

ON A SECOND ORDER RATIONAL DIFFERENCE EQUATION

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Abstract

In this paper, we investigate the stability of the following difference equation

$$x_{n+1} = \frac{ax_n^4 + bx_nx_{n-1}^3 + cx_n^2x_{n-1}^2 + dx_n^3x_{n-1} + ex_{n-1}^4}{Ax_n^4 + Bx_nx_{n-1}^3 + Cx_n^2x_{n-1}^2 + Dx_n^3x_{n-1} + Ex_{n-1}^4},$$
$$n = 0, 1, \dots,$$

where the parameters $a, b, c, d, e, A, B, C, D, E$ are positive real numbers and the initial values x_0, x_{-1} are arbitrary positive numbers.

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1. Introduction and preliminaries

Consider the following second-order difference equation

$$(1.1) \quad x_{n+1} = \frac{ax_n^4 + bx_nx_{n-1}^3 + cx_n^2x_{n-1}^2 + dx_n^3x_{n-1} + ex_{n-1}^4}{Ax_n^4 + Bx_nx_{n-1}^3 + Cx_n^2x_{n-1}^2 + Dx_n^3x_{n-1} + Ex_{n-1}^4}, \quad n = 0, 1, \dots,$$

where the initial conditions $x_0, x_{-1} \in (0, \infty)$ and the parameters $a, b, c, d, e, A, B, C, D, E \in (0, \infty)$. In this paper we study the global stability of the unique positive equilibrium point, and the boundedness and the convergence of the solutions of Equation (1.1).

Nonlinear difference equations appear naturally, for example, from certain models in ecology, economy, automatic control theory, and they are of great importance in applications where the $(n+1)^{st}$ state of the model depends on the previous k states.

Recently, there has been a lot of attention given to studying the global behavior of nonlinear difference equations by many authors, See for example [1-3,5-19] and the references cited therein.

Now, we review some definitions (see for example [11-12]), which will be useful in the sequel.

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Let I be an interval of real numbers and let $F : I \times I \rightarrow I$ be a continuously differentiable function. Consider the difference equation

$$(1.2) \quad x_{n+1} = F(x_n, x_{n-1})$$

with initial values $x_{-1}, x_0 \in I$.

1.1. Definition. A point $\bar{x} \in I$ is called an *equilibrium point* of (1.2) if

$$\bar{x} = F(\bar{x}, \bar{x}).$$

1.2. Definition. Let \bar{x} be an equilibrium point of (1.2).

- The equilibrium \bar{x} is called *locally stable* if for every $\epsilon > 0$, there exist $\delta > 0$ such that for all $x_{-1}, x_0 \in I$ with $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \epsilon$, for all $n \geq -1$.
- The equilibrium \bar{x} is called *locally asymptotically stable* if it is locally stable, and if there exists $\gamma > 0$ such that if $x_{-1}, x_0 \in I$ and $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$ then

$$\lim_{n \rightarrow +\infty} x_n = \bar{x}.$$

- The equilibrium \bar{x} is called a *global attractor* if for all $x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow +\infty} x_n = \bar{x}.$$

- The equilibrium \bar{x} is called *global asymptotically stable* if it is locally stable and a global attractor.
- The equilibrium \bar{x} is called *unstable* if it is not stable.
- Let $p = \frac{\partial F}{\partial x}(\bar{x}, \bar{x})$ and $q = \frac{\partial F}{\partial y}(\bar{x}, \bar{x})$. Then the equation

$$(1.3) \quad y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \dots$$

is called the *linearized equation of (1.2) about the equilibrium point \bar{x}* .

The next result, which was given by Clark [4], provides a sufficient condition for the locally asymptotically stability of (1.2).

1.3. Theorem. Consider the difference equation (1.3). Then, $|p| + |q| < 1$ is a sufficient condition for the locally asymptotically stability of (1.2). \square

1.4. Definition. The difference equation (1.2) is said to be *permanent* if there exist numbers α, β with $0 < \alpha \leq \beta < \infty$ such that for any initial values $x_{-1}, x_0 \in I$ there exists a positive integer N which depends on the initial conditions such that $\alpha \leq x_n \leq \beta$ for all $n \geq N$.

2. Main results

Let us define the following real numbers: $r_1 = aB - bA$, $r_2 = aC - cA$, $r_3 = aD - dA$, $r_4 = aE - eA$, $r_5 = bE - eB$, $r_6 = cB - bC$, $r_7 = cE - eC$, $r_8 = dB - bD$, $r_9 = dC - cD$, $r_{10} = dE - eD$.

