ON GENERALIZATIONS OF
THE CONCEPT OF CONVEXITY

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Abstract

In this article, the concept of convexity is considered. Generalizations of this concept, that were examined by different authors, are briefly given. In this paper, a study of two forms of abstract convexity is undertaken: one of which is B-convexity which was defined and analyzed by W. Briec and C. Horvath, the other being B^{-1}-convexity which is introduced and examined by us.

Keywords: Abstract convexity, B-convexity, B^{-1}-convexity.

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1. Introduction

Convexity is one of the most important topics in mathematics. It has many fields of applications. Convex functions play a central role in many branches of applied mathematics, especially optimization theory and the theory of inequalities. A large number of optimization methods use properties of convex functions. However most real life problems use a nonconvex mathematical model, since they represent the reality more accurately. In many cases, in spite of the nonconvexity of these functions, they retain some of the nice properties and characteristics of convex functions. This led to generalizations of the concept of convexity. A lot of authors has given and examined different generalizations of the classical concept of convexity (see [1]-[7] ,[10]-[12] and references therein). These generalizations has found important applications specially related to applied mathematics, economics, etc.

In this paper, we refer to abstract convexity which is a generalization of convexity. Abstract convexity is determined by two forms: Topological abstract convexity, which based on the fulfilment of some conditions related to a family of functions on a given set and its image set, Functional abstract convexity which is based on separability. In this

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work, B-convexity and B^{-1}-convexity are considered. B-convexity, which was introduced in [4] and studied in [4, 5, 6] by Briec W. and Horvath C., is one of these concepts. The other one is B^{-1}-convexity, that is defined and examined by us.

In section 2, some important definitions and theorems of classic convexity are given. In section 3, in general, two ways of abstracting convexity are discussed. In the fourth section, there are vital theorems and definitions associated with B-convexity. In section 5, the concept of B^{-1}-convexity is defined and theorems for this concept are proved.

2. Classic convexity

Let X be a vector space and $x_1, \ldots, x_m \in X$. A vector sum

$$\lambda_1 x_1 + \cdots + \lambda_m x_m$$

is called a *convex combination* of $x_1, \ldots, x_m$ if the coefficients λ_i are any non-negative real numbers and $\lambda_1 + \cdots + \lambda_m = 1$.

2.1. Definition. A subset of X is *convex* if it contains all the convex combinations of its elements.

Let f be a function whose values are real or $\pm\infty$ and whose domain is a subset S of X. The set

$$\{(x, \mu) : x \in S, \mu \in \mathbb{R}, \mu \geq f(x)\}$$

is called the *epigraph* of f and is denoted by epi(f). f is defined to be a *convex function* on S if epi(f) is convex as a subset of $X \times \mathbb{R}$.

2.2. Theorem. [9] *Let f be a function from S to $(-\infty, +\infty]$, where S is a convex subset of X. Then f is convex on S if, and only if,*

$$f\left((1 - \lambda)x + \lambda y\right) \leq (1 - \lambda)f(x) + \lambda f(y), \ 0 < \lambda < 1,$$

for every x and y in S.   □

2.3. Theorem. [9, Theorem 12.1] *A closed convex function (i.e. epi(f) is a closed set in $\mathbb{R}^{n+1}$) f is the pointwise supremum of the collection of all affine functions h such that $h \leq f$.*   □

3. Abstractions of convexity

There are two large branches of abstract convexity: functional abstract convexity and topological abstract convexity. Here, only a short description of these notions is given (see [12] for the definition, details and examples).

