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CONSEQUENCES OF ALLEE EFFECTS ON STABILITY ANALYSIS OF THE POPULATION MODEL $X_{t+1} = \lambda X_t f(X_{t-3})$

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Abstract

The stability conditions of equilibrium points of the population model $X_{t+1} = \lambda X_t f(X_{t-3})$ with and without Allee effects are investigated. It is assumed that the Allee effect occurs at low population density. Analysis and numerical simulations show that Allee effects have both stabilizing and destabilizing effects on population dynamics with delay.

Keywords: Stability, Allee effect, Population model, Equilibrium point, Difference equation, Time delay.

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1. Introduction.

Studies on population dynamics have been attractive for ecologists, biologists and mathematicians. Today, many of the population phenomenon are modelled by scientists using discrete and continuous dynamical systems. Although a discrete-time population model represents a richer dynamical picture (specifically, in terms of numerical simulation), continuous-time population models are more appropriate to nature except for non-overlapping generations. These models sometimes consist of a delay term if they depend on past history.

Formerly, scientists believed that a population achieves its equilibrium density when everything is sufficient (food, place, finding mates, etc.). However, Allee in 1931 [1] introduced a new idea which represents a negative density dependence when the population growth rate is reduced to a low population size. This may be due to a number of sources including difficulties in finding mates, social dysfunction at small population sizes, inbreeding depression, food exploitation, predator avoidance of defence. Such effects may be observed on different organisms including vertebrates, invertebrates and plants. The

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effect usually saturates or vanishes as the population size gets larger (see, for example, [2, 3, 8, 18, 21]).

Allee effects have been recently studied intensively in population dynamics. Former studies have demonstrated that Allee effects can have important dynamical effects on the local stability analysis of population models. It may have either a destabilizing role or a stabilizing role in the system (see, for example, [4, 6, 8, 10, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22]). The local stability of an equilibrium point may be changed from stable to unstable or vice versa. It is also possible that for a population subject to an Allee effect, the system may take a much longer time to reach its steady state even though it is stable at an equilibrium point (see, for instance, [6, 17, 22]).

In this work, the following general discrete-time population model with delay

$$X_{t+1} = \lambda X_t f(X_{t-T}) \tag{1.1}$$

is considered. Here, λ is the per capita growth rate which is always positive, X_t represents population density at time t and T is the time for sexual maturity. The function $f(X_{t-T})$ describes interactions (competitions) among mature individuals. It will be assumed that the function f satisfies the following conditions:

(i) $\frac{d}{dX}(f(X)) < 0$ for $X \in [0, \infty)$; that is; as density increases, the competition among mature individuals decreases;
(ii) $f(0)$ is a positive finite number;
(iii) $\lim_{X\to\infty} f(X) = 0$.

In addition to Eq (1.1) the model below which includes Allee effects

$$X_{t+1} = \lambda^* X_t a f(X_{t-T}) \tag{1.2}$$

is also considered. In this model, the function f again satisfies the conditions (i), (ii), (iii) and $\lambda^* > 0$ is the per capita growth rate. Here, $a := a(X_{t-T}, X_{t-(T-1)}, \dots, X_t)$ represents the Allee effect. Biological facts lead to the following assumptions on the Allee function a:

(iv) There is no reproduction without partners, that is, if $X = 0$, then $a(X) = 0$;
(v) $\frac{d}{dX}(a(X)) > 0$ for $X > 0$ (but the Allee effect decreases as density increases);
(vi) The Allee effect vanishes at high densities, that is, $lim_{X\to\infty} a(X) = 1$.

The goal of this work is to study the stability of Eq (1.1) and Eq (1.2) when $T = 3$. By comparing the stability conditions, the consequences of Allee effects will be discussed. A similar analysis has been studied by Celik *et al.* ([4]) for $T = 1$ and by Merdan and Gumus ([16]) for $T = 2$.

This paper is organized as follows: Section 2 involves a characterization of the stability of the equilibrium points of Eq (1.1) when $T = 3$ and an investigation of their stability behavior as the parameter λ increases. In Section 3, we work on the stability analysis of the equilibrium points of Eq (1.2) with $T = 3$. Section 4 consists of some numerical simulations and finally the last section of the paper includes conclusions and remarks. Throughout the paper, the derivative with respect to the population density will be denoted by $' := \frac{d}{dX}$.

2. Stability analysis of Eq (1.1)

In this section, we will obtain conditions on the local stability of a positive equilibrium point of Eq (1.1) when $T = 3$. Let us denote an equilibrium point by X^*. Clearly, $X^* = 0$ is an equilibrium point of Eq (1.1) (and also Eq (1.2)). A positive equilibrium point can be obtained by solving the equations $\lambda f(X^*) = 1$ for Eq (1.1) and $\lambda^* a f(X^*) = 1$ for

Eq (1.2) (see, for example, [19]). Here, we are mainly interested in positive equilibrium points due to the biological facts.

Then, one has the following result.

2.1. Theorem. *The (positive) equilibrium point X^* is locally stable if and only if the inequality*

$$X^* \frac{f'(X^*)}{f(X^*)} > x_0 \approx -0.4450 \tag{2.1}$$

holds.

