ when R is $(m-1)!$-torsion free. In [15] T.K. Lee and J.S. Lin proved Bresar's result without the $(m-1)!$-torsion free assumption on R. They studied the Lie ideal case and, for the prime case, they showed that if R is a prime ring with a derivation $d \neq 0$, L a Lie ideal of R, $a \in R$ such that $\mathrm{ad}(u)^m = 0$, for all $u \in L$, where m is fixed, then $\mathrm{ad}(L) = 0$ unless the case when $\mathrm{char}(R) = 2$ and $\dim_C RC = 4$. In addition, if $[L, L] \neq 0$, then $\mathrm{ad}(R) = 0$.

In [4] C.M. Chang and T.K. Lee established a unified version of the previous results for prime rings. More precisely they proved the following theorem: let R be a prime ring, ϱ a nonzero right ideal of R, d a nonzero derivation of R, $a \in R$ such that $\mathrm{ad}([x, y])^m \in Z(R)$ $(d([x, y])^m a \in Z(R))$. If $[\varrho, \varrho]\varrho \neq 0$ and $\dim_C RC > 4$, then either $\mathrm{ad}(\varrho) = 0$ $(a = 0$ resp.) or d is the inner derivation induced by some $q \in U$ such that $q\varrho = 0$.

Recently in the first part of [3], C. M. Chang generalized above results by proving that if R is a prime ring with extended centroid C, I a non-zero right ideal of R, d a non-zero derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, $a \in R$ and $m \geq 1$ a fixed integer such that $\mathrm{ad}(f(r_1 \ldots, r_n))^m = 0$ for all $r_1 \ldots, r_n \in I$, then either $aI = d(I)I = (0)$ or $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I.

In [7] the second author obtained some results under the assumption that I is a nonzero right ideal of a noncommutative prime ring R, G is a generalized derivation of R, m is a fixed positive integer, $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over C such that $aG(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in I$. In this case one of the following holds:

(1) $aI = aG(I) = (0)$;
(2) $G(x) = qx$, for some $q \in U$ and $aqI = 0$;
(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;
(4) $G(x) = cx + [q, x]$ for all $x \in R$, where $c, q \in U$ such that $cI = 0$ and $[q, I]I = 0$.

Motivated by the above results we will prove:

1.1. Theorem. *Let R be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R.*

Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C, $m \geq 1$ a fixed integer, a a fixed element of R, G a non-zero generalized derivation of R. If $aG(f(r_1, \ldots, r_n))^m \in Z(R)$ for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(1) $aI = aG(I) = (0)$;
(2) $G(x) = qx$, *for some* $q \in U$ *and* $aqI = 0$;
(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ *is an identity for* I;
(4) $G(x) = cx + [q, x]$ *for all* $x \in R$, *where* $c, q \in U$ *such that* $cI = 0$ *and* $[q, I]I = 0$;
(5) $\dim_C(RC) \leq 4$;
(6) $G(x) = \alpha x$, *for some* $\alpha \in C$; *moreover* $a \in C$ *and* $f(x_1, \ldots, x_n)^m$ *is central valued on* R.

In order to prove our Theorem we will use frequently the theory of generalized polynomial identities and differential identities (see [1, 9, 13, 17]). In particular we need to recall the following:

1.2. Remark. In [11], T. K. Lee proved that every generalized derivation G of R can be uniquely extended to a generalized derivation of U. In particular, there exists $a \in U$ and a derivation d of U such that $G(x) = ax + d(x)$ for all $x \in U$ [11, Theorem 3].

1.3. Remark. We need to recall the following notation:

$$f(x_1, \ldots, x_n) = x_1 x_2 \cdot \ldots \cdot x_n + \sum_{\sigma \in S_n, \sigma \neq 1} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma \cdot 1)$. Thus, for d a usual derivation, we write $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$, for all $r_1, \ldots, r_n \in R$.

Finally we also recall the following:

1.4. Definition. By a *differential polynomial* $f(d_j(x_i))$ over U we mean a generalized polynomial with coefficients in U and with variables acted on by derivation words, that is, $f(z_{ij})$ is a generalized polynomial in variables z_{ij} and with coefficients in U, and each d_j is either a derivation word or the identity map of R.

In particular in this note we consider the differential polynomial

$$f(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)),$$

that is, we will consider the case when a derivation d and the identity map act on the variables.

We say that the differential polynomial $f(d_j(x_i))$ is a *central differential identity* (central DI) for a right ideal ϱ of R if $f(z_{ij})$ has no constant term and $f(d_j(r_i)) \in C$ for all $r_1, \ldots, r_n \in \varrho$, but there exist $s_1, \ldots, s_n \in \varrho$ such that $f(d_j(s_i)) \neq 0$ (for more details we refer the reader to [4]).