Topological abstract convexity: Let C be a set. For each positive integer $m \geq 2$, consider a set $V_m \subset \mathbb{R}^m$. Let a family of functions $\phi_m : C^m \times V_m \to C$ be given. It is assumed that there are some links between V_m with different m and between ϕ_m with different m. A set $U \subset C$ is called *abstract convex with respect to the family ϕ_m* if

$$(x_1, \ldots, x_m \in U, (\alpha_1, \ldots, \alpha_m) \in V_m) \implies \phi_m(x_1, \ldots, x_m, \alpha_1, \ldots, \alpha_m) \in U,$$
$$m = 2, 3, \ldots.$$

Functional abstract convexity: Let C be a set and H a set of functions $h : C \to \mathbb{R}$. A set $U \subset C$ is called *abstract convex with respect to H* if each point x that does not belong to U can be separated from U by a function from H, that is, there exist $h \in H$ such that $h(x) > sup_{u \in U} h(u)$. A function f is called *abstract convex with respect to H* if this function can be represented as the upper envelope of a subset of H. The set H is called a set of *elementary functions* in such a setting.

While defining the H-convexity of functions, the H-convexity of sets is used.

3.1. Definition. Let H be a set of functions $h : X \times \mathbb{R} \to \mathbb{R}$ and $f : X \to \mathbb{R}$ a function, where X is a vector space. If the set of epi$(f) = \{(x, \mu) : \mu \geq f(x),\ \mu \in \mathbb{R}\}$ is convex with respect to H, then f is called H-*convex*.

Using the Theorem 2.3, the concept of convexity of functions can be generalized as follows:

3.2. Definition. [10] Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and let H be a nonempty set of functions $h : X \to \overline{\mathbb{R}}$. A function $f : X \to \overline{\mathbb{R}}$ is called *abstract convex with respect to H* (or H-*convex*) if there exists a set $V \subset H$ such that f is the upper envelope of this set:

$$f(x) = \sup \{h(x) : h \in V\}$$

for all $x \in X$.

The concept of convexity is generalized in other ways and they can be represented as the above statements. In this sense, the concept of B-convexity, which is examined by W. Briec and C. Horvath, and B^{-1}-convexity, which is defined and analyzed by G. Adilov and I. Yesilce, are given in detail below.

4. B-convexity

For all $r \in N$ the map $x \mapsto \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}$ to itself; $x = (x_1, \ldots, x_n) \mapsto \Phi_r(x) = (\varphi_r(x_1), \ldots, \varphi_r(x_n))$ is a homeomorphism from $\mathbb{R}^n$ to itself. For a finite nonempty set $A = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n$, the r-convex hull of A, which is denoted by $\mathrm{Co}^r(A)$, is given by

$$\mathrm{Co}^r(A) = \left\{ \Phi_r^{-1} \left(\sum_{i=1}^m t_i \Phi_r(x_i) \right) : t_i \geq 0,\ \sum_{i=1}^m t_i = 1 \right\}.$$

The structure of B-convex sets, shortly to be defined, will involve the order structure with respect to the positive cone of $\mathbb{R}^n$; denoted by $\bigvee_{i=1}^m x_i$, the least upper bound of $x_1, \ldots, x_m \in \mathbb{R}^n$, that is:

$$\bigvee_{i=1}^m x_i = (\max \{x_{1,1}, \ldots, x_{m,1}\}, \ldots, \max \{x_{1,n}, \ldots, x_{m,n}\}).$$

The limit hull of a finite set A is defined as the Kuratowski–Painleve upper limit of the sequence of sets $\{\mathrm{Co}^r(A)\}_{r \in N}$ (The Kuratowski–Painleve upper limit of the sequence of sets $\{A_n\}$ is $\bigcap_n \bigcup_k \overline{A_{n+k}}$; it is also the set of points p for which there exists an increasing sequence $\{n_k\}_{k \in N}$ and points $p_{n_k} \in A_{n_k}$ such that $p = \lim_{k \to \infty} p_{n_k}$) [8].

4.1. Definition. The Kuratowski–Painleve upper limit of the sequence of sets $(\mathrm{Co}^r(A))_{r \in N}$, where A is finite, will be denoted by $\mathrm{Co}^\infty(A)$, and $\mathrm{Co}^\infty(A)$ is called the B-*polytope* of A.