Proof. Remembering first that due to the definition of an equilibrium point we have the following identity

$$1 = \lambda f(X^*). \tag{2.2}$$

The linearization of Eq (1.1) around X^* is

$$U_{t+1} = U_t + \lambda f'(X^*) X^* U_{t-3} \tag{2.3}$$

so that its characteristic polynomial is

$$p(\mu) = \mu^4 - \mu^3 - \lambda f'(X^*) X^*. \tag{2.4}$$

By the Schur-Cohn criterion (see Appendix and the references [2, 7] for more details of this criterion, and also see the papers [5, 11] for the stability theorem when $T = 1$), the first and the second conditions (see Appendix (A4)-(A8)) lead to the following inequalities

$$\lambda f'(X^*) X^* < 0, \tag{2.5}$$

$$\lambda f'(X^*) X^* < 2, \tag{2.6}$$

$$-1 < \lambda f'(X^*) X^* < 1. \tag{2.7}$$

Since f is a decreasing function for $X \in [0, \infty)$ the term $\lambda f'(X^*) X^*$ is always negative. Then, these three inequalities reduce to the inequality

$$\lambda f'(X^*) X^* > -1. \tag{2.8}$$

On the other hand, the last condition yields the following inequality

$$|-2\lambda f'(X^*) X^* - [-\lambda f'(X^*) X^*]^3| < 1 - [-\lambda f'(X^*) X^*]^2. \tag{2.9}$$

Solving the last inequality yields

$$x_0 < \lambda f'(X^*) X^* < -x_0. \tag{2.10}$$

Now combining inequalities (2.8) and (2.10) leads to the result that the equilibrium point X^* is stable if and only if

$$X^* \frac{f'(X^*)}{f(X^*)} > x_0, \tag{2.11}$$

as claimed. □

Notice that the equilibrium points depend on the per capita growth rate, namely λ. The following theorem answers how the variation of the parameter λ effects the stability of a fixed point.

2.2. Theorem. *Suppose that $X^{(i)}$ is an equilibrium point of Eq (1.1) corresponding to the per capita growth rate $\lambda_{(i)}$ for $i = 1, 2$. Then, increasing the per capita growth rate decreases the local stability of the equilibrium points of Eq (1.1) if the following inequality*

$$\int_{X^{(1)}}^{X^{(2)}} \left[X(\log f(X))'\right]' dX < 0 \tag{2.12}$$

holds. In other words, the local stability of $X^{(2)}$ is weaker than $X^{(1)}$ as long as the condition (2.12) is fulfilled.

Proof. Suppose that λ_1 and λ_2 are positive numbers such that $\lambda_1 < \lambda_2$, and let $X^{(1)}$ and $X^{(2)}$ be the corresponding positive equilibrium points of Eq (1.1) with respect to λ_1 and λ_2, respectively. By the definition of equilibrium point, it is easy to see that

$$1 = \lambda_1 f(X^{(1)}) \text{ and } 1 = \lambda_2 f(X^{(2)}). \tag{2.13}$$

Since $\lambda_1 < \lambda_2$, it follows from (2.13) that $f(X^{(1)}) > f(X^{(2)})$. Also, since f is a decreasing function, we have $X^{(1)} < X^{(2)}$. Thus, by Theorem 2.1, for each $i = 1, 2$, the equilibrium point $X^{(i)}$ is locally stable if and only if

$$X^{(i)} \frac{f'(X^{(i)})}{f(X^{(i)})} > x_0. \tag{2.14}$$

If the condition (2.12) holds, then one has

$$X^{(1)} \frac{f'(X^{(1)})}{f(X^{(1)})} > X^{(2)} \frac{f'(X^{(2)})}{f(X^{(2)})}. \tag{2.15}$$

Considering (2.14) and (2.15) together yields the result we are looking for. □

2.3. Theorem. *Delay decreases the stability of the equilibrium points of Eq (1.1). In other words, the longer the delay is, the greater the destabilizing effect will be.*

Proof. Suppose that X^* is a positive equilibrium point of Eq (1.1). By [16, Theorem 1] it is stable for $T = 2$ if and only if the following inequality

$$X^* \frac{f'(X^*)}{f(X^*)} > \frac{1-\sqrt{5}}{2} \approx -0.6180 \tag{2.16}$$

holds. On the other hand, the same equilibrium point is stable for $T = 3$ if and only if the following inequality

$$X^* \frac{f'(X^*)}{f(X^*)} > x_0 \approx -0.4450 \tag{2.17}$$

holds so from the comparison (2.16) and (2.17) we conclude the result claimed. □

3. Stability analysis under Allee effects

This section consists of the local stability analysis of the equilibrium points of Eq (1.2) when $T = 3$ in which the equation involves Allee effects. We will consider three cases depending on the definition of the function a, namely, the Allee effect at time $t-3$, t and $(t, t-3)$, respectively.

By the conditions (i)-(vi), Eq (1.2) has at most two positive equilibrium points (see the paper [19]). Assume that Eq (1.2) has exactly two equilibrium points, say X_1^* and X_2^* (without loss of generality, $X_1^* < X_2^*$). By the definition of equilibrium point, for each $i = 1, 2$, they satisfy the following identity

$$\lambda^* a(X_i^*) f(X_i^*) = 1. \tag{3.1}$$

Let us define the function $\Psi(X)$ as

$$\Psi(X) := \lambda^* a(X) f(X).$$

By the Mean Value Theorem there exists a critical point X_c such that $\Psi'(X_c) = 0$ where $X_1^* < X_c < X_2^*$. One then has the following results.

(i) Allee effect at time $t-3$.

First, let us take $a := a(X_{t-3})$ so that Eq (1.2) has the form

$$X_{t+1} = \lambda^* X_t a(X_{t-3}) f(X_{t-3}). \tag{3.2}$$

We then have the following theorem for the equilibrium points X_1^* and X_2^*.