2.1. Remark. Equation (1.1) has a unique positive equilibrium point which is given by

$$\bar{x} = \frac{a + b + c + d + e}{A + B + C + D + E}.$$

Let $f : (0, +\infty)^2 \rightarrow (0, +\infty)$ be the function defined by

$$f(x, y) = \frac{ax^4 + bxy^3 + cx^2y^2 + dx^3y + ey^4}{Ax^4 + Bxy^3 + Cx^2y^2 + Dx^3y + Ey^4}.$$

2.2. Lemma.

(1) Assume that

- (a) $\frac{a}{A} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (b) $\frac{e}{E} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (c) $3r_1 + r_9 \geq 0,$
- (d) $2r_4 + r_8 \geq 0,$
- (e) $r_6 + 3r_{10} \geq 0.$

Then f is increasing in x for each y and it is decreasing in y for each x .

(2) Assume that

- (a) $\frac{a}{A} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (b) $\frac{e}{E} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (c) $3r_1 + r_9 \leq 0,$
- (d) $2r_4 + r_8 \leq 0,$
- (e) $r_6 + 3r_{10} \leq 0.$

Then f is decreasing in x for each y and it is increasing in y for each x .

Proof. (1) We have, $3r_1 + r_9 \geq 0, 2r_4 + r_8 \geq 0$ and $r_6 + 3r_{10} \geq 0$. Using the fact that, $\frac{a}{A} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B})$ and $\frac{e}{E} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B})$, we get: $r_2, r_3, r_5, r_7 \geq 0$. Now, the result follows from the formulaes

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{r_3yx^6 + 2r_2y^2x^5 + (r_9 + 3r_1)y^3x^4 + (4r_4 + 2r_8)y^4x^3 + (3r_{10} + r_6)y^5x^2 + (2r_7)y^6x + r_5y^7}{(Ax^4 + Bxy^3 + Cx^2y^2 + Dx^3y + Ey^4)^2}, \\ \frac{\partial f}{\partial y}(x, y) &= \frac{-r_3x^7 - 2r_2yx^6 - (r_9 + 3r_1)y^2x^5 - (4r_4 + 2r_8)y^3x^4 - (3r_{10} + r_6)y^4x^3 - 2r_7y^5x^2 - r_5y^6x}{(Ax^4 + Bxy^3 + Cx^2y^2 + Dx^3y + Ey^4)^2}. \end{aligned}$$

(2) The proof of (2) is similar and will be omitted. □

The locally stability of the positive equilibrium point $\bar{x} = \frac{a+b+c+d+e}{A+B+C+D+E}$ of (1.1) is described in the following theorem.

2.3. Theorem. Assume that

$$2 \frac{|3r_1 + 2r_2 + r_3 + 4r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}|}{(a + b + c + d + e)(A + B + C + D + E)} < 1.$$

Then the positive equilibrium point $\bar{x} = \frac{a+b+c+d+e}{A+B+C+D+E}$ of (1.1) is locally asymptotically stable.

Proof. The linearized equation of (1.1) about $\bar{x} = \frac{a+b+c+d+e}{A+B+C+D+E}$ is

$$x_{n+1} = px_n + qx_{n-1}$$

where

$$p = \frac{3r_1 + 2r_2 + r_3 + 4r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}}{(a + b + c + d + e)(A + B + C + D + E)}$$

and

$$q = -\frac{3r_1 + 2r_2 + r_3 + 4r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}}{(a + b + c + d + e)(A + B + C + D + E)}.$$

By using Theorem 1.3, we get that \bar{x} is locally asymptotically stable if

$$2 \frac{|3r_1 + 2r_2 + r_3 + 4r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}|}{(a + b + c + d + e)(A + B + C + D + E)} < 1. \quad \square$$

The next theorem is devoted to the permanence of the difference equation (1.1).