It can be shown that in $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_i \geq 0,\ i = \overline{1, n}\}$, the upper-limit is in fact a limit and that elements of $\mathrm{Co}^\infty(A)$ have a simple analytic description:

4.2. Theorem. *For all nonempty finite subset $A = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n_+$ we have*

$$\mathrm{Co}^\infty(A) = \mathrm{Lim}_{r \to \infty} \mathrm{Co}^r(A) = \left\{ \bigvee_{i=1}^m t_i x_i : t_i \in [0, 1],\ \max_{1 \leq i \leq m} \{t_i\} = 1 \right\}. \qquad \square$$

4.3. Definition. A subset S of $\mathbb{R}^n$ is B-*convex* if for all finite subsets $A \subset S$ the B-polytope $\mathrm{Co}^\infty(A)$ is contained in S.

The following theorem can be proved by using Theorem 4.2 (see [5]) and consequently, for subsets of $\mathbb{R}^n_+$, B-convexity can be defined in a different way.

4.4. Theorem. *A subset S of $\mathbb{R}^n_+$ is B-convex if, and only if, for all $x_1, x_2 \in S$ and all $\lambda \in [0,1]$ one has $\lambda x_1 \vee x_2 \in S$.*

In the following, there are some properties of B-convex sets:

(1) The empty set, $\mathbb{R}^n$, as well as all the singletons are B-convex;
(2) If $\{S_\lambda : \lambda \in \Lambda\}$ is an arbitrary family of B-convex sets, then $\bigcap_\lambda S_\lambda$ is B-convex;
(3) If $\{S_\lambda : \lambda \in \Lambda\}$ is a family of B-convex sets such that $\forall \lambda_1, \lambda_2 \in \Lambda, \exists \lambda_3 \in \Lambda$ such that $S_{\lambda_1} \cup S_{\lambda_2} \subset S_{\lambda_3}$, then $\bigcup_\lambda S_\lambda$ is B-convex.

Given a set $S \subset \mathbb{R}^n$, the intersection of all the B-convex subsets of $\mathbb{R}^n$ containing S is called the *B-convex hull of S*. A large number of the theorems about the B-convex hull have been proved in [4].

Furthermore, some topological properties of B-convex sets are studied in [4]:

4.5. Theorem. *If S is a B-convex subset of $\mathbb{R}^n_+$, then $U_\infty(S, \delta) = \{x : \|y - x\|_\infty < \delta, y \in S\}$ is also B-convex.* $\qquad\square$

It can be shown that the closure of a B-convex subset of $\mathbb{R}^n_+$ is B-convex by using Theorem 4.5. Also a theorem about the interior of a B-convex set is proved [4]:

4.6. Theorem. *The interior of a B-convex subset of $\mathbb{R}^n_+$ is B-convex.*

5. B^{-1}-convexity

For $r \in Z^-$, where Z^- is the set of negative integers, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $K = \mathbb{R} \backslash \{0\}$ to itself; likewise $x = (x_1, x_2, \ldots, x_n) \mapsto \Phi_r(x) = (\varphi_r(x_1), \varphi_r(x_2), \ldots, \varphi_r(x_n))$ is homeomorphism from K^n to itself.

For a finite nonempty set $A = \{x_1, x_2, \ldots, x_m\} \subset K^n$, the Φ_r-*convex hull* (shortly, *r-convex hull*) of A, which we denote by $\mathrm{Co}^r(A)$ is given by

$$\mathrm{Co}^r(A) = \left\{ \Phi_r^{-1}\left(\sum_{i=1}^m t_i \Phi_r(x_i) \right) : t_i \geq 0, \ \sum_{i=1}^m t_i = 1 \right\}.$$

We denote by $\bigwedge_{i=1}^m x_i$ the greatest lower bound of $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$, that is:

$$\bigwedge_{i=1}^m x_i = \left(\min\{x_{1,1}, \ldots, x_{m,1}\}, \ldots, \min\{x_{1,n}, \ldots, x_{m,n}\} \right).$$

Now let us define the B^{-1}-polytopes.