3.1. Theorem. *The equilibrium point X_1^* of Eq (3.2) is unstable. Also, X_2^* is locally stable if and only if the inequality*

$$X_2^* \left(\frac{a'(X_2^*)}{a(X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) > x_0 \tag{3.3}$$

holds.

Proof. The linearization of Eq (3.2) around X^* will be

$$U_{t+1} = U_t + X^* \left(\frac{a'(X^*)}{a(X^*)} + \frac{f'(X^*)}{f(X^*)} \right) U_{t-3}. \tag{3.4}$$

Using Eq (3.1), its characteristic polynomial can be obtained as

$$p(\mu) = \mu^4 - \mu^3 - X^* \left(\frac{a'(X^*)}{a(X^*)} + \frac{f'(X^*)}{f(X^*)} \right). \tag{3.5}$$

By the Schur-Cohn criterion, the first condition leads to the following inequality

$$-2 < X^* \left(\frac{a'(X^*)}{a(X^*)} + \frac{f'(X^*)}{f(X^*)} \right) < 0 \tag{3.6}$$

and the second condition yields the following inequality

$$-1 < X^* \left(\frac{a'(X^*)}{a(X^*)} + \frac{f'(X^*)}{f(X^*)} \right) < 1. \tag{3.7}$$

Since $\Psi'(X) > 0$ for all $X \in (0, X_c)$, the inequality (3.6) is not satisfied so that X_1^* is unstable ([19]).

On the other hand, $\Psi'(X) < 0$ for $X \in (X_c, \infty)$. For the fixed point X_2^*, the right hand sides of the inequalities (3.6) and (3.7) are always true. From the left hand sides, one can obtain

$$X^* \left(\frac{a'(X^*)}{a(X^*)} + \frac{f'(X^*)}{f(X^*)} \right) > -1. \tag{3.8}$$

Now, the last condition yields the following inequality which is the same as in Theorem 2.1:

$$|-2\lambda f'(X^*)X^* - [-\lambda f'(X^*)X^*]^3| < 1 - [-\lambda f'(X^*)X^*]^2. \tag{3.9}$$

Now considering this with together the inequality (3.8), one concludes that the equilibrium point X_2^* is stable if and only if

$$X_2^* \left(\frac{a'(X_2^*)}{a(X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) > x_0. \tag{3.10}$$

This completes the proof. □

3.2. Theorem. *Allee effects increase the stability of the equilibrium points; that is, the local stability of the equilibrium points of Eq (3.2) is stronger than those of Eq (1.1).*

Proof. Let us choose λ in Eq (1.1) as $\lambda = \lambda^* a(X_2^*)$. In this case, X_2^* is a positive equilibrium point of Eq (1.1), too. Furthermore, since $a(X_2^*) < 1$ and $\lambda = \lambda^* a(X_2^*)$, it is easy to see that $\lambda < \lambda^*$. Also, the definition of the Allee function a leads to the result that $X_2^* \left(\frac{a'(X_2^*)}{a(X_2^*)}\right) > 0$ for $i = 1, 2$ so that one can easily obtain

$$X_2^* \frac{f'(X_2^*)}{f(X_2^*)} < X_2^* \left(\frac{a'(X_2^*)}{a(X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)}\right). \tag{3.11}$$

Now, considering this inequality with together Theorem 2.1 and Theorem 2.3 leads to the result that the local stability of the equilibrium points of Eq (3.2) is stronger than those of Eq (1.1). □

(ii) Allee effect at time t.

Let us now consider Eq (1.2) of the form

$$X_{t+1} = \lambda^* X_t a(X_t) f(X_{t-3}), \tag{3.12}$$

where $a := a(X_t)$ represents the Allee effect here. Again assume that this equation has two equilibrium points namely, X_1^* and X_2^*, $(X_1^* < X_2^*)$, satisfying Eq (3.1). One then has the following theorems.

3.3. Theorem. *X_1^* is an unstable equilibrium point of Eq (3.12). On the other hand, X_2^* is locally stable if and only if the following inequalities hold:*

$$X_2^* \frac{f'(X_2^*)}{f(X_2^*)} > -1, \tag{3.13}$$

$$\left(X_2^* \frac{f'(X_2^*)}{f(X_2^*)}\right)^3 + \left(X_2^* \frac{f'(X_2^*)}{f(X_2^*)}\right)^2 - \left[\left(1 + X_2^* \frac{a'(X_2^*)}{a(X_2^*)}\right)^2 + 1\right]\left(X_2^* \frac{f'(X_2^*)}{f(X_2^*)}\right) - 1 < 0, \tag{3.14}$$

$$\left(X_2^* \frac{f'(X_2^*)}{f(X_2^*)}\right)^3 - \left(X_2^* \frac{f'(X_2^*)}{f(X_2^*)}\right)^2 - \left[\left(1 + X_2^* \frac{a'(X_2^*)}{a(X_2^*)}\right)^2 + 1\right]\left(X_2^* \frac{f'(X_2^*)}{f(X_2^*)}\right) + 1 > 0. \tag{3.15}$$

Proof. By linearizing of Eq (3.12) around X^* one obtains

$$\begin{aligned} X_{t+1} = (\lambda^* a(X^*) f(X^*) + \lambda^* X^* a'(X^*) f(X^*)) X_t \\ + \lambda^* X^* a(X^*) f'(X^*) X_{t-3}. \end{aligned} \tag{3.16}$$