Proof. Firstly we prove Theorem 1.1. We consider $G(x) = cx + d(x)$, for some $c \in U$ and a derivation d on U. If $aG(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in I$, the result follows from [7]. Hence we suppose there exist $s_1, \ldots, s_n \in I$ such that $aG(f(s_1, \ldots, s_n))^m \neq 0$. Therefore $aG(f(x_1, \ldots, x_n))^m \in Z(R)$ is a central DI for I, then by [4, Theorem 1], R is a PI-ring. Thus by Posner's Theorem (see for example [18, Theorem 1.7.9]), RC is a finite-dimensional central simple algebra over C and $RC \cong M_k(F)$, the ring of $k \times k$ matrices over F, for some integer k and some finite-dimensional central division algebra F over C. We note that in this case a is invertible, therefore $aG(f(x_1, \ldots, x_n))^m \in Z(R)$ if and only if $G(f(x_1, \ldots, x_n))^m a \in Z(R)$. By [13, Theorem 2], $G(f(r_1, \ldots, r_n))^m a \in C$

for all $r_1, \ldots, r_n \in IC$. In order to prove our result we may replace R with RC and I with IC, so that we assume without loss of generality that $R \cong M_k(F)$. Since I satisfies

$$\Big(cf(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_{i=1}^{n} f(x_1, \ldots, d(x_i), \ldots, x_n)\Big)^m a \in C,$$

then for all $y \in R$, I also satisfies

$$\Big(cf(x_1y, \ldots, x_n) + f^d(x_1y, \ldots, x_n) + f(d(x_1)y + x_1d(y), x_2 \ldots, x_n) + \sum_{i=2}^{n} f(x_1y, \ldots, d(x_i), \ldots, x_n)\Big)^m a \in C.$$

In the light of Kharchenko's theory [9], we divide the proof into two cases:

If the derivation d is not inner, I satisfies

$$\Big(cf(x_1y, \ldots, x_n) + f^d(x_1y, \ldots, x_n) + f(d(x_1)y + x_1z, x_2 \ldots, x_n) + \sum_{i=2}^{n} f(x_1y, \ldots, d(x_i), \ldots, x_n)\Big)^m a \in C,$$

where the variable z falls in R. In particular, for $y = 0$, I satisfies $f(x_1z, \ldots, x_n)^m a \in C$ for all $z \in R$, that is, I satisfies $f(x_1, \ldots, x_n)^m a \in C$.

In case there are $z_1, \ldots, z_n \in I$ such that $f(z_1, \ldots, z_n)^m a \neq 0$, then by [14, Theorem 1], $f(x_1, \ldots, x_n)^m$ is central valued on R and also $a \in C$. Thus $I = R$ and $G(f(x_1, \ldots, x_n))^m$ is central valued on R. Hence, by [19], either $f(x_1, \ldots, x_n)$ is central valued on R, or R satisfies s_4 the standard identity of degree 4, or there exists $\alpha \in C$ such that $G(x) = \alpha x$. In any case we are done.

On the other hand, if I satisfies $f(x_1, \ldots, x_n)^m a$. Then, by [6] we get the conclusion that either $a = 0$ or $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I.

Let now d be the inner derivation induced by $q \in U$, namely $d(x) = [q, x]$, then we have $G(x) = (c + q)x - xq$. In this case I satisfies

$$(1.1) \qquad \Big((c + q)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)(-q)\Big)^m a \in C.$$

Denote by K the algebraic closure of F if F is infinite, otherwise let $K = F$. Then $M_k(F) \otimes_C K \cong M_l(K)$ for some $l \geq 2$. By [12, Lemma 2] and [10, Proposition], it follows that $((c+q)f(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)(-q))^m a \in Z(M_l(K))$ for all $r_1, \ldots, r_n \in IC \otimes_C K$. Also in this case we assume, without loss of generality, that $R = M_l(K)$ and $I = \sum_{i=1}^{t} e_{ii}R$, where $t \leq l$.

