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ON THE "WEIGHTED" SCHRÖDINGER OPERATOR WITH POINT δ-INTERACTIONS[†]

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Abstract

The number of negative eigenvalues of the "weighted" Schrödinger operator with point δ-interactions are found and by means of the Floquet theory, stability or instability of the solutions to periodic "weighted" equations with δ-interactions are determined.

Keywords: "Weighted" Schrödinger operator, Point δ-interactions, Negative eigenvalues.

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1. Introduction and main results

Problems on the study of the Schrödinger operator with short interaction potential (of δ-function type) have appeared in the physical literature. Mathematical investigations of appropriate physical models were initiated at the beginning of the sixties in the papers [2, 9]. This theme has developed intensively in the last three decades. There is a monograph [1] where one can be acquainted with details of the Berezin-Minlos-Faddeev theory in its contemporary state and other new directions arising from this theory. In the same place, one can find a wide bibliography.

We use the following notation: $\mathbb{C}^{(n)}(a,b)$ is the linear space of scalar complex-valued functions which are n-times continuously differentiable on (a,b), $L_2(a,b)$ is the linear space of scalar complex-valued functions on (a,b), which have square summable modules, m is in $\mathbb{N}$ and fixed, $x_0 = -\infty$, and $x_{m+1} = +\infty$.

The "weighted" one-dimensional Schrödinger operator L^q_ρ(or $L^q_{X,\alpha}$) with a point δ-interaction on a finite set $X = \{x_1, x_2, \ldots, x_m\}$ with intensities $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is defined by the differential expression

$$(1.1) \qquad \ell^q_\rho[y] \equiv \frac{1}{\rho(x)} \frac{d}{dx}\left(\rho(x)\frac{dy}{dx}\right) + q(x)y$$

[†]Dedicated to the memory of M. G. Gasymov.

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on functions $y(x)$ that belong to the space $L_2(-\infty,\infty)$, where the "weighted" function $\rho(x) = 1 + \sum\limits_{k=1}^{m} \alpha_k \delta(x - x_k)$ and $q(x)$ is a scalar real-valued nonnegative function on $(-\infty,\infty)$ such that

$$\int\limits_{-\infty}^{\infty} \left(1 + x^2\right) q(x)\, dx < \infty.$$

In this formula, $\alpha_k > 0$, $x_k (x_1 < x_2 < \cdots < x_m)$ $(k = 1, 2, , m = \overline{1, m})$ are real numbers. Note that in [8, 5] the inverse problem of the operator $L^0_\rho(q(x) \equiv 0)$, with the function $\rho(x) = \rho_1^2(x)$ satisfying $\frac{\rho_1'(x)}{\rho_1(x)} \in L_2(0, 1)$ is investigated.

The operator L^q_ρ is self-adjoint on $L_2(-\infty,\infty)$.

Here, the approach is based on the idea of approximation of the generalized "weight" with smooth "weight"s.

Consider the differential expression

$$\ell^q_{\rho_\varepsilon}[y] \equiv -\frac{1}{\rho_\varepsilon(x)} \frac{d}{dx} \left(\rho_\varepsilon(x) \frac{dy}{dx}\right) + q(x)y,$$

where the density function

$$\rho_\varepsilon(x) = 1 + \frac{1}{\varepsilon} \sum_{k=1}^{m} \alpha_k \chi_\varepsilon(x - x_k),$$

is defined using the characteristic function

$$\chi_\varepsilon(x) = \begin{cases} 1, & \text{for } x \in [0,\varepsilon], \\ 0, & \text{for } x \notin [0,\varepsilon], \end{cases} \qquad \varepsilon < \min_{i=\overline{2,m}} \{x_i - x_{i-1}\}.$$