2.4. Theorem. *Let $\{x_n\}_{n=-1}^{+\infty}$ be a positive solution of equation (1.1).*

- (1) *Assume that*
 - (a) $\frac{a}{A} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
 - (b) $\frac{e}{E} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}).$

Then,

$$\frac{e}{E} \leq x_n \leq \frac{a}{A}$$

for all $n \geq 1.$

- (2) *Assume that*
 - (a) $\frac{a}{A} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
 - (b) $\frac{e}{E} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}).$

Then,

$$\frac{a}{A} \leq x_n \leq \frac{e}{E}$$

for all $n \geq 1.$

Proof. (1) We have

$$x_{n+1} - \frac{a}{A} = \frac{-r_3x_{n-1}x_n^3 - r_2x_{n-1}^2x_n^2 - r_1x_{n-1}^3x_n - r_4x_{n-1}^4}{A(Ax_n^4 + Bx_nx_{n-1}^3 + Cx_n^2x_{n-1}^2 + Dx_n^3x_{n-1} + Ex_{n-1}^4)},$$

$$x_{n+1} - \frac{e}{E} = \frac{r_4x_n^4 + r_{10}x_{n-1}x_n^3 + r_7x_{n-1}^2x_n^2 + r_5x_{n-1}^3x_n}{E(Ax_n^4 + Bx_nx_{n-1}^3 + Cx_n^2x_{n-1}^2 + Dx_n^3x_{n-1} + Ex_{n-1}^4)}.$$

Now, it suffices to get $r_1, r_2, r_3, r_4, r_5, r_7, r_{10} \geq 0,$ which result from $\frac{a}{A} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B})$ and $\frac{e}{E} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}).$

(2) Similarly we can easily prove (2). □

Here we study the global asymptotic stability of equation (1.1).

2.5. Theorem. *Let*

$$p_1 = (A + D)e + Ab - Ea,$$

$$p_2 = (A + C + D)e - Ed + Ac + (A + D)b - (B + E)a,$$

$$p_3 = (A + B + C + D)e + (A - B - E)d + (A + D - E)c + (A + C + D)b - (B + C + E)a,$$

$$p_4 = (A + B + C + D + E)e + (A - B - C + D - E)d + (A - B + C + D - E)c$$

$$+ (A + B + C + D - E)b + (A - B - C - D - E)a.$$

Assume that

- (1) $\frac{a}{A} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (2) $\frac{e}{E} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (3) $3r_1 + r_9 \geq 0,$
- (4) $2r_4 + r_8 \geq 0,$
- (5) $r_6 + 3r_{10} \geq 0,$
- (6) $2 \frac{|3r_1+2r_2+r_3+4r_4+r_5+r_6+2r_7+2r_8+r_9+3r_{10}|}{(a+b+c+d+e)(A+B+C+D+E)} < 1,$
- (7) $p_1, p_2, p_3, p_4 \geq 0.$

Then the equilibrium point $\bar{x} = \frac{a+b+c+d+e}{A+B+C+D+E}$ of (1.1) is globally asymptotically stable.

Proof. Let $\{x_n\}_{n=-1}^{+\infty}$ be a solution of equation (1.1). In view of theorem 2.3 we need only to prove that \bar{x} is a global attractor.

Let

$$m = \lim_{n \rightarrow +\infty} \inf x_n$$

and

$$M = \limsup_{n \rightarrow +\infty} x_n.$$

To prove that

$$\lim_{n \rightarrow +\infty} x_n = \bar{x},$$

it suffices to show that $m = M$.

Let $\epsilon \in]0, m[$ then there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we get

$$m - \epsilon \leq x_n \leq M + \epsilon.$$

Thus by using Lemma 2.2, Part (1); we get for all $n \geq n_0 + 1$

$$\begin{aligned} x_{n+1} &\geq \frac{a(m - \epsilon)^4 + b(m - \epsilon)(M + \epsilon)^3 + c(m - \epsilon)^2(M + \epsilon)^2 + d(m - \epsilon)^3(M + \epsilon) + e(M + \epsilon)^4}{A(m - \epsilon)^4 + B(m - \epsilon)(M + \epsilon)^3 + C(m - \epsilon)^2(M + \epsilon)^2 + D(m - \epsilon)^3(M + \epsilon) + E(M + \epsilon)^4}, \\ x_{n+1} &\leq \frac{a(M + \epsilon)^4 + b(M + \epsilon)(m - \epsilon)^3 + c(M + \epsilon)^2(m - \epsilon)^2 + d(M + \epsilon)^3(m - \epsilon) + e(m - \epsilon)^4}{A(M + \epsilon)^4 + B(M + \epsilon)(m - \epsilon)^3 + C(M + \epsilon)^2(m - \epsilon)^2 + D(M + \epsilon)^3(m - \epsilon) + E(m - \epsilon)^4}. \end{aligned}$$