Using Eq (3.1), one can write Eq (3.16) as follows:

$$X_{t+1} - \left(1 + X^* \frac{a'(X^*)}{a(X^*)}\right) X_t - \left(X^* \frac{f'(X^*)}{f(X^*)}\right) X_{t-3} = 0 \tag{3.17}$$

whose characteristic polynomial is

$$p(\mu) = \mu^4 - \left(1 + X^* \frac{a'(X^*)}{a(X^*)}\right) \mu^3 - \left(X^* \frac{f'(X^*)}{f(X^*)}\right). \tag{3.18}$$

Using now the Schur-Cohn criterion (see Appendix (A4)-(A8)) one can obtain the inequalities (3.13)–(3.15). □

3.4. Theorem. *The Allee effect decreases the stability of a fixed point X^* if the inequality*

$$\left[\left(X^* \frac{f'(X^*)}{f(X^*)}\right)^2 - 1\right] \left[\left(X^* \frac{f'(X^*)}{f(X^*)}\right) + 1\right] > \left(1 + X^* \frac{a'(X^*)}{a(X^*)}\right)^2 X^* \frac{f'(X^*)}{f(X^*)} \tag{3.19}$$

holds. That is, the local stability of the equilibrium points of Eq (1.1) is stronger than those of Eq (3.12).

Proof. Let x and y denote the following terms

$$x := X^* \frac{f'(X^*)}{f(X^*)} \text{ and } y := X^* \frac{a'(X^*)}{a(X^*)} \tag{3.20}$$

so that the inequality (3.19) (in terms of x and y) has the form

$$(x^2 - 1)(x + 1) > (1 + y)^2 x. \tag{3.21}$$

Clearly, $x < 0$ and $y > 0$ by the conditions (i)-(vi) on f and a. The proof will be done in two parts.

First, we consider $x \le x_0$. In this case, the equilibrium points of Eq (1.1) will be not stable by Theorem 2.1, that is, the inequality (2.1) is unfulfilled. Under this assumption, we need to show that at least one of the conditions, namely (3.13), (3.14) and (3.15), in Theorem 3.3 is not satisfied.

If these conditions are written in terms of x and y, one has from Theorem 3.3 that X_2^* is stable $\iff$ $x > -1$, $x^3+x^2-\left[(1+y)^2+1\right]x-1 < 0$ and $x^3-x^2-\left[(1+y)^2+1\right]x+1 > 0$, respectively. These will be examined in two cases as follows:

Case 1: Assume first that $x = x_0$. Our aim is to show that one of these conditions is not fulfilled under this assumption. Suppose that the inequality (3.14) is fulfilled. Adding $-2x$ both side of (3.14) we get that $x^3 + x^2 - 2x - 1 - \left[(1 + y)^2 + 1\right] x < -2x$. Remembering that x_0 is one of the zeros of $x^3 + x^2 - 2x - 1$ one has $x - (1 + y)^2 x < 0$ so that $1 - (1 + y)^2 > 0 \iff y < 0$ (since $x < 0$). But, this contradicts the fact that y is always positive. Therefore, the equilibrium points of Eq (3.12) are not stable either.

Case 2: It is obvious that when $x \le -1$, (3.13) is not satisfied. We now assume that $-1 < x < x_0$. On this interval, we know that $x^3 + x^2 - 2x - 1 > 0$ so

$$x^3 + x^2 - 1 > 2x. \tag{3.22}$$

On the other hand, since $y > 0$ and $x < 0$, one can easily get that $(1 + y)^2 > 1$ and $-(1 + y)^2 x + x > 0$ so that adding $2x$ to both sides one has

$$-\left[(1 + y)^2 + 1\right] x + 2x > 0. \tag{3.23}$$

Combining now (3.22) and (3.23) one can see that $x^3 + x^2 - \left[(1 + y)^2 + 1\right] x - 1 > 0$. As a result, the condition (3.14) is not satisfied once again so that the equilibrium points of Eq (3.12) are unstable.

Second, suppose that $x_0 < x < 0$. By Theorem 2.1, it is clear that the equilibrium points of Eq (2.1) will be stable. For comparison we need to check whether each condition in Theorem 3.3 is satisfied under this assumption.

Since $x > -1$, the condition (3.13) is fulfilled. Also, notice that, on this interval

$$-x^3 + x^2 + 2x - 1 < 0 \iff x^3 - x^2 + 1 > 2x \tag{3.24}$$

so that utilizing (3.23) one obtains

$$x^3 - x^2 - \left[(1 + y)^2 + 1\right] x + 1 > 0 \tag{3.25}$$

that means (3.15) is always true.

Finally, we will examine whether the condition (3.14) is satisfied under the assumption of $x > x_0$. From the hypothesis, we have

$$\begin{aligned} & (x^2 - 1)(x + 1) > (1 + y)^2 x \\ \iff\ & x^2(x + 1) - (x + 1) > (1 + y)^2 x, \end{aligned} \tag{3.26}$$

$$\begin{aligned} & x^3 + x^2 - x - 1 > (1 + y)^2 x \\ \iff\ & x^3 + x^2 - \left[(1 + y)^2 + 1\right] x - 1 > 0. \end{aligned} \tag{3.27}$$

Hence, the condition (3.14) is not satisfied under the condition (3.19) so that the equilibrium points of Eq (3.12) are unstable while those of Eq (2.1) are stable. This completes the proof. □

3.5. Note. Numerical simulations show that there exist such fixed points satisfying (3.19).

(iii) Allee effect at time t and $t-3$. We finally consider Eq (1.2) in the form

$$X_{t+1} = \lambda^* X_t a(X_t, X_{t-3}) f(X_{t-3}). \tag{3.28}$$

Here, the Allee effect has the form $a := a(X_t, X_{t-3})$. We then state the following theorem for the equilibrium points, namely X_1^* and X_2^* ($X_1^* < X_2^*$), satisfying Eq (3.1).