Notice that the density function $\rho_\varepsilon(x)$ is chosen so that it converges to $\rho(x)$ as $\varepsilon \to 0^+$ (see [12]). Therefore, the approximation equation is of the form:

$$\ell^q_{\rho_\varepsilon}[y] = \lambda y \ \ -\infty < x < \infty. \tag{1.2}$$

Agree that the solution of equation (1.2) is any function $y(x)$ determined on $(-\infty,\infty)$ for which the following conditions are fulfilled:

1) $y(x) \in \mathbb{C}^2(x_k, x_k + \varepsilon) \cap \mathbb{C}^2(x_k + \varepsilon, x_{k+1})$ for $k = \overline{0, m}$;
2) $-y''(x) + q(x)y(x) = \lambda y(x)$ $\ x \in (x_k, x_k + \varepsilon) \cup (x_k + \varepsilon, x_{k+1})$, $k = \overline{0, m}$;
3) $y(x_k^+) = y(x_k^-)$, $(1 + \alpha_k \frac{1}{\varepsilon})y'(x_k^+) = y'(x_k^-)$ for $k = \overline{1, m}$;
4) $y((x_k+\varepsilon)^+) = y((x_k+\varepsilon)^-)$, $y'((x_k+\varepsilon)^+) = (1+\alpha_k \frac{1}{\varepsilon})y'((x_k+\varepsilon)^-)$ for $k = \overline{1, m}$.

These conditions guarantee that the functions $y(x)$ and $\rho_\varepsilon(x)y'(x)$ are continuous at the points x_k and $x_k + \varepsilon$, $(k = \overline{1, m})$.

The paper comprises two sections. Section 2 determines the spectrum operator L^q_ρ. Section 3 cover the basic Floquet theory, properties of the discriminant and the existence of the stability and instability intervals.

2. Nature of the spectrum of the operator L^q_ρ

In connection with important applications to problems of Quantum Mechanics (see [1]) it is of interest to study the spectral characteristics of the operator L^q_ρ.

It is well-known (see [10]) that the equation

$$-y''(x) + q(x)y(x) = \lambda y(x),\ x \in (-\infty,\infty)$$

has two linear independent solutions $\varphi_1(x,\lambda)$, $\varphi_2(x,\lambda)$, any solution of the equation $y(x,\lambda)$ has the following representation

$$y(x,\lambda) = C_1\varphi_1(x,\lambda) + C_2\varphi_2(x,\lambda),$$

where C_1, C_2 are some numbers, moreover for $\mathrm{Im}\lambda \neq 0$ or $\lambda < 0$

$$\int_{-\infty}^{0} |\varphi_2(x,\lambda)|^2\,dx < \infty, \quad \int_{0}^{\infty} |\varphi_1(x,\lambda)|^2\,dx < \infty,$$

$$\int_{-\infty}^{0} |\varphi_1(x,\lambda)|^2\,dx = \int_{0}^{\infty} |\varphi_2(x,\lambda)|^2\,dx = \infty.$$

Then we can write any solution of equation (1.2) in the form

$$y^\varepsilon(x,\lambda) = \begin{cases} C_1^\varepsilon\varphi_1(x,\lambda) + C_2^\varepsilon\varphi_2(x,\lambda), & \text{if } x \in (-\infty, x_1), \\ C_{4k-1}^\varepsilon\varphi_1(x,\lambda) + C_{4k}^\varepsilon\varphi_2(x,\lambda), & \text{if } x \in (x_k, x_k+\varepsilon),\ (k=\overline{1,m}), \\ C_{4k+1}^\varepsilon\varphi_1(x,\lambda) + C_{4k+2}^\varepsilon\varphi_2(x,\lambda), & \text{if } x \in (x_k+\varepsilon, x_{k+1}),\ (k=\overline{1,m-1}), \\ C_{4m+1}^\varepsilon\varphi_1(x,\lambda) + C_{4m+2}^\varepsilon\varphi_2(x,\lambda), & \text{if } x \in (x_m+\varepsilon, \infty), \end{cases}$$

where C_k^ε $(k=\overline{1,4m+2})$ are some constant numbers such that for $y(x,\lambda)$ conditions 3) and 4) are fulfilled.