Then we get the following inequalities

$$\begin{aligned} m &\geq \frac{a(m - \epsilon)^4 + b(m - \epsilon)(M + \epsilon)^3 + c(m - \epsilon)^2(M + \epsilon)^2 + d(m - \epsilon)^3(M + \epsilon) + e(M + \epsilon)^4}{A(m - \epsilon)^4 + B(m - \epsilon)(M + \epsilon)^3 + C(m - \epsilon)^2(M + \epsilon)^2 + D(m - \epsilon)^3(M + \epsilon) + E(M + \epsilon)^4}, \\ M &\leq \frac{a(M + \epsilon)^4 + b(M + \epsilon)(m - \epsilon)^3 + c(M + \epsilon)^2(m - \epsilon)^2 + d(M + \epsilon)^3(m - \epsilon) + e(m - \epsilon)^4}{A(M + \epsilon)^4 + B(M + \epsilon)(m - \epsilon)^3 + C(M + \epsilon)^2(m - \epsilon)^2 + D(M + \epsilon)^3(m - \epsilon) + E(m - \epsilon)^4}. \end{aligned}$$

These inequalities yield

$$\begin{aligned} m &\geq \frac{am^4 + bmM^3 + cm^2M^2 + dm^3M + eM^4}{Am^4 + BmM^3 + Cm^2M^2 + Dm^3M + EM^4}, \\ M &\leq \frac{aM^4 + bMm^3 + cM^2m^2 + dM^3m + em^4}{AM^4 + BMm^3 + CM^2m^2 + DM^3m + Em^4}. \end{aligned}$$

So,

$$\begin{aligned} mM &\geq M \frac{am^4 + bmM^3 + cm^2M^2 + dm^3M + eM^4}{Am^4 + BmM^3 + Cm^2M^2 + Dm^3M + EM^4}, \\ mM &\leq m \frac{aM^4 + bMm^3 + cM^2m^2 + dM^3m + em^4}{AM^4 + BMm^3 + CM^2m^2 + DM^3m + Em^4}. \end{aligned}$$

Hence

$$M \frac{am^4 + bmM^3 + cm^2M^2 + dm^3M + eM^4}{Am^4 + BmM^3 + Cm^2M^2 + Dm^3M + EM^4} - m \frac{aM^4 + bMm^3 + cM^2m^2 + dM^3m + em^4}{AM^4 + BMm^3 + CM^2m^2 + DM^3m + Em^4} \leq 0$$

which can be written

$$\begin{aligned} &(M - m) \\ &\times \frac{eA(m^8 + M^8) + p_1mM(m^6 + M^6) + p_2m^2M^2(M^4 + m^4) + p_3m^3M^3(M^2 + m^2) + p_4m^4M^4}{(Am^4 + BmM^3 + Cm^2M^2 + Dm^3M + EM^4)(AM^4 + BMm^3 + CM^2m^2 + DM^3m + Em^4)} \\ &\leq 0. \end{aligned}$$

Since

$$\frac{eA(m^8 + M^8) + p_1mM(m^6 + M^6) + p_2m^2M^2(M^4 + m^4) + p_3m^3M^3(M^2 + m^2) + p_4m^4M^4}{(Am^4 + BmM^3 + Cm^2M^2 + Dm^3M + EM^4)(AM^4 + BMm^3 + CM^2m^2 + DM^3m + Em^4)} > 0$$

we get

$$M \leq m.$$

So, $m = M = \bar{x}$. □

By the same arguments, we can prove the following theorem.

2.6. Theorem. *Let*

$$\begin{aligned} q_1 &= -Ae + Ed + (B + E)a, \\ q_2 &= -(A + D)e + (B + E)d + Ec - Ab + (C + B + E)a, \\ q_3 &= -(A + C + D)e + (B + C + E)d + (-A + B + E)c + (-A - D + E)b \\ &\quad + (B + C + D + E)a, \\ q_4 &= (-A - B - C - D + E)e + (-A + B + C + D + E)d + (-A + B + C - D + E)c \\ &\quad + (-A + B - C - D + E)b + (A + B + C + D + E)a. \end{aligned}$$

Assume that

- (1) $\frac{a}{A} \leq \min(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (2) $\frac{e}{E} \geq \max(\frac{d}{D}, \frac{c}{C}, \frac{b}{B}),$
- (3) $3r_1 + r_9 \leq 0,$
- (4) $2r_4 + r_8 \leq 0,$
- (5) $r_6 + 3r_{10} \leq 0,$
- (6) $2 \frac{|3r_1+2r_2+r_3+4r_4+r_5+r_6+2r_7+2r_8+r_9+3r_{10}|}{(a+b+c+d+e)(A+B+C+D+E)} < 1,$
- (7) $q_1, q_2, q_3, q_4 \geq 0.$