5.1. Definition. The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r \in Z^-}$, where A is finite, will be denoted by $\mathrm{Co}^{-\infty}(A)$; that is the set of points $x \in K^n$ for which there exist a decreasing sequence $\{r_k\}_{k \in N}$ and points $x_{r_k} \in \mathrm{Co}^{r_k}(A)$ such that $x = \lim_{k \to \infty} x_{r_k}$. And $\mathrm{Co}^{-\infty}(A)$ is called the B^{-1}-*polytope* of A.

For B^{-1}-convexity, Theorem 4.2 has the following form.

5.2. Theorem. *For all nonempty finite subsets $A = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n_{++}$ we have*

$$\mathrm{Co}^{-\infty}(A) = \lim_{r \to -\infty} Co^r(A) = \left\{ \bigwedge_{i=1}^m t_i x_i : t_i \geq 1, \ \min_{1 \leq i \leq m} t_i = 1 \right\}.$$

To prove this theorem, we use Lemma 5.3.

5.3. Lemma.

a) *For any decreasing sequence of negative integer numbers* $\{r_k\}_{k\in N}$ *and for any* $a \in K^m$

$$\lim_{k\to\infty} \left(\sum_{i=1}^{m} |a_i|^{r_k} \right)^{\frac{1}{r_k}} = \min_{i=1,\ldots,m} |a_i|.$$

b) *If* $\left\{ a^{(k)} \right\}_{k\in N}$ *is a sequence in* K^m *which converges to* $a = (a_1, a_2, \ldots, a_m) \in K^m$ *and if* $\{r_k\}_{k\in N}$ *is a decreasing sequence of negative integers, then*

$$\lim_{k\to\infty} \left(\sum_{i=1}^{m} \left| a_i^{(k)} \right|^{r_k} \right)^{\frac{1}{r_k}} = \min_{i=1,\ldots,m} |a_i|.$$

c) *If* $x_1, x_2, \ldots, x_m \in \mathbb{R}^n_{++}$, *then given a convergent sequence of positive real numbers* $\{\lambda_{k,i}\}_{k\in N}$, $i = 1, \ldots, m$, *whose limits are respectively* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *and a decreasing sequence of negative integers* $\{r_k\}_{k\in N}$,

$$\lim_{k\to\infty} \left(\lambda_{k,1} x_1 \overset{r_k}{+} \lambda_{k,2} x_2 \overset{r_k}{+} \cdots \overset{r_k}{+} \lambda_{k,m} x_m \right) = \bigwedge_{i=1}^{m} \lambda_i x_i.$$

This lemma can be shown easily.

Proof of Theorem 5.2 First we show that

$$\left\{ \bigwedge_{i=1}^{m} t_i x_i : t_i \geq 1, \ \min_{1\leq i\leq m} t_i = 1 \right\} \subset Li_{r\to-\infty} \mathrm{Co}^r(A).$$

With $\lambda_1, \lambda_2, \ldots, \lambda_m \in [1, \infty)$ and $\min_{1\leq i\leq m} \{\lambda_i\} = 1$, let $x = \lambda_1 x_1 \wedge \lambda_2 x_2 \wedge \cdots \wedge \lambda_m x_m$. If

$$y_r = \frac{1}{\lambda_1 \overset{r}{+} \lambda_2 \overset{r}{+} \cdots \overset{r}{+} \lambda_m} \left(\lambda_1 \overset{r}{\cdot} x_1 \overset{r}{+} \lambda_2 \overset{r}{\cdot} x_2 \overset{r}{+} \cdots \overset{r}{+} \lambda_m \overset{r}{\cdot} x_m \right)$$

then $y_r \in \mathrm{Co}^r(A)$. Since $x_1, x_2, \ldots, x_m \in \mathbb{R}^n_{++}$ and (see Lemma 5.3(a))