Define the operator $L_{\rho_\varepsilon}^q$ generated in the Hilbert space $L_2(-\infty,\infty)$ by the differential expression $\ell_{\rho_\varepsilon}^q[y]$. The domain of definition of the operator $L_{\rho_\varepsilon}^q$ is the set of all functions belonging to $L_2(-\infty,\infty)$ and satisfying the conditions 1)–4).

Let R_λ^ε be the resolvent of the operator $L_{\rho_\varepsilon}^q$, and R_λ the resolvent of the operator $L_1^q(\alpha_k \equiv 0,\ k=\overline{1,m})$.

2.1. Theorem. *Let* $\mathrm{Im}\lambda \neq 0$*, then* $R_\lambda^\varepsilon - R_\lambda$ *is a finite-dimensional operator whose rank doesn't exceed 2m.*

Proof. We construct the resolvent of the operator $L_{\rho_\varepsilon}^q$ for $\mathrm{Im}\lambda \neq 0$. For that we solve in $L_2(-\infty,\infty)$ the problem

$$\text{(2.1)}\qquad \begin{cases} -y''(x) + q(x)y(x) = \lambda y(x) + F(x), x \neq x_k, x_k+\varepsilon\ \left(k=\overline{1,m}\right) \\ y(x_k^+) = y(x_k^-),\ (1+\alpha_k\frac{1}{\varepsilon})y'(x_k^+) = y'(x_k^-)\ (k=\overline{1,m}) \\ y((x_k+\varepsilon)^+) = y((x_k+\varepsilon)^-)\ \left(k=\overline{1,m}\right), \\ y'((x_k+\varepsilon)^+) = (1+\alpha_k\frac{1}{\varepsilon})y'((x_k+\varepsilon)^-)\ \left(k=\overline{1,m}\right), \end{cases}$$

where $F(x)$ is an arbitrary function belonging to $L_2(-\infty,\infty)$.

By the Lagrange method (see [10]) the solution of problem (2.1) takes the form

$$y^\varepsilon(x,\lambda) = -\frac{1}{W[\varphi_1,\varphi_2]}\int_{-\infty}^{\infty} R(x,t;\lambda)F(t)\,dt$$
$$-\frac{1}{W[\varphi_1,\varphi_2]}\begin{cases} b_2^\varepsilon\varphi_2(x,\lambda), & -\infty < x < x_1 \\ b_{4k-1}^\varepsilon\varphi_1(x,\lambda) + b_{4k}^\varepsilon\varphi_2(x,\lambda), & x_k < x < x_k+\varepsilon,\ \left(k=\overline{1,m}\right) \\ b_{4k+1}^\varepsilon\varphi_1(x,\lambda) + b_{4k+2}^\varepsilon\varphi_2(x,\lambda), & x_k+\varepsilon < x < x_{k+1},\ \left(k=\overline{1,m-1}\right) \\ b_{4m+1}^\varepsilon\varphi_1(x,\lambda), & x_m < x < \infty, \end{cases}$$

where

$$R(x,,t;\lambda)=\begin{cases}\varphi_1(x,\lambda)\varphi_2(t,\lambda), & t\le x\\ \varphi_1(t,\lambda)\varphi_2(x,\lambda), & t\ge x\end{cases}$$

and b_j^ε $(j=\overline{2,4m+1})$ are arbitrary numbers.

