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ON THE OSTROWSKI-GRÜSS TYPE INEQUALITY FOR TWICE DIFFERENTIABLE FUNCTIONS

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Abstract

In this paper we obtain some new Ostrowski-Grüss type inequalities containing twice differentiable functions.

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1. Introduction

In [11], Ostrowski proved the following inequality.

1.1. Theorem. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of I and $a, b \in I^o$, $a < b$. If $|f'| \leq M$, $\forall\, t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\, M, \tag{1.1}$$

for $x \in [a, b]$.

In the past several years there has been considerable interest in the study of Ostrowski type inequalities. In [12], Özdemir *et al.* proved Ostrowski's type inequalities for (α, m)-convex functions and in [15], an Ostrowski type inequality was given by Sarıkaya. However, some new types of inequality are established, for example inequalities of Ostrowski-Grüss type and inequalities of Ostrowski-Chebyshev type. In [9], Milovanović and Pečarić gave a generalization of Ostrowski's inequality and some related applications.

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An Ostrowski-Grüss type inequality was given for the first time by Dragomir and Wang in [4]. In [8], Matić *et al.*, generalized and improved this inequality. For generalizations, improvements and recent results see the papers [1]–[10], [13], [14], [16] and [18]. Recently, in [17], Ujević proved following theorems;

1.2. Theorem. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of I and $a, b \in I^o$, $a < b$. If there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le f'(t) \le \Gamma$, $\forall\, t \in [a, b]$ and $f' \in L_1[a, b]$, then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{(b-a)}{2} (S - \gamma) \tag{1.2}$$

and

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{(b-a)}{2} (\Gamma - S), \tag{1.3}$$

where $S = \frac{f(b)-f(a)}{b-a}$.

1.3. Theorem. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior of I with $f'' \in L_2[a, b]$ and $a, b \in I^o$, $a < b$. Then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{(b-a)^{\frac{3}{2}}}{2\pi\sqrt{3}} \left\| f'' \right\|_2, \tag{1.4}$$

for $x \in [a, b]$.

The main purpose of this paper is to prove Ostrowski-Grüss type inequalities similar to above but now involving twice differentiable mappings.

2. Main Results

2.1. Theorem. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice differentiable mapping in the interior of I and $a, b \in I^o$, $a < b$. If there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \le f''(t) \le \Gamma$, $\forall t \in [a, b]$ and $f'' \in L_2[a, b]$, then we have*

$$\begin{aligned} &\left| f(x) - x f'(x) - \frac{a^2 f'(a) - b^2 f'(b)}{2(b-a)} \right. \\ &\qquad \left. - \left(\frac{x^2}{2} - \frac{a^2 + ab + b^2}{3} \right) \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{(b-a)^2}{3} (S - \gamma) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\left| f(x) - x f'(x) - \frac{a^2 f'(a) - b^2 f'(b)}{2(b-a)} \right. \\ &\qquad \left. - \left(\frac{x^2}{2} - \frac{a^2 + ab + b^2}{3} \right) \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{(b-a)^2}{3} (\Gamma - S) \end{aligned} \tag{2.2}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. We can define a mapping $K(x,t)$ as follows:

$$K(x,t)=\begin{cases}\frac{t}{2}(t-2a), & t\in[a,x],\\ \frac{t}{2}(t-2b), & t\in(x,b].\end{cases}$$

By using this mapping and integrating by parts, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt\\ &\qquad=\frac{1}{b-a}\left[\int_a^x \frac{t}{2}(t-2a)\,f''(t)\,dt+\int_x^b \frac{t}{2}(t-2b)\,f''(t)\,dt\right]\\ &\qquad=xf'(x)-f(x)+\frac{a^2f'(a)-b^2f'(b)}{2(b-a)}+\frac{1}{b-a}\int_a^b f(t)\,dt.\end{aligned}\tag{2.3}$$

By a simple computation, we have

$$\frac{1}{b-a}\int_a^b K(x,t)\,dt=\frac{x^2}{2}-\frac{a^2+ab+b^2}{3}\tag{2.4}$$

and

$$\int_a^b f''(t)\,dt=f'(b)-f'(a).\tag{2.5}$$

Using (2.3), (2.4) and (2.5), we get

$$\begin{aligned}&xf'(x)-f(x)+\frac{a^2f'(a)-b^2f'(b)}{2(b-a)}\\ &\qquad-\left(\frac{x^2}{2}-\frac{a^2+ab+b^2}{3}\right)\frac{f'(b)-f'(a)}{b-a}+\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt-\frac{1}{(b-a)^2}\int_a^b f''(t)dt\int_a^b K(x,t)\,dt.\end{aligned}$$

We set

$$R_n(x)=\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt-\frac{1}{(b-a)^2}\int_a^b f''(t)dt\int_a^b K(x,t)\,dt.$$

If we write $R_n(x)$ as follows with $C\in\mathbb{R}$ an arbitrary constant, then we have

$$R_n(x)=\frac{1}{b-a}\int_a^b \big(f''(t)-C\big)\left[K(x,t)-\frac{1}{b-a}\int_a^b K(x,s)\,ds\right]dt\tag{2.6}$$

We know that

$$\int_a^b\left[K(x,t)-\frac{1}{b-a}\int_a^b K(x,s)\,ds\right]dt=0\tag{2.7}$$

So, if we choose $C = \gamma$ in (2.6). Then we get

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f''(t) - \gamma\right) \left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right] dt$$

and

$$|R_n(x)| \le \frac{1}{b-a}\max_{t\in[a,b]}\left|K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3}\right)\right| \int_a^b \left|f''(t) - \gamma\right| dt. \tag{2.8}$$

Since

$$\max_{t\in[a,b]}\left|K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3}\right)\right| = \frac{(b-a)^2}{3}$$

and

$$\int_a^b \left|f''(t) - \gamma\right| dt = f'(b) - f'(a) - \gamma(b-a)$$
$$= (S-\gamma)(b-a),$$

from (2.8) we have

$$|R_n(x)| \le \frac{(b-a)^2}{3}(S-\gamma), \tag{2.9}$$

which gives (2.1).

Secondly, if we choose $C = \Gamma$ in (2.6) then by a similar argument we get

$$|R_n(x)| \le \frac{1}{b-a}\max_{t\in[a,b]}\left|K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3}\right)\right| \int_a^b \left|f''(t) - \Gamma\right| dt \tag{2.10}$$

and

$$\int_a^b \left|f''(t) - \Gamma\right| dt = \Gamma(b-a) - f'(b) + f'(a) \tag{2.11}$$
$$= (\Gamma - S)(b-a)$$

so from (2.10) and (2.11), we get (2.2). □

2.2. Theorem. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior of I with $f'' \in L_2[a,b]$ and $a, b \in I^o$, $a < b$. Then we have*

$$\left|f(x) - xf'(x) - \frac{a^2f'(a) - b^2f'(b)}{2(b-a)} - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3}\right)\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.12}$$
$$\le \frac{(b-a)^2}{3}\left(S - f''\left(\frac{a+b}{2}\right)\right),$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. Let $R_n(x)$ be defined as in the equality (2.6) with $C \in \mathbb{R}$ an arbitrary constant. If we choose $C = f''\left(\frac{a+b}{2}\right)$, we get

$$|R_n(x)| \leq \frac{1}{b-a} \max_{t\in[a,b]} \left| K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3} \right) \right| \int_a^b \left| f''(t) - f''\left(\frac{a+b}{2}\right) \right| dt.$$

By a simple computation, we get the required result. □

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