3.6. Theorem. *X_1^* is an unstable equilibrium point of Eq (3.28). On the other hand, X_2^* is locally stable if and only if the following inequalities hold*

$$X_2^* \left(\frac{a_{X_t}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} - \frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} - \frac{f'(X_2^*)}{f(X_2^*)} \right) > -2 \tag{3.29}$$

$$-1 < \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right] < 1 \tag{3.30}$$

$$\begin{aligned} &- \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right]^3 - \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right]^2 \\ &\quad + \left[1 + \left(1 + X_2^* \frac{a_{X_t}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} \right)^2 \right] \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right] > -1. \end{aligned} \tag{3.31}$$

$$\begin{aligned} &- \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right]^3 + \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right]^2 \\ &\quad + \left[1 + \left(1 + X_2^* \frac{a_{X_t}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} \right)^2 \right] \left[X_2^* \left(\frac{a_{X_{t-3}}(X_2^*, X_2^*)}{a(X_2^*, X_2^*)} + \frac{f'(X_2^*)}{f(X_2^*)} \right) \right] < 1. \end{aligned} \tag{3.32}$$

Proof. By linearizing of Eq (3.28) around X^* one obtains

$$\begin{aligned} X_{t+1} = {} & (\lambda^* a(X^*, X^*) f(X^*) + \lambda^* X^* a_{X_t}(X^*, X^*) f(X^*)) X_t \\ & + (\lambda^* X^* a_{X_{t-3}}(X^*, X^*) f(X^*) + \lambda^* X^* a(X^*, X^*) f'(X^*)) X_{t-3}. \end{aligned} \tag{3.33}$$

Using now Eq (3.1) one can write Eq (3.33) as

$$\begin{aligned} X_{t+1} - \left(1 + X^* \frac{a_{X_t}(X^*, X^*)}{a(X^*, X^*)} \right) X_t - \left(X^* \frac{a_{X_{t-3}}(X^*, X^*)}{a(X^*, X^*)} + X^* \frac{f'(X^*)}{f(X^*)} \right) X_{t-3} \\ = 0 \end{aligned} \tag{3.34}$$

whose characteristic polynomial is

$$p(\mu) = \mu^4 - \left(1 + X^* \frac{a_{X_t}(X^*, X^*)}{a(X^*, X^*)} \right) \mu^3 - X^* \left(\frac{a_{X_{t-3}}(X^*, X^*)}{a(X^*, X^*)} + \frac{f'(X^*)}{f(X^*)} \right). \tag{3.35}$$

Now, applying the Schur-Cohn criterion to (3.35) one can complete the proof. □

4. Numerical simulations

We now report on numerical simulations of the fourth-order Discrete Logistic Equation ([2]) with and without Allee effects. We used the MATLAB ODE package for these computations. The equation has the form

$$X_{t+1} = \lambda X_t (1 - X_{t-3}/K), \ \lambda > 0 \text{ and } K > 0 \tag{4.1}$$

with the initial values X_{-3}, X_{-2}, X_{-1} and X_0.

If we choose $0 < X_{-3}, X_{-2}, X_{-1}, X_0 < K$, then we can guarantee that $X_t > 0$ for any time t with appropriate $\lambda > 0$. In each simulation, we take the Allee effect as $a(z) = z/(\alpha + z)$, where α is a positive constant.

Figure 1 (a–c) show that the local stability of the fixed points decreases as the parameter λ increases. In Figure 2, we graph the trajectories of the population dynamics model (4.1) with or without Allee effect. Here, the Allee effect is taken as $a(X_{t-3}) = X_{t-3}/(\alpha + X_{t-3})$, where $\alpha = 0.03$. Simulation supports Theorem 3.2. As we can see from the graph that when we impose the Allee effect at time $t-3$ into the model (4.1), the local stability of the fixed point increases and the trajectories approach to the corresponding fixed point faster. Figure 3 and Figure 4 show that more delay decreases the stability of the fixed points.

Finally, in Figure 5 and Figure 6, we take the Allee effect as $a(X_t) = X_t/(\alpha + X_t)$, where α is a positive constant. As we illustrated in Theorem 3.4, when $\lambda = 1.4$ the condition (3.19) is unsatisfied so that the Allee effect at time t does not change the stability of the fixed point (see Figure 6). However, when $\lambda = 1.43$, the condition (3.19) is satisfied, hence, the Allee effect at time t decreases the stability as one can see from Figure 5.

Figure 1. Density-time graphs of the model $X_{t+1} = \lambda X_t(1 - X_{t-3})$ with the initial conditions: $X_{-3} = 0.14$, $X_{-2} = 0.15$, $X_{-1} = 0.16$ and $X_0 = 0.17$. (a) $\lambda = 1.2$ (b) $\lambda = 1.3$ and (c) $\lambda = 1.4$.