Write

$$\varphi_{j,k}=\varphi_j(x_k,\lambda),\ \varphi'_{j,k}=\varphi'_j(x_k,\lambda),\ \varphi'_{j,k+\varepsilon}=\varphi_j(x_k+\varepsilon,\lambda),\ \varphi'_{j,k+\varepsilon}=\varphi'_j(x_k+\varepsilon,\lambda);$$

$$R'_h(F)=\begin{cases}\int\limits_{-\infty}^{\infty} R(x_k,\,,t;\lambda)F(t)\,dt, & \text{if } h=2k-1,\\ \int\limits_{-\infty}^{\infty} R(x_k+\varepsilon,t;\lambda)F(t)\,dt, & \text{if } h=2k,\end{cases}$$

$$A_h=\begin{cases}-\alpha_k, & h=2k-1,\\ \alpha_k, & h=2k,\end{cases}\quad \left(k=\overline{1,m}\right);\ D^\varepsilon(\lambda)=\det(M^\varepsilon_{4m}(\lambda)),$$

where $M^\varepsilon_{4m}(\lambda)=$

$$\begin{bmatrix}
-\varphi_{2,1} & \varphi_{1,1} & \varphi_{2,1} & 0 & 0 & \cdots\\
-\varphi'_{2,1} & (1+\frac{\alpha_1}{\varepsilon})\varphi'_{1,1} & (1+\frac{\alpha_1}{\varepsilon})\varphi'_{2,1} & 0 & 0 & \cdots\\
0 & -\varphi_{1,1+\varepsilon} & -\varphi_{2,1+\varepsilon} & \varphi_{1,1+\varepsilon} & \varphi_{2,1+\varepsilon} & \cdots\\
0 & -(1+\frac{\alpha_1}{\varepsilon})\varphi'_{1,1+\varepsilon} & -(1+\frac{\alpha_1}{\varepsilon})\varphi'_{2,1+\varepsilon} & -\varphi'_{1,1+\varepsilon} & \varphi_{2,1+\varepsilon} & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & -\varphi_{1,m} & -\varphi_{2,m} & \varphi_{1,m} & \varphi_{2,m} & 0\\
\cdots & -\varphi'_{1,m} & -\varphi'_{2,m} & (1+\frac{\alpha_m}{\varepsilon})\varphi'_{1,m} & (1+\frac{\alpha_m}{\varepsilon})\varphi'_{2,m} & 0\\
\cdots & 0 & 0 & -\varphi_{1,m+\varepsilon} & -\varphi_{2,m+\varepsilon} & \varphi_{1,m+\varepsilon}\\
\cdots & 0 & 0 & -(1+\frac{\alpha_m}{\varepsilon})\varphi'_{1,m+\varepsilon} & -(1+\frac{\alpha_m}{\varepsilon})\varphi'_{2,m+\varepsilon} & \varphi'_{1,m+\varepsilon}
\end{bmatrix}$$

Then for defining the number b_j^ε, from the conditions of problem (2.1) we get the system $M^\varepsilon_{4m}(\lambda)B^\varepsilon=\frac{1}{\varepsilon}AR'$, where $B^\varepsilon=\operatorname{col}(b_2^\varepsilon,b_3^\varepsilon,\dots,b^\varepsilon_{4m+1})$ $AR'=\operatorname{col}(0,A_1R'_1,0,A_2R'_2,\dots,$ $0,A_{2m}R'_{2m})$.

Define the set $\Gamma=\{\lambda:\operatorname{Im}\lambda\ne 0,\ D^\varepsilon(\lambda)=0\}$. For $\lambda\notin\Gamma$ we have

$$b_j^\varepsilon=\frac{1}{\varepsilon D^\varepsilon(\lambda)}\sum_{p=1}^{2m}A_pR'_pM^\varepsilon_{4m,2p,j}(\lambda),$$

where $M^\varepsilon_{4m,2p,j}(\lambda)$ is an algebraic complement of the element $m_{i,j}$ of the matrix $M^\varepsilon_{4m}(\lambda)=(m_{i,j})_{4m\times 4m}$. If we introduce the denotation