$$\lim_{r\to-\infty} \left(\lambda_1 \overset{r}{+} \lambda_2 \overset{r}{+} \cdots \overset{r}{+} \lambda_m \right) = \lim_{r\to-\infty} \left(\sum_{i=1}^{m} \lambda_i^{2r+1} \right)^{\frac{1}{2r+1}} = \min_{1\leq i\leq m} \lambda_i$$

we obtain that (see Lemma 5.3(c))

$$\lim_{r\to-\infty} y_r = \lim_{r\to-\infty} \left(\lambda_1 \overset{r}{\cdot} x_1 \overset{r}{+} \lambda_2 \overset{r}{\cdot} x_2 \overset{r}{+} \cdots \overset{r}{+} \lambda_m \overset{r}{\cdot} x_m \right)$$

$$= \lambda_1 x_1 \wedge \lambda_2 x_2 \wedge \cdots \wedge \lambda_m x_m = x.$$

Next, let us verify that

$$Ls_{r\to-\infty} \mathrm{Co}^r(A) \subset \left\{ \bigwedge_{i=1,\ldots,m} t_i x_i : t_i \in [1, \infty), \ \min_{1\leq i\leq m} t_i = 1 \right\}.$$

Take $x \in Ls_{r\to-\infty} \mathrm{Co}^r(A)$; there is a decreasing sequence $\{r_k\}_{k\in N}$ and a sequence $\{p_k\}_{k\in N}$ of points such that $p_k \in \mathrm{Co}^{r_k}(A)$ and $\lim_{k\to\infty} p_k = x$, therefore

$$p_k = (p_{k,1}, p_{k,2}, \ldots, p_{k,n}) = \lambda_{k,1} \overset{r_k}{\cdot} x_1 \overset{r_k}{+} \lambda_{k,2} \overset{r_k}{\cdot} x_2 \overset{r_k}{+} \cdots \overset{r_k}{+} \lambda_{k,m} \overset{r_k}{\cdot} x_m$$

$$= \left(\left[\sum_{i=1}^{m} \lambda_{k,i}^{2r_k+1} x_{i,1}^{2r_k+1} \right]^{\frac{1}{2r_k+1}}, \ldots, \left[\sum_{i=1}^{m} \lambda_{k,i}^{2r_k+1} x_{i,n}^{2r_k+1} \right]^{\frac{1}{2r_k+1}} \right).$$

Since $\lambda_k = (\lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,m}) \in [1, \infty)^m$ we can presume that the sequence $\{\lambda_k\}_{k \in N}$ converges to a point $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \in [1, \infty)^m$. Furthermore

$$\lim_{k \to \infty} \left(\sum_{i=1}^{m} \lambda_{k,i}^{2r_k+1} \right)^{\frac{1}{2r_k+1}} = \min_{1 \leq i \leq m} \{\lambda_i^*\}$$

and, from $\sum_{i=1}^{m} \lambda_{k,i}^{2r_k+1} = 1$ we have

$$\min_{1 \leq i \leq m} \{\lambda_i^*\} = 1.$$

Consequently, (see Lemma 5.3(b))

$$\lim_{k \to \infty} p_{k,j} = \lim_{k \to \infty} \left(\sum_{i=1}^{m} \lambda_{k,i}^{2r_k+1} x_{i,j}^{2r_k+1} \right)^{\frac{1}{2r_k+1}} = \min_{1 \leq i \leq m} \{\lambda_i^* x_{i,j}\},$$

we have shown that $x = \bigwedge_{i=1}^{m} \lambda_i^* x_i$, with $\min_{1 \leq i \leq m} \{\lambda_i^*\} = 1$. Hence, we obtain that

$$Ls_{r \to -\infty} Co^r(A) \subset \left\{ \bigwedge_{i=1,..,m} t_i x_i : t_i \in [1, \infty), \min_{1 \leq i \leq m} t_i = 1 \right\} \subset Li_{r \to -\infty} Co^r(A).$$

Since $Li_{r \to -\infty} Co^r(A) \subset Ls_{r \to -\infty} Co^r(A)$ is always valid, the theorem is proved.   □

We can define B^{-1}-convex sets as follows:

5.4. Definition. A subset S of K^n is called B^{-1}-*convex* if for all finite subsets $A \subset S$ the B^{-1}-polytope $Co^{-\infty}(A)$ is contained in S.