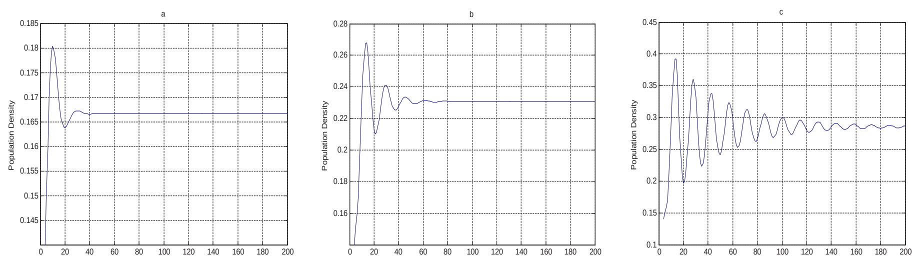


Figure 2. Density-time graphs of the models $X_{t+1} = \lambda X_t(1 - X_{t-3}/K)$ and $X_{t+1} = \lambda^* X_t a(X_{t-3})(1 - X_{t-3}/K)$ with $K = 1$, $\lambda = 1.4$, $a(X_{t-3}) = X_{t-3}/(\alpha + X_{t-3})$, $\alpha = 0.03$, $\lambda = \lambda^* a(X^*)$ and the initial conditions $X_{-3} = 0.36$, $X_{-2} = 0.37$, $X_{-1} = 0.38$ and $X_0 = 0.39$.

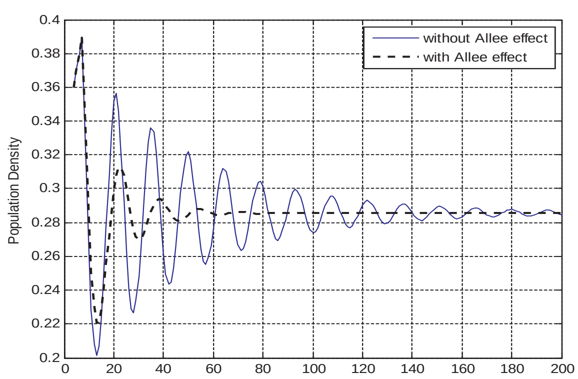


Figure 3. Density-time graphs of the models $X_{t+1} = \lambda X_t(1 - X_{t-T}/K)$ when $T = 2$ and $T = 3$ with $K = 1$, $\lambda = 1.4$, $a(X_{t-T}) = X_{t-T}/(\alpha + X_{t-T})$, $\alpha = 0.03$, $\lambda = \lambda^* a(X^*)$ and the initial conditions $X_{-3} = 0.36$, $X_{-2} = 0.37$, $X_{-1} = 0.38$ and $X_0 = 0.39$

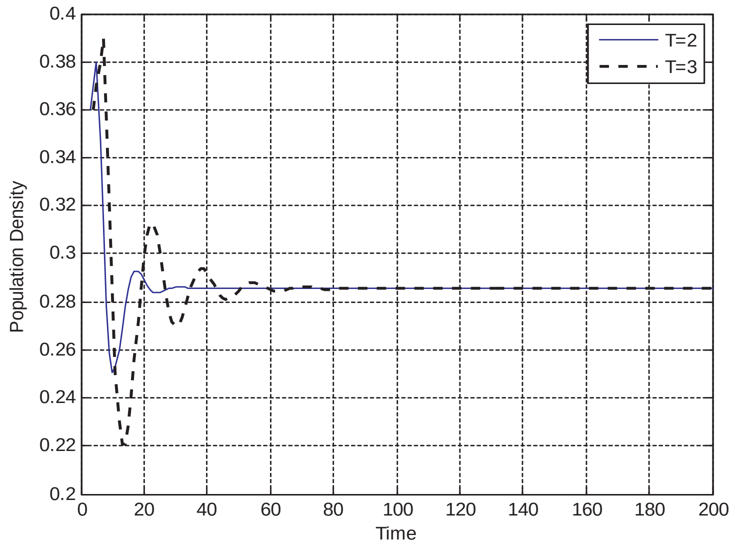


Figure 4. Density-time graphs of the models $X_{t+1} = \lambda X_t(1 - X_{t-T}/K)$ when $T = 2$ and $T = 3$ with $K = 1$, $\lambda = 1.4$, $a(X_t) = X_t/(\alpha + X_t)$, $\alpha = 0.03$, $\lambda = \lambda^* a(X^*)$ and the initial conditions $X_{-3} = 0.36$, $X_{-2} = 0.37$, $X_{-1} = 0.38$ and $X_0 = 0.39$