$$X_p^\varepsilon(x,\lambda)=\begin{cases}A_1M^\varepsilon_{4m,2p,1}(\lambda)\,\varphi_2(x,\lambda), & x\in(-\infty,x_1),\\ A_k\left[M^\varepsilon_{4m,2p,4k-2}(\lambda)\,\varphi_1(x,\lambda)\right.\\ \qquad\left.+M^\varepsilon_{4m,2p,4k-1}(\lambda)\,\varphi_2(x,\lambda)\right], & x\in(x_k,x_k+\varepsilon)\left(n=\overline{1,m}\right),\\ A_k\left[M^\varepsilon_{4m,2p,4k}(\lambda)\,\varphi_1(x,\lambda)\right.\\ \qquad\left.+M^\varepsilon_{4m,2p,4k+1}(\lambda)\,\varphi_2(x,\lambda)\right], & x\in(x_k+\varepsilon,x_{k+1})\left(n=\overline{1,m-1}\right),\\ A_mM^\varepsilon_{4m,2p,4m}(\lambda)\,\varphi_1(x,\lambda), & x\in(x_m+\varepsilon,\infty),\end{cases}$$

for $p=\overline{1,m}$, then the solution of problem (2.1) takes the form

$$R^\varepsilon_\lambda(F)\equiv y^\varepsilon(x,\lambda)$$

$$=-\frac{1}{W[\varphi_1,\varphi_2]}\left[\int\limits_{-\infty}^{\infty}R(x,t;\lambda)\,F(t)\,dt+\frac{1}{\varepsilon D^\varepsilon(\lambda)}\sum_{p=1}^{2m}X_p^\varepsilon(x,\lambda)\,R_p(F)\right],\tag{2.2}$$

where

$$X_p^\varepsilon(\,.\,,\lambda)\in L^2(-\infty,\infty)\left(p=\overline{1,2m}\right),\operatorname{Im}\lambda\ne 0,\lambda\notin\Gamma$$

as $\varepsilon \to 0^+$ form expression (2.2) the finite-dimensionality of the operator $R_\lambda^\varepsilon - R_\lambda$ follows and its rank does not exceed $2m$. □

Since the operator L_ρ^q is self-adjoint, consequently its spectrum is real.

2.2. Theorem. *Let all intensities of the δ-interactions $\alpha_k > 0, k = \overline{1,m}$. Then the spectrum of the operator L_ρ^q consists of the absolutely continuous part $[0,+\infty)$ and has exactly m distinct eigenvalues on the negative half-line, that are determined as roots of the equation $D^\varepsilon(\lambda) = 0$, $(\varepsilon \to 0^+)$.*

Proof. By the conditions

$$\int_{-\infty}^{\infty} \left(1+x^2\right) q(x)\, dx < \infty \text{ and } q(x) \geq 0,$$

the spectrum of the operator $L_1^q \left(\alpha_k \equiv 0,\ k = \overline{1,m}\right)$ is absolutely continuous and coincides with the set $[0,+\infty)$. Since the operator $R_\rho^q - R_1^q$ is finite dimensional then according to the known results of [3, 6], the absolutely continuous part of the spectrum of the operator L_ρ^q coincides with the absolutely continuous part of the spectrum of the operator $L_1^q \left(\alpha_k \equiv 0,\ k = \overline{1,m}\right)$, i.e. with $[0,+\infty)$. According to [7], the spectrum of the operator L_ρ^q may differ from the spectrum of the operator $L_1^q \left(\alpha_k \equiv 0, k = \overline{1,m}\right)$ only by finitely many negative eigenvalues. Furthermore, the number of these eigenvalues is exactly m. □

3. On Floquet's solutions for a periodic "weight" equation

In this section we will state the Floquet theory (see [4]) for the equation

$$\ell_\rho^q[y] = \lambda y,\ -\infty < x < \infty \tag{3.1}$$

that clarifies the structure of the space of solutions of this equation for each complex value of the parameter λ. Notice that the "weight" function $\rho(x) = 1 + \alpha \sum_{n=-\infty}^{\infty} \delta(x - Nn)$ and the coefficient $q(x)$ is a real valued periodic continuous function with a period equal to N, $\alpha \neq 0$ and $N \geq 1$ are real and natural numbers, respectively. The spectral analysis of this equation in the case $\alpha \equiv 0$ was stated in detail in [4, 11].

