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ON SOME NEW HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract

In this paper, we establish some Hadamard-type inequalities based on co-ordinated quasi-convexity. Also we define a new mapping associated with co-ordinated convexity and prove some properties of this mapping.

Keywords: Quasi-convex functions, Hölder Inequality, Power mean inequality, Co-ordinates, Lipschitzian function.

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1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well-known in the literature as Hadamard's inequality. We recall some definitions;

1.1. Definition. (See [4]) A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}, \quad \text{(QC)}$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. In [1], Dragomir defined convex functions on the co-ordinates as follows:

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1.2. Definition. Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called *convex on the co-ordinates* if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

In [1], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

1.3. Theorem. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\ &\qquad \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\ &\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}. \end{aligned} \tag{1.2}$$

The above inequalities are sharp.

Similar results for co-ordinated m-convex and (α,m)-convex functions can be found in [3]. In [1], Dragomir considered a mapping closely connected with the above inequalities and established the main properties of this mapping as follows.

Now, for a mapping $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ convex on the co-ordinates on Δ, we can define the mapping $H : [0,1]^2 \to \mathbb{R}$ by

$$H(t,s) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right)dx\,dy.$$

1.4. Theorem. *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on $\Delta = [a,b] \times [c,d]$. Then:*

(i) *The mapping H is convex on the co-ordinates on $[0,1]^2$.*
(ii) *We have the bounds*

$$\sup_{(t,s)\in[0,1]^2} H(t,s) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy = H(1,1),$$

$$\inf_{(t,s)\in[0,1]^2} H(t,s) = f\left(\frac{a+b}{2},\frac{c+d}{2}\right) = H(0,0).$$

(iii) *The mapping H is monotonic nondecreasing on the co-ordinates.*

1.5. Definition. Consider a function $f : V \to \mathbb{R}$ defined on a subset V of $\mathbb{R}_n$, $n \in \mathbb{N}$. Let $L = (L_1, L_2, \ldots, L_n)$ where $L_i \geq 0$, $i = 1,2,\ldots,n$. We say that f is a L-Lipschitzian function if

$$|f(x)-f(y)| \leq \sum_{i=1}^{n} L\,|x_i-y_i|$$

for all $x, y \in V$.

In [2], Özdemir *et al.* defined quasi-convex function on the co-ordinates as follows:

1.6. Definition. A function $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to be a *quasi-convex function on the co-ordinates on* Δ if the following inequality

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \leq \max\{f(x,y), f(z,w)\}$$

holds for all (x,y), $(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Let us consider a bidimensional interval $\Delta := [a,b]\times[c,d]$. Then $f:\Delta\to\mathbb{R}$ will be called *co-ordinated quasi-convex on the co-ordinates* if the partial mappings

$$f_y:[a,b]\to\mathbb{R},\ f_y(u)=f(u,y)$$

and

$$f_x:[c,d]\to\mathbb{R},\ f_x(v)=f(x,v)$$

are quasi-convex where defined for all $y\in[c,d]$ and $x\in[a,b]$. We denote by QC(Δ) the class of quasi-convex functions on the co-ordinates on Δ.

A formal definition of quasi-convex functions on the co-ordinates as follows:

1.7. Definition. A function $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is said to be a *quasi-convex function on the co-ordinates on* Δ if the following inequality

$$f(\lambda x+(1-\lambda)y, \lambda u+(1-\lambda)v) \leq \max\{f(x,u), f(x,v), f(y,u), f(y,v)\}$$

holds for all $(x,u),\ (x,v),\ (y,u),\ (y,v)\in\Delta$ and $\lambda\in[0,1]$.

In [5], Sarıkaya *et al.* proved the following Lemma and established some inequalities for co-ordinated convex functions.

1.8. Lemma. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\\&\qquad-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right]\\&=\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1(1-2t)(1-2s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\,dt\,ds.\end{aligned}$$

The main purpose of this paper is to obtain some inequalities for co-ordinated quasi-convex functions by using Lemma 1.8 and elementary analysis.