By Theorem 5.2, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$ as follows:

5.5. Theorem. *A subset S of $\mathbb{R}^n_{++}$ is B^{-1}-convex if, and only if, for all $x_1, x_2 \in S$ and all $\lambda \in [1, \infty)$ one has $\lambda x_1 \wedge x_2 \in S$.*   □

The proofs of the following obvious, but nonetheless important propositions, are left to the reader.

5.6. Theorem.
   (a) *The emptyset, K^n, as well as the singletons are B^{-1}-convex;*
   (b) *If $\{S_\lambda : \lambda \in \Lambda\}$ is an arbitrary family of B^{-1}-convex sets, then $\bigcap_\lambda S_\lambda$ is B^{-1}-convex;*
   (c) *If $\{S_\lambda : \lambda \in \Lambda\}$ is a family of B^{-1}-convex sets such that $\forall \lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_3 \in \Lambda$ such that $S_{\lambda_1} \bigcup S_{\lambda_2} \subset S_{\lambda_3}$, then $\bigcup_\lambda S_\lambda$ is B^{-1}-convex.*   □

5.7. Definition. Given a set $S \subset K^n$, the intersection of all the B^{-1}-convex subsets of K^n containing S is called the B^{-1}-*convex hull* of S and we denote it by $\mathbb{B}^{-1}[S]$.

The next theorem can be shown easily.

5.8. Theorem. *The following properties hold:*
   (a) *$\mathbb{B}^{-1}[\emptyset] = \emptyset$, $\mathbb{B}^{-1}[K^n] = K^n$ for all $x \in K^n$, $\mathbb{B}^{-1}[\{x\}] = \{x\}$;*
   (b) *For all $S \subset K^n$, $S \subset \mathbb{B}^{-1}[S]$ and $\mathbb{B}^{-1}\left[\mathbb{B}^{-1}[S]\right] = \mathbb{B}^{-1}[S]$;*
   (c) *For all $S_1, S_2 \subset K^n$, if $S_1 \subset S_2$ then $\mathbb{B}^{-1}[S_1] \subset \mathbb{B}^{-1}[S_2]$;*
   (d) *For all $S \subset K^n$, $\mathbb{B}^{-1}[S] = \bigcup \{\mathbb{B}^{-1}[A] : A$ is a finite subset of $S\}$;*
   (e) *A subset $S \subset K^n$ is B^{-1}-convex if, and only if, for all finite subsets A of S, $\mathbb{B}^{-1}[A] \subset S$.*   □

We show some topological properties of B^{-1}-convex sets:

If S is B-convex, then all δ neighborhoods $U_\infty(S, \delta)$ of S are B-convex (Theorem 4.5). This theorem does not hold for B^{-1}-convexity. That is, the δ neighborhood $U_\infty(S, \delta)$ of a B^{-1}-convex set $S \subset \mathbb{R}^n_{++}$ need not be a B^{-1}-convex set, as can be seen from the following example:

5.9. Example. Let $S \subset \mathbb{R}^2_{++}$ be the line segment between $(2, 2)$ and $(5, 5)$ and $U_\infty(S, \delta)$ the δ neighborhood of S for $\delta = 1.1$. Then, S is B^{-1}-convex set but $U_\infty(S, \delta)$ is not a B^{-1}-convex set. Indeed, $x = (1, 3) \in U_\infty(S, \delta)$ and $y = (4, 6) \in U_\infty(S, \delta)$; however for $\lambda = 2$, $z = \lambda x \wedge y = (2, 6) \wedge (4, 6) = (2, 6) \notin U_\infty(S, \delta)$.