3.1. Definition. For the given real value of the parameter λ, equation (3.1) is said to be *stable* if all its solutions are bounded on the axis $(-\infty,\infty)$, *instable* if all its solutions are not bounded on the axis $(-\infty,\infty)$, *conditionally stable* if the has at least one non-trivial solution bounded on the whole of the axis $(-\infty,\infty)$.

Consider the differential expression

$$\ell_{\rho_\varepsilon}^q[y] \equiv -\frac{1}{\rho_\varepsilon(x)} \frac{d}{dx}\left(\rho_\varepsilon(x) \frac{dy}{dx}\right) + q(x)\, y.$$

Here, the density of the function

$$\rho_\varepsilon(x) = 1 + \frac{\alpha}{\varepsilon} \sum_{n=-\infty}^{\infty} \chi_\varepsilon(x - Nn)$$

is determined by means of the characteristic function

$$\chi_\varepsilon(x) = \begin{cases} 1, & \text{for } x \in [0,\varepsilon], \\ 0, & \text{for } x \notin [0,\varepsilon],\ \varepsilon \ll N. \end{cases}$$

Notice that the density of the function $\rho_\varepsilon(x)$ is chosen so that as $\varepsilon \to 0^+$ it approaches the function $\rho(x)$. The approximation equation is of the form:

(3.2) $\ell^q_{\rho_\varepsilon}[y] = \lambda y, -\infty < x < \infty.$

Agree that a solution of equation (3.2) is any function $y(x,\lambda)$ determined on $(-\infty, \infty)$ for which the following conditions are fulfilled.

(1) $y(x) \in \mathbb{C}^2(Nn, Nn+\varepsilon) \cap \mathbb{C}^2(Nn+\varepsilon, N(n+1))$ for $n \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$;
(2) $-y''(x) + q(x)\,y(x) = \lambda y(x)$ for $x \in (Nn, Nn+\varepsilon) \cup (Nn+\varepsilon, N(n+1)), n \in \mathbb{Z}$;
(3) $y\left((Nn)^+\right) = y\left((Nn)^-\right), \left(1+\frac{\alpha}{\varepsilon}\right) y\left((Nn)^+\right) = y\left((Nn)^-\right)$ for $n \in \mathbb{Z}$;
(4) $y\left((Nn+\varepsilon)^+\right) = y\left((Nn+\varepsilon)^-\right)$, $y'\left((Nn+\varepsilon)^+\right) = \left(1+\frac{\alpha}{\varepsilon}\right) y'\left((Nn+\varepsilon)^-\right)$, for $n \in \mathbb{Z}$.

These conditions guarantee that $y(x)$ and $\rho_\varepsilon(x)\,y'(x)$ are continuous functions at the points Nn and $Nn+\varepsilon\,(n \in \mathbb{Z})$.

If $y(x)$ is a solution of equation (3.1), it follows from the periodicity of the functions $\rho(x)$ and $q(x)$ that $y(x+N)$ will be also a solution of this equation. However, generally speaking, $y(x) \neq y(x+N)$. We will show that there always exists a non-zero number $p = p(\lambda)$ and a non-trivial solution $\psi(x,\lambda)$ of equation (3.2), such that

$$\begin{aligned} &\psi(0,\lambda) = p\psi(N,\lambda), \left(1+\frac{\alpha}{\varepsilon}\right)\psi'(N,\lambda) = p\psi'(N,\lambda) \\ &\psi\left(\varepsilon^+,\lambda\right) = \psi\left(\varepsilon^-,\lambda\right), \psi'\left(\varepsilon^+,\lambda\right) = \left(1+\frac{\alpha}{\varepsilon}\right)\psi'\left(\varepsilon^-,\lambda\right). \end{aligned} \tag{3.3}$$