2. Main Results

2.1. Theorem. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is quasi-convex on the co-ordinates on Δ, then one has the inequality:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx-A\right|\\&\qquad\leq\frac{(b-a)(d-c)}{16}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}\end{aligned}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]\,dx+\frac{1}{d-c}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]\,dy\right].$$

Proof. From Lemma 1.8, we can write

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)\,dy\,dx-A\right|\\&\quad\leq\frac{\left(b-a\right)\left(d-c\right)}{4}\\&\qquad\times\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)\right|\,dt\,ds.\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$is quasi-convex on the co-ordinates on Δ, we have

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)\,dy\,dx-A\right|\\&\quad\leq\frac{\left(b-a\right)\left(d-c\right)}{4}\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\\&\qquad\times\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}\,dt\,ds.\end{aligned}$$

On the other hand, we have

$$\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|dtds=\frac{(b-a)(d-c)}{16}.$$

The proof is completed. □

2.2. Theorem. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$ is a quasi-convex function on the co-ordinates on Δ, then one has the inequality:*

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)\,dy\,dx-A\right|\\&\quad\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}\end{aligned}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]\,dx+\frac{1}{d-c}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]\,dy\right]$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1.8 and using Hölder's inequality, we get

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4}$$
$$\times\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dt\,ds$$
$$\leq\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1|(1-2t)(1-2s)|^p\,dtds\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q dt\,ds\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is quasi-convex on the co-ordinates on Δ, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1|(1-2t)(1-2s)|^p\,dt\,ds\right)^{\frac{1}{p}}$$
$$\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}$$
$$\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.$$

So, the proof is completed. □

2.3. Corollary. *Since* $\frac{1}{4}<\frac{1}{(p+1)^{\frac{2}{p}}}<1$*, for* $p>1$ *we have the following inequality;*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4}$$
$$\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.$$

2.4. Theorem. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$*,* $q\geq 1$*, is a quasi-convex function on the co-ordinates on* Δ*, then one has the inequality:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|$$
$$\leq\frac{(b-a)(d-c)}{16}$$
$$\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]\,dx+\frac{1}{d-c}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]\,dy\right].$$

Proof. From Lemma 1.8 and using the Power Mean inequality, we can write

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)\,dy\,dx-A\right|\\&\quad\le\frac{\left(b-a\right)\left(d-c\right)}{4}\\&\qquad\times\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)\right|\,dt\,ds\\&\quad\le\frac{\left(b-a\right)\left(d-c\right)}{4}\left(\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\,dt\,ds\right)^{1-\frac{1}{q}}\\&\qquad\times\left(\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)\right|^q\,dt\,ds\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is quasi-convex on the co-ordinates on Δ, we have

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)\,dy\,dx-A\right|\\&\quad\le\frac{\left(b-a\right)\left(d-c\right)}{4}\\&\qquad\times\left(\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\,dt\,ds\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\,dt\,ds\right)^{\frac{1}{q}}\\&\qquad\quad\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}\\&\quad=\frac{(b-a)(d-c)}{16}\\&\qquad\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}},\end{aligned}$$

which completes the proof. □

2.5. Remark. Since $\frac{1}{4}<\frac{1}{(p+1)^{\frac{2}{p}}}<1$, for $p>1$, the estimation in Theorem 2.4 is better than that in Theorem 2.2.

Now, for a mapping $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ that is convex on the co-ordinates on Δ, we can define a mapping $G:[0,1]^2\to\mathbb{R}$ by

$$\begin{aligned}G(t,s):=\frac{1}{4}\Bigg[&f\left(ta+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\&+f\left(tb+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\&+f\left(ta+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\\&+f\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\Bigg].\end{aligned}$$

We will now give the following theorem which contains some properties of this mapping.

2.6. Theorem. *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates on $\Delta = [a,b] \times [c,d]$. Then:*

(i) *The mapping G is convex on the co-ordinates on $[0,1]^2$.*
(ii) *We have the bounds*

$$\inf_{(t,s)\in[0,1]^2} G(t,s) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = G(0,0),$$
$$\sup_{(t,s)\in[0,1]^2} G(t,s) = \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} = G(1,1);$$

(iii) *If f satisfies the Lipschitzian conditions, then the mapping G is L-Lipschitzian on $[0,1] \times [0,1]$;*
(iv) *The following inequality holds:*

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$\le \frac{1}{4}\left[\frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4} + \frac{f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)}{2} + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right].$$