5.10. Theorem. *The closure of a B^{-1}-convex subset of $\mathbb{R}^n_{++}$ is B^{-1}-convex.*

Proof. Let S be a B^{-1}-convex subset of $\mathbb{R}^n_{++}$ and let $\bar{S}$ be the closure of S. If $x = \lim_{k \to \infty} x_k$ and $y = \lim_{k \to \infty} y_k$ with $x_k, y_k \in S$ for all $k \in N$, then, since $(x, y) \mapsto \lambda x \wedge y$ is continuous, for all $\lambda \in [1, \infty)$, $\lambda x \wedge y = \lim_{k \to \infty} \lambda x_k \wedge y_k \in \bar{S}$. $\qquad\square$

5.11. Theorem. *The interior of a B^{-1}-convex subset of $\mathbb{R}^n_{++}$ is B^{-1}-convex.*

Proof. Let S be a B^{-1}-convex subset of $\mathbb{R}^n_{++}$ with nonempty interior; if x_1, x_2 are in $\text{int}(S)$ there are open sets W_1, W_2 of $\mathbb{R}^n_{++}$ such that $x_i \in W_i \subset S, i = 1, 2$. Fix $\rho_1, \rho_2 \in [1, \infty)$ such that $\min\{\rho_1, \rho_2\} = 1$ and let

$$\rho_1 W_1 \wedge \rho_2 W_2 = \{\rho_1 z_1 \wedge \rho_2 z_2 : z_1 \in W_1, z_2 \in W_2\}.$$

We have $\rho_1 x_1 \wedge \rho_2 x_2 \in \rho_1 W_1 \wedge \rho_2 W_2 \subset S$; we show that $\rho_1 W_1 \wedge \rho_2 W_2$ is open.

$\rho_1 W_1$ and $\rho_2 W_2$ are open, since $x \mapsto \rho_i x, i = 1, 2$ are homeomorphisms of $\mathbb{R}^n_{++}$ onto themselves, let $\rho_1 W_1 = U$ and $\rho_2 W_2 = V$. We have reduced the general proof to the proof of the following statement: if U and V are open subsets of $\mathbb{R}^n_{++}$ then $U \wedge V$ is also open in $\mathbb{R}^n_{++}$. Let us show that this is the case. Let $x = y \wedge z$ with $y \in U$ and $z \in V$; there exists $\delta > 0$ such that $U_\infty(y, \delta) \subset U$ and $U_\infty(z, \delta) \subset V$; we have to find $\epsilon > 0$ such that $U_\infty(x, \epsilon) \subset U \wedge V$.

For all $i = 1, \ldots, n$ we have $x_i = \min\{y_i, z_i\}$; we distinguish two cases (three by symmetry):

(1) $x_i = y_i = z_i$. If $|x'_i - x_i| < \delta$ we can find y'_i and z'_i such that $|y'_i - y_i| < \delta$, $|z'_i - z_i| < \delta$ and $x'_i = \min\{y'_i, z'_i\}$; simply take $x'_i = y'_i = z'_i$.

(2) $x_i = y_i < z_i$. If $|x'_i - x_i| < \min\{\delta, 2^{-1}(z_i - y_i)\}$, then with $y'_i = x'_i$ and $z'_i = z_i$ we have $x'_i = \min\{y'_i, z'_i\}$, $|y'_i - y_i| < \delta$ and $|z'_i - z_i| < \delta$.

Put $I(y) = \{i : y_i < z_i\}$, $I(z) = \{i : z_i < y_i\}$ and

$$\epsilon = \min\{\delta, 2^{-1}(z_i - y_i), 2^{-1}(y_j - z_j) : (i, j) \in I(y) \times I(z)\};$$

we have shown that $U_\infty(x, \epsilon) \subset U_\infty(y, \delta) \wedge U_\infty(z, \delta) \subset U \wedge V$. $\qquad\square$

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