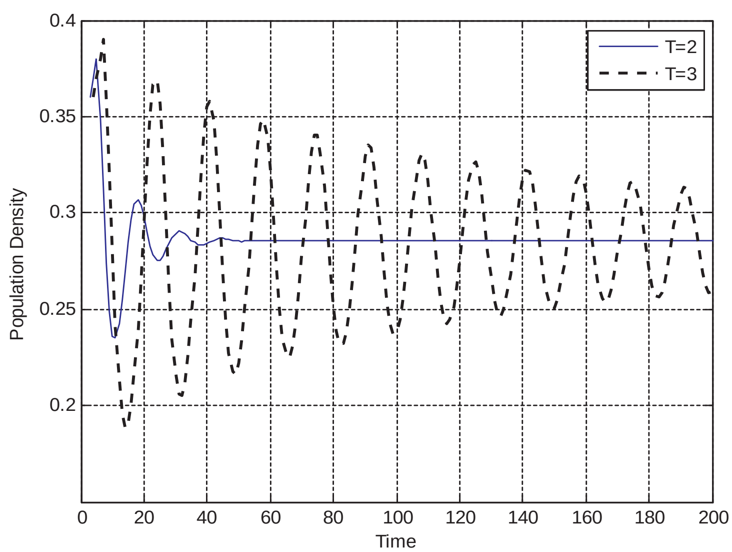


Figure 5. Density-time graphs of the models $X_{t+1} = \lambda X_t(1 - X_{t-3}/K)$ **and** $X_{t+1} = \lambda^* X_t a(X_t)(1 - X_{t-3}/K)$ **with** $K = 1$, $\lambda = 1.43$, $a(X_t) = X_t/(\alpha + X_t)$, $\alpha = 0.03$, $\lambda = \lambda^* a(X^*)$ **and the initial conditions** $X_{-3} = 0.29$, $X_{-2} = 0.295$, $X_{-1} = 0.296$ **and** $X_0 = 0.297$

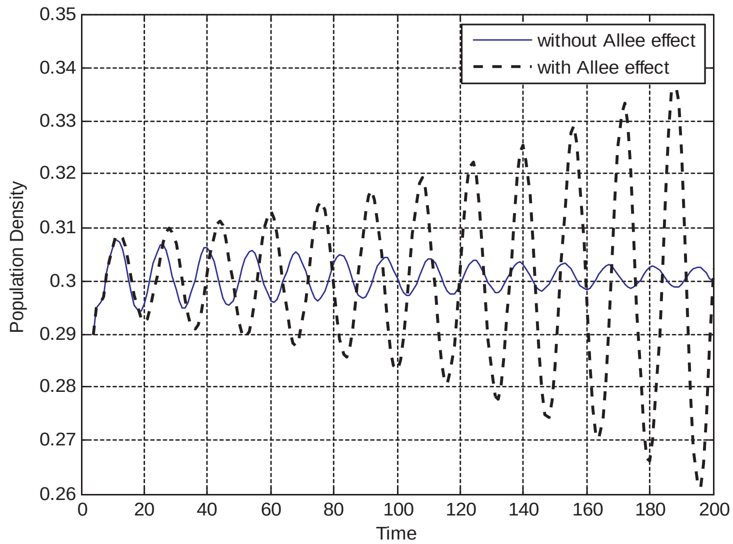
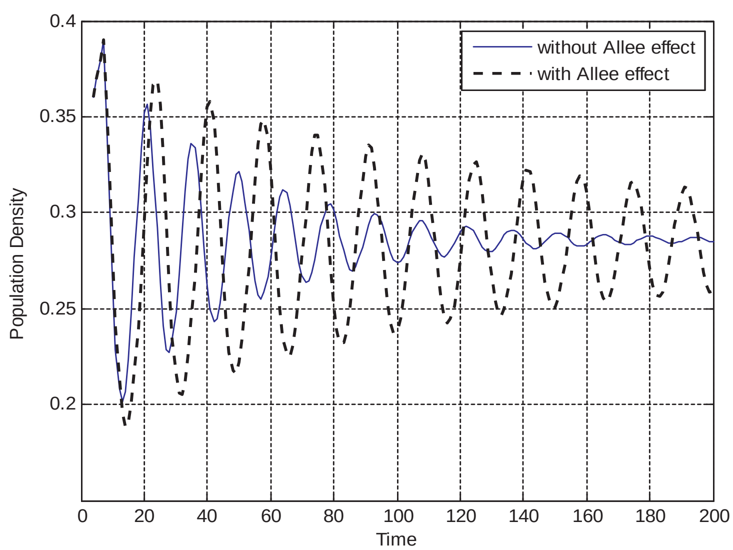


Figure 6. Density-time graphs of the models $X_{t+1} = \lambda X_t(1 - X_{t-3}/K)$ **and** $X_{t+1} = \lambda^* X_t a(X_t)(1 - X_{t-3}/K)$ **with** $K = 1$, $\lambda = 1.4$, $a(X_t) = X_t/(\alpha + X_t)$, $\alpha = 0.03$, $\lambda = \lambda^* a(X^*)$ **and the initial conditions** $X_{-3} = 0.36$, $X_{-2} = 0.37$, $X_{-1} = 0.38$ **and** $X_0 = 0.39$



5. Conclusions and remarks

Previous studies demonstrate that Allee effects play an important role in the stability analysis of equilibrium points of a population dynamics model (see, for example, [4, 6, 8, 10, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22]). Allee effect may have a stabilizing or a destabilizing effect on population dynamics. Even if the system is stable at an equilibrium point, the system subject to an Allee effect may reach to its stable state in a much longer time.

This paper focused on the stability analysis of a fourth-order discrete-time population model involving delay, where increasing per capita growth rate decreases the stability of the fixed points. First, we characterized the stability of fixed point(s) of the model. Imposing different Allee effects into the system, the stability of equilibrium points were also studied. Using the same normalized growth rate we compared the stability of the equilibrium points corresponding to the models with and without Allee effect defined by equations (1.1) and (1.2), respectively. Theorem 3.4 shows that an Allee effect at time t decreases the stability of the equilibrium points of Eq (1.1). This result differs from the former cases $T = 1$ and $T = 2$ (see the reference [4, 16]). On the other hand, this work shows that an Allee effect at time $t - 2$ increases stability of equilibrium points, which is a good agreement with the former studies. As a conclusion, Allee effects play an important role in stability.