Proof. (i) Let $s \in [0,1]$. For all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, then we have

$$\begin{aligned}
&G(\alpha t_1 + \beta t_2, s)\\
&= \frac{1}{4}\left[f\left((\alpha t_1 + \beta t_2)\,a + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\right.\\
&\quad + f\left((\alpha t_1 + \beta t_2)\,b + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\\
&\quad + f\left((\alpha t_1 + \beta t_2)\,a + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right)\\
&\quad \left. + f\left((\alpha t_1 + \beta t_2)\,b + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right)\right]\\
&= \frac{1}{4}\left[f\left(\alpha\left(t_1 a + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2 a + (1-t_2)\frac{a+b}{2}\right), sc + (1-s)\frac{c+d}{2}\right)\right.\\
&\quad + f\left(\alpha\left(t_1 b + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2 b + (1-t_2)\frac{a+b}{2}\right), sc + (1-s)\frac{c+d}{2}\right)\\
&\quad + f\left(\alpha\left(t_1 a + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2 a + (1-t_2)\frac{a+b}{2}\right), sd + (1-s)\frac{c+d}{2}\right)\\
&\quad \left. + f\left(\alpha\left(t_1 b + (1-t_1)\frac{a+b}{2}\right) + \beta\left(t_2 b + (1-t_2)\frac{a+b}{2}\right), sd + (1-s)\frac{c+d}{2}\right)\right].
\end{aligned}$$

Using the convexity of f, we obtain

$$\begin{aligned}
G(\alpha t_1+\beta t_2,s) \le \frac{1}{4}\Bigg[&\alpha\Bigg(f\left(t_1a+(1-t_1)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_1b+(1-t_1)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_1a+(1-t_1)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_1b+(1-t_1)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\Bigg)\\
&+\beta\Bigg(f\left(t_2a+(1-t_2)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_2b+(1-t_2)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_2a+(1-t_2)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_2b+(1-t_2)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\Bigg)\Bigg]\\
=\ &\alpha G(t_1,s)+\beta G(t_2,s)
\end{aligned}$$

if $s\in[0,1]$. For all $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and $s_1,s_2\in[0,1]$, then we also have

$$G(t,\alpha s_1+\beta s_2)\le \alpha G(t,s_1)+\beta G(t,s_2),$$

and the statement is proved.

(ii) It is easy to see that by taking $t=s=0$ and $t=s=1$, respectively, in G, we have the bounds

$$\begin{aligned}
\inf_{(t,s)\in[0,1]^2} G(t,s) &= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=G(0,0),\\
\sup_{(t,s)\in[0,1]^2} G(t,s) &= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}=G(1,1).
\end{aligned}$$

(iii) Let $t_1,t_2,s_1,s_2\in[0,1]$. Then we have:

$$\begin{aligned}
&|G(t_2,s_2)-G(t_1,s_1)|\\
&=\frac{1}{4}\Bigg|f\left(t_2a+(1-t_2)\frac{a+b}{2},s_2c+(1-s_2)\frac{c+d}{2}\right)\\
&+f\left(t_2b+(1-t_2)\frac{a+b}{2},s_2c+(1-s_2)\frac{c+d}{2}\right)\\
&+f\left(t_2a+(1-t_2)\frac{a+b}{2},s_2d+(1-s_2)\frac{c+d}{2}\right)\\
&+f\left(t_2b+(1-t_2)\frac{a+b}{2},s_2d+(1-s_2)\frac{c+d}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&- f\left(t_1 a+(1-t_1)\frac{a+b}{2}, s_1 c+(1-s_1)\frac{c+d}{2}\right)\\
&- f\left(t_1 b+(1-t_1)\frac{a+b}{2}, s_1 c+(1-s_1)\frac{c+d}{2}\right)\\
&- f\left(t_1 a+(1-t_1)\frac{a+b}{2}, s_1 d+(1-s_1)\frac{c+d}{2}\right)\\
&- f\left(t_1 b+(1-t_1)\frac{a+b}{2}, s_1 d+(1-s_1)\frac{c+d}{2}\right)\Bigg| .
\end{aligned}
$$