To this end, we consider a fundamental system of solutions $\theta(x,\lambda), \varphi(x,\lambda)$ of the equation $-y'' + q(x)\,y = \lambda y$ that will be determined by means of the initial conditions:

(3.4) $\theta(0,\lambda) = \varphi'(0,\lambda) = 1,\ \theta'(0,\lambda) = \varphi(0,\lambda) = 0$

The general solution of equation (3.2) will be of the form:

$$\psi^\varepsilon(x,\lambda) = \begin{cases} c_1^\varepsilon\theta(x,\lambda) + c_2^\varepsilon\varphi(x,\lambda), & \text{for } 0 < x < \varepsilon, \\ c_3^\varepsilon\theta(x,\lambda) + c_4^\varepsilon\varphi(x,\lambda), & \text{for } \varepsilon < x < N. \end{cases} \tag{3.5}$$

Substituting (3.5) in (3.3), for the definition of the constants $C_i^\varepsilon,\ i = \overline{1,4}$ in (3.5) we get a homogeneous linear system of equations whose non-trivial solvability condition is the relation

$$\begin{vmatrix} 1 & 0 & -p\theta(N,\lambda) & -p\varphi(N,\lambda) \\ 0 & 1+\frac{\alpha}{\varepsilon} & -p\theta(N,\lambda) & -p\varphi'(N,\lambda) \\ \theta(\varepsilon,\lambda) & \varphi(\varepsilon,\lambda) & -\theta(\varepsilon,\lambda) & -\varphi(\varepsilon,\lambda) \\ \left(1+\frac{\alpha}{\varepsilon}\right)\theta'(\varepsilon,\lambda) & \left(1+\frac{\alpha}{\varepsilon}\right)\varphi'(\varepsilon,\lambda) & -\theta(\varepsilon,\lambda) & -\varphi'(\varepsilon,\lambda) \end{vmatrix} = 0 \tag{3.6}$$

By (3.4) we have the identity

(3.7) $\theta(x,\lambda)\,\varphi'(x,\lambda) - \theta'(x,\lambda)\,\varphi(x,\lambda) = 1$

According to (3.7), as $\varepsilon \to 0^+$, equation (3.6) is arranged in the form

(3.8) $p^2 - \left[\theta(N,\lambda) + \varphi'(N,\lambda) - \alpha\lambda\varphi(N,\lambda)\right]p + 1 = 0$

Since, this equation has always the root p, and obviously its roots are non-zero, reduced reasoning proves the existence of a non-trivial solution $\psi(x,N)$ of equation (3.1) possessing the property $\psi(x,\lambda) = p\psi(x+N,\lambda)$.

Introducing the function

$$F(\lambda) = \frac{1}{2}\left[\theta(N,\lambda) + \varphi'(N,\lambda) - \alpha\lambda\varphi(N,\lambda)\right]$$

with parameter λ we rewrite equation (3.8) in the form

(3.9) $p^2 - 2F(\lambda)\,p + 1 = 0$

The function $F(\lambda)$ is said to be a *discriminant*, the roots of the equation (3.9) the *multiplicators* of equation (3.1).

From Definition 3.1 and results in [4], we obtain the following theorem.

3.2. Theorem. *For fixed* $\lambda \in (-\infty, \infty)$, *the equation (3.1) is instable if* $|F(\lambda)| > 1$ *and stable if* $|F(\lambda)| < 1$ *and also stable if* $|F(\lambda)| = 1$ *and* $\theta'(N, \lambda) = 2\alpha\lambda$, $\varphi(N, \lambda) = 0$. *Finally if* $|F(\lambda)| = 1$ *and* $\theta'(N, \lambda) \neq 2\alpha\lambda$ *or* $\varphi(N, \lambda) \neq 0$ *then (3.1) is conditionally stable.* □

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