Appendix

Linear scalar equations

Consider the kth-order difference equation

$$x_{n+k} + a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_k x_n = 0, \tag{A1}$$

where the a_i's are real numbers. The solution of Eq (A1) is asymptotically stable if and only if $|\mu| < 1$ holds for all characteristic roots μ of Eq (A1), that is, for every zero μ of the characteristic polynomial

$$p(\mu) = \mu^k + a_1 \mu^{k-1} + \cdots + a_k. \tag{A2}$$

One of the main tools that provides necessary and sufficient conditions for the zeros of a kth-degree polynomial, such as (A2), to lie inside the unit disk is the Schur-Cohn criterion. This is useful in order to study the stability of the solution of Eq (A1). Furthermore, this criterion may be utilized to investigate the stability of a k-dimensional system of the form

$$x_{n+1} = A x_n, \tag{A3}$$

where $p(\mu)$ is the characteristic polynomial of the coefficient matrix A.

Definition. A matrix $B = (b_{ij})$ is said to be *positive innerwise* if the determinants of all of its inners are positive. The *inners of the matrix B* are the matrix itself and all the matrices obtained by omitting successively the first and last rows and the first and last columns.

Schur-Cohn Criterion: The roots of the characteristic polynomial (A2) lie inside the unit disk if and only if the following hold:

(i) $p(1) > 0$,
(ii) $(-1)^k p(-1) > 0$,

(iii) The $(k-1)\times(k-1)$ matrices

$$B^{\pm}_{k-1} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ a_1 & 1 & \cdots & & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{k-3} & & & 1 & 0 \\ a_{k-2} & a_{k-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_k \\ 0 & 0 & \cdots & a_k & a_{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_k & & a_4 & a_3 \\ a_k & a_{k-1} & \cdots & a_3 & a_2 \end{pmatrix}$$

are each positive innerwise.

Using the Schur-Cohn criterion one may obtain necessary and sufficient conditions on the coefficients a_i such that the solution of Eq (A1) is asymptotically stable. For example, for a fourth-order difference equation

$$x_{n+4} + a_1 x_{n+3} + a_2 x_{n+2} + a_3 x_{n+1} + a_4 x_n = 0 \tag{A4}$$

the corresponding characteristic polynomial becomes

$$p(\mu) = \mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu + a_4. \tag{A5}$$

By the Schur-Cohn criterion, the necessary and sufficient conditions on the coefficients a_i for the solution of Eq (A4) to be asymptotically stable are

$$|a_1 + a_3| < 1 + a_2 + a_4, \tag{A6}$$

$$|a_4| < 1, \tag{A7}$$

$$\begin{aligned}\big|a_2 + a_4 - a_2a_4 & \\ - a_1a_3 - a_4^3 + a_1^2a_4\big| &< 1 + a_2a_4 - a_3^2 + a_1a_3a_4 - a_4^2a_2 - a_4^2.\end{aligned} \tag{A8}$$

Solution of the inequalities

In Theorem 2.1, we want to solve the inequality (2.9). To do this, we substitute $\lambda = \frac{1}{f(X^*)}$ into (2.9) so that we have

$$|-2\frac{f'(X^*)}{f(X^*)}X^* + [\frac{f'(X^*)}{f(X^*)}X^*]^3| < 1 - [\frac{f'(X^*)}{f(X^*)}X^*]^2.$$

This inequality yields the following two inequalities:

$$\left[\frac{f'(X^*)}{f(X^*)}X^*\right]^3 + \left[\frac{f'(X^*)}{f(X^*)}X^*\right]^2 - 2\frac{f'(X^*)}{f(X^*)}X^* - 1 < 0,$$
$$-\left[\frac{f'(X^*)}{f(X^*)}X^*\right]^3 + \left[\frac{f'(X^*)}{f(X^*)}X^*\right]^2 + 2\frac{f'(X^*)}{f(X^*)}X^* - 1 < 0.$$

If we set $x := \frac{f'(X^*)}{f(X^*)}X^*$ and write these expressions into MATLAB, we can find the roots of $-x^3 + x^2 + 2x - 1 = 0$ and $x^3 + x^2 - 2x - 1 = 0$:

MatLab Command: `solve('-x^3+x^2+2*x-1')`

First root

$$\begin{aligned}&= \frac{1}{6}\frac{28 + (84\sqrt{3}i - 28)^{\frac{2}{3}} + 2(84\sqrt{3}i - 28)^{\frac{1}{3}}}{(84\sqrt{3}i - 28)^{\frac{1}{3}}} \\ &\approx 1.801937735804838,\end{aligned}$$

Second root

$$= \frac{1}{12}\frac{4(84\sqrt{3}i-28)^{\frac{1}{3}}-(84\sqrt{3}i-28)^{\frac{2}{3}}-28\sqrt{3}i+\sqrt{3}i(84\sqrt{3}i-28)^{\frac{2}{3}}-28}{(84\sqrt{3}i-28)^{\frac{1}{3}}}$$
$$\approx -1.246979603717467,$$

Third root

$$= \frac{1}{12}\frac{28\sqrt{3}i-\sqrt{3}i(84\sqrt{3}i-28)^{\frac{2}{3}}-(84\sqrt{3}i-28)^{\frac{2}{3}}-28+4(84\sqrt{3}i-28)^{\frac{1}{3}}}{(84\sqrt{3}i-28)^{\frac{1}{3}}}$$
$$\approx 0.445041867912629 := x_0.$$

Then one can easily see that the solution of the inequalities must be

$$X^*\frac{f'(X^*)}{f(X^*)} > x_0 \approx -0.4450.$$

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