By using the triangle inequality, we get

$$
\begin{aligned}
&|G(t_2,s_2)-G(t_1,s_1)|\\
&\le \frac{1}{4}\left| f\left(t_2 a+(1-t_2)\frac{a+b}{2}, s_2 c+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 a+(1-t_1)\frac{a+b}{2}, s_1 c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&+\left| f\left(t_2 b+(1-t_2)\frac{a+b}{2}, s_2 c+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 b+(1-t_1)\frac{a+b}{2}, s_1 c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&+\left| f\left(t_2 a+(1-t_2)\frac{a+b}{2}, s_2 d+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 a+(1-t_1)\frac{a+b}{2}, s_1 d+(1-s_1)\frac{c+d}{2}\right)\right|\\
&+\left| f\left(t_2 b+(1-t_2)\frac{a+b}{2}, s_2 d+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 b+(1-t_1)\frac{a+b}{2}, s_1 d+(1-s_1)\frac{c+d}{2}\right)\right| .
\end{aligned}
$$

By using that f satisfies the Lipschitzian conditions, then we obtain

$$
\begin{aligned}
&\frac{1}{4}\left| f\left(t_2 a+(1-t_2)\frac{a+b}{2}, s_2 c+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 a+(1-t_1)\frac{a+b}{2}, s_1 c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&+\left| f\left(t_2 b+(1-t_2)\frac{a+b}{2}, s_2 c+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 b+(1-t_1)\frac{a+b}{2}, s_1 c+(1-s_1)\frac{c+d}{2}\right)\right|\\
&+\left| f\left(t_2 a+(1-t_2)\frac{a+b}{2}, s_2 d+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 a+(1-t_1)\frac{a+b}{2}, s_1 d+(1-s_1)\frac{c+d}{2}\right)\right|\\
&+\left| f\left(t_2 b+(1-t_2)\frac{a+b}{2}, s_2 d+(1-s_2)\frac{c+d}{2}\right)\right.\\
&\left. -f\left(t_1 b+(1-t_1)\frac{a+b}{2}, s_1 d+(1-s_1)\frac{c+d}{2}\right)\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{4}\left[L_1(b-a)\left|t_2-t_1\right|+L_2(d-c)\left|s_2-s_1\right|\right.\\
&\quad+L_3(b-a)\left|t_2-t_1\right|+L_4(d-c)\left|s_2-s_1\right|\\
&\qquad+L_5(b-a)\left|t_2-t_1\right|+L_6(d-c)\left|s_2-s_1\right|\\
&\qquad\quad\left.+L_7(b-a)\left|t_2-t_1\right|+L_8(d-c)\left|s_2-s_1\right|\right]\\
&=\frac{1}{4}\left[\left(L_1+L_2+L_3+L_4\right)(b-a)\left|t_2-t_1\right|\right.\\
&\quad\left.+\left(L_5+L_6+L_7+L_8\right)(d-c)\left|s_2-s_1\right|\right]
\end{aligned}$$

which implies that the mapping G is L-Lipschitzian on $[0,1]\times[0,1]$.

(iv) By using the convexity of G on $[0,1]\times[0,1]$, we have

$$\begin{aligned}
&f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\\
&\quad+f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\\
&\qquad+f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\\
&\qquad\quad\left.+f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\quad\leq tsf(a,c)+t(1-s)f\left(a,\frac{c+d}{2}\right)+(1-t)sf\left(\frac{a+b}{2},c\right)\\
&\qquad+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+tsf(b,c)+t(1-s)f\left(b,\frac{c+d}{2}\right)\\
&\qquad\quad+(1-t)sf\left(\frac{a+b}{2},c\right)+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\qquad\quad\quad+tsf(a,d)+t(1-s)f\left(a,\frac{c+d}{2}\right)+(1-t)sf\left(\frac{a+b}{2},d\right)\\
&\qquad\quad\quad\quad+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+tsf(b,d)+t(1-s)f\left(b,\frac{c+d}{2}\right)\\
&\qquad\quad\quad\quad\quad+(1-t)sf\left(\frac{a+b}{2},d\right)+(1-t)(1-s)f\left(\frac{a+b}{2},\frac{c+d}{2}\right).
\end{aligned}$$

By integrating both sides of the above inequality and taking into account the change of variables, we obtain

$$\begin{aligned}
\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \leq \frac{1}{4}&\left[\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\right.\\
&\left.+\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{2}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right].
\end{aligned}$$

This completes the proof. □

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