Then the equilibrium point $\bar{x} = \frac{a+b+c+d+e}{A+B+C+D+E}$ of (1.1) is globally asymptotically stable. □

2.7. Theorem. *Assume that $q_1, q_2, q_3, q_4 \geq 0$. Then, Equation (1.1) has no positive solution of period two.*

Proof. For the sake of contradiction, assume that there exist distinct positive real numbers α, β , such that

$$\dots, \alpha, \beta, \alpha, \beta, \dots$$

is a period two solution of (1.1). Then,

$$\alpha = f(\beta, \alpha), \beta = f(\alpha, \beta).$$

Thus, we have

$$\beta f(\beta, \alpha) = \alpha f(\alpha, \beta),$$

which implies

$$(\beta - \alpha) \frac{aE(\alpha^8 + \beta^8) + q_1\alpha\beta(\alpha^6 + \beta^6) + q_2\alpha^2\beta^2(\alpha^4 + \beta^4) + q_3\alpha^3\beta^3(\alpha^2 + \beta^2) + q_4\alpha^4\beta^4}{(A\beta^4 + B\beta\alpha^3 + C\beta^2\alpha^2 + D\beta^3\alpha + E\alpha^4)(A\alpha^4 + B\alpha\beta^3 + C\alpha^2\beta^2 + D\alpha^3\beta + E\beta^4)} = 0.$$

Since

$$\frac{aE(\alpha^8 + \beta^8) + q_1\alpha\beta(\alpha^6 + \beta^6) + q_2\alpha^2\beta^2(\alpha^4 + \beta^4) + q_3\alpha^3\beta^3(\alpha^2 + \beta^2) + q_4\alpha^4\beta^4}{(A\beta^4 + B\beta\alpha^3 + C\beta^2\alpha^2 + D\beta^3\alpha + E\alpha^4)(A\alpha^4 + B\alpha\beta^3 + C\alpha^2\beta^2 + D\alpha^3\beta + E\beta^4)} > 0,$$

we get $\alpha = \beta$, which is a contradiction. □

2.8. Remark. It follows from (1.1), when $r_1 = \dots = r_{10} = 0$ that $x_{n+1} = \alpha$ for all $n \geq -1$ for some constant α .

3. Numerical examples

In order to illustrate our results and to support our theoretical discussions, we consider numerical examples in this section.

3.1. Example. Let $(a, b, c, d, e, A, B, C, D, E) = (3.8, 2.2, 3.3, 3, 5, 1.5, 2, 3.5, 3.2, 5.5)$ and $(x_{-1}, x_0) = (5, 2.9)$. Then, all the conditions of Theorem 2.5 are satisfied and we have the following results:

n	$ x_n - \bar{x} $	n	$ x_n - \bar{x} $	n	$ x_n - \bar{x} $
1	0.1285600589	21	$0.5070 \cdot 10^{-5}$	31	$0.27 \cdot 10^{-7}$
2	0.1665995958	22	$0.3589 \cdot 10^{-5}$	32	$0.14 \cdot 10^{-7}$
3	0.014511762	23	$0.504 \cdot 10^{-6}$	33	$0.5 \cdot 10^{-8}$
4	0.062906849	24	$0.1394 \cdot 10^{-5}$	34	$0.6 \cdot 10^{-8}$

3.2. Example. Let $(a, b, c, d, e, A, B, C, D, E) = (1.5, 2, 3.5, 3.2, 5.5, 3.8, 2.2, 3.3, 3, 5)$ and $(x_{-1}, x_0) = (0.15, 21)$. Then, all the conditions of Theorem 2.6 are satisfied and we have the following results:

n	$ x_n - \bar{x} $	n	$ x_n - \bar{x} $	n	$ x_n - \bar{x} $
1	0.5089804983	31	$0.2260828 \cdot 10^{-3}$	97	$0.2 \cdot 10^{-9}$
2	0.1908955041	32	$0.1757875 \cdot 10^{-3}$	98	$0.2 \cdot 10^{-9}$
3	0.3068138396	33	$0.1369202 \cdot 10^{-3}$	99	$0.6 \cdot 10^{-9}$
4	0.1270735371	34	$0.1065022 \cdot 10^{-3}$	100	$0.1 \cdot 10^{-9}$

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