If $l = 2$ we are done, thus we suppose that $l \geq 3$. By [3, Lemma 3], if $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not an identity for I, then for all $\alpha \in F$, $i \leq l$ and $j \neq i$ there exist $r_1, \ldots, r_n \in I$ such that $f(r_1, \ldots, r_n) = \alpha e_{ij}$. Without loss of generality we may consider $f(r_1, \ldots, r_n) = e_{ij}$. Therefore $((c + q)e_{ij} + e_{ij}(-q))^m a \in Z(M_l(K))$. Since $((c + q)e_{ij} + e_{ij}(-q))^m a$ has rank ≤ 2, then it is zero in $M_l(K)$, hence $((c + q)e_{ij} + e_{ij}(-q))^m = 0$, since a is invertible. This means both $e_{ij}((c + q)e_{ij} + e_{ij}(-q))^m = 0$ and $((c + q)e_{ij} + e_{ij}(-q))^m e_{ij}$=0. Therefore the (j, i)-entries of the matrices q and c is zero, so that $qI \subseteq I$ and $cI \subseteq I$. This means that $G(I) \subseteq I$ and so $G(f(r_1, \ldots, r_n))^m a \in I \cap K$, for all $r_1, \ldots, r_n \in I$, implies $I = R = M_l(K)$. Therefore R satisfies (1).

In the light of this, we may repeat the previous argument, for any $i \neq j$ and with no assumption on i and j. There are $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n) = e_{ij}$ and $((c+q)e_{ij} + e_{ij}(-q))^m a \in Z(M_l(K))$. As above we have that $((c+q)e_{ij} + e_{ij}(-q))^m = 0$. Since it holds for all $i \neq j$, it follows that both c and q are diagonal matrices in R and a standard argument shows that both c and q are central matrices in R. Thus $G(x) = cx$

for $c \in C$, and $(c^m)af(x_1,\dots,x_n)^m \in C$ is satisfied by R. Consider the following subset of R:

$$A = \{x \in R : xf(r_1,\dots,r_n)^m \in C,\ \forall r_1,\dots,r_n \in R\}.$$

Of course A is a subgroup of R which is invariant under the action of all the inner K-automorphisms. By [5] either $A \subseteq Z(R)$ or $[R,R] \subseteq A$. In the first case $a \in Z(R)$ and $f(x_1,\dots,x_n)^m$ is central valued on R. In the second one, for all $i \neq j$, $e_{ij}f(x_1,\dots,x_n)^m \in Z(R)$. By commuting this last with e_{ij} we get

$$0 = [e_{ij}f(r_1,\dots,r_n)^m, e_{ij}] = e_{ij}f(r_1,\dots,r_n)^m e_{ij},$$

for all $r_1,\dots,r_n \in R$. This means that $f(r_1,\dots,r_n)^m$ is a diagonal matrix on R, and as above we obtain that $f(r_1,\dots,r_n)^m$ is a central matrix, for all $r_1,\dots,r_n \in R$. As a consequence, once again $a \in Z(R)$. □

As a consequence of the previous theorem we also have the following:

1.5. Corollary. *Let R be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R. Let $m \geq 1$ be a fixed integer, a a fixed element of R, G a generalized derivation of R. If $aG(r)^m \in Z(R)$ for all $r \in I$, then one of the following holds:*

(1) $aI = aG(I) = (0)$;
(2) $G(x) = qx$, *for some* $q \in U$ *and* $aqI = 0$;
(3) $[x_1,x_2]x_3$ *is an identity for* I;
(4) $G(x) = cx + [q,x]$ *for all* $x \in R$, *where* $c,q \in U$ *such that* $cI = 0$ *and* $[q,I]I = 0$;
(5) $\dim_C(RC) \leq 4$. □

We would like to conclude this note with the following results, which are easy reductions of the previous ones:

1.6. Corollary. *Let R be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring U and extended centroid C, I a non-zero two-sided ideal of R. Let $f(x_1,\dots,x_n)$ be a non-central multilinear polynomial over C, $m \geq 1$ a fixed integer, a a non-zero fixed element of R, G a non-zero generalized derivation of R. If $aG(f(r_1,\dots,r_n))^m \in Z(R)$ for all $r_1,\dots,r_n \in I$, then one of the following holds:*

(1) $G(x) = qx$, *for some* $q \in U$ *and* $aq = 0$;
(2) R *satisfies* s_4, *the standard identity of degree* 4;
(3) $G(x) = \alpha x$, *for some* $\alpha \in C$; *moreover* $a \in C$ *and* $f(x_1,\dots,x_n)^m$ *is central valued on* R. □

1.7. Corollary. *Let R be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring U and extended centroid C, I a non-zero two-sided ideal of R. Let $m \geq 1$ be a fixed integer, a a non-zero fixed element of R, G a non-zero generalized derivation of R. If $aG(r)^m \in Z(R)$ for all $r \in I$, then one of the following holds:*

(1) $G(x) = qx$, *for some* $q \in U$ *and* $aq = 0$;
(2) R *satisfies* s_4, *the standard identity of degree* 4. □

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