

PRICING AND COMPLETION IN A LÉVY MARKET MODEL WITH TEUGEL MARTINGALES

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Abstract

We consider a geometric Lévy market model. Since these markets are generally incomplete, we cannot find a unique martingale measure. There are many ways to handle this problem. In this paper, we choose the completion technique, firstly introduced in J. M. Corcuera, D. Nualart and W. Schoutens (*Completion of a Lévy market by power-jump assets*, Journal of Computational Finance, **7**, 1–49, 2004), which employs special artificial assets called power-jump assets. The price processes of power-jump assets are based on an orthogonalized family of Teugel martingales. By using the Gram-Schmidt process and obtaining the coefficients we express the price process of the power-jump assets in terms of Teugel martingales. Afterwards, we derive pricing formulas for European call options by using two methods: the martingale pricing approach and the characteristic formula approach which is performed via the fast Fourier transform (FFT). Throughout the pricing and application to real market price data, jump sizes are assumed to have a particular distribution.

Keywords: Chaotic representation, Lévy processes, Market completion, Teugel Martingales.

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1. Introduction

During the past few decades financial analysts experienced significant success by using simple diffusion models to approximate the returns of assets. The Black-Scholes (BS) model [4], where the price dynamics are driven by the Wiener process, is the most popular model in option pricing theory. However some well-known contradictions arising in Black-Scholes revealed that a simple geometric Brownian motion (GBM) process fails to represent some important features of the data. Assuming continuous paths for price processes neglects unexpected movements. Also as Cont *et. al.* discuss in [7], various empirical results show that sudden downward jumps have been observed in stock price processes. To capture these essential characteristics and improve on the pricing and hedging performance of the Black-Scholes model, recently the majority of the research has been done on alternative pricing models such as the models where the jumps are taken into account. Actually, it is possible to consider the models that incorporate jumps in two separate classes. Processes that have almost surely a finite number of jumps in every compact, and the processes with infinite jumps. In the first category, called jump-diffusion models ([1] and [14]), the evolution of prices is given by a diffusion process, intervened by jumps at random times where the jumps represent rare events. It must contain a Brownian component and the distribution of jump sizes is known. In this case the process is called a compound Poisson process with Brownian component. Infinite activity models, which have infinitely many jumps in any finite time interval, fall into the second category. For more detailed information on different types of geometric Lévy models in finance see [7] and [12]. Because of the reasons we mentioned above, we model the market by a geometric Lévy process. Due to the uncertainty of the jump size distribution in the infinity activity models, we restrict ourselves to a jump-diffusion model in pricing and application.

An important feature of the standard Black-Scholes model is that the market is complete, that is, any contingent claim can be replicated by a self-financing portfolio. Incompleteness arises in a market when the sources of randomness are more than the number of assets traded. In this work, we model the market with a general Lévy process and denote this process and the jump size at time t under equivalent martingale measure \mathbb{Q} as \tilde{X}_t and $\Delta\tilde{X}_s = \tilde{X}_s - \tilde{X}_{s-}$ for $0 \leq t \leq T$ respectively. Since the dynamics of the market model is driven by a general Lévy process, the market becomes incomplete. In the literature, there are different approaches towards the completion of a Lévy market. We choose the completion technique, firstly introduced in Corcuera *et. al.* [9], where the market is enlarged with special artificial assets called *power-jump assets* which will be discussed in detail in Section 4. The orthogonalized price processes of power-jump assets, $\bar{H}^{(i)} = (\bar{H}_t^{(i)})_{0 \leq t \leq T}$, are based on an orthogonalized family of Teugel martingales, defined as follows:

$$\bar{H}_t^{(i)} = e^{rt} \tilde{T}_t^{(i)}, i \geq 2.$$

Here r is the risk-free interest rate and the orthogonalized family of Teugel martingales $\tilde{T}^{(i)} = (\tilde{T}_t^{(i)})_{0 \leq t \leq T}$ are derived by the following relation:

$$\tilde{T}^{(i)} = a_{i,i} \tilde{Y}^{(i)} + a_{i,i-1} \tilde{Y}^{(i-1)} + \dots + a_{i,1} \tilde{Y}^{(1)}, i \geq 1,$$

where $a_{n,k}$, $k = 1, 2, \dots, n-1$, $n = 2, 3, \dots$ are constant coefficients with $a_{i,i} = 1$ for all $i \geq 1$. Here, $\tilde{Y}_t^{(i)} := \tilde{X}_t^{(i)} - \mathbb{E}_{\mathbb{Q}}[\tilde{X}_t^{(i)}]$, $0 \leq t \leq T$ with $\tilde{X}_t^{(i)} = \sum_{0 < s \leq t} (\Delta\tilde{X}_s)^i$ are Teugel martingales under the equivalent martingale measure \mathbb{Q} . One of our main goals is to find the coefficients $a_{n,k}$, $k = 1, 2, \dots, n-1$, $n = 2, 3, \dots$ explicitly, so that we acquire the orthogonalized price processes of the power-jump assets. By its definition the power-jump process of order two is just the quadratic variation process (see, e.g., [2] and [3]), and is

closely linked with the so-called realized variance. Contracts on realized variance have managed to get into OTC markets and are traded regularly. Higher order power-jump processes have a similar relationship with realized skewness and realized kurtosis. Hence, obtaining the coefficients plays an important role in pricing these assets.

The paper is organized as follows. The second section deals with some basic concepts of Lévy processes. In Section 3, we introduce the framework of the geometric Lévy market model. In Section 4, the market completion is examined under the conditions for a unique equivalent martingale measure. In Section 5, we obtain the price of European call options by the martingale pricing approach and the characteristic formula approach which is performed via the fast Fourier transform (FFT). Finally, in Section 6 we conclude with an application.

2. Lévy processes

In this section, we concisely recall some concepts of Lévy processes. For detailed information, see [7] and [16]. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ where the filtration satisfies the following *usual hypotheses*:

- (i) \mathcal{F}_0 contains all the \mathbb{P} -null sets of $(\mathcal{F}_t)_{t \geq 0}$,
- (ii) $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u, \forall t, 0 \leq t \leq \infty$.

Throughout this study we will work on this complete filtered probability space. A real-valued, adapted stochastic process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ a.s. is said to be a *Lévy process* if it has independent, stationary increments and is stochastically continuous. A Lévy process have a càdlàg modification [16] and throughout this paper, we only use the càdlàg version. Let us denote the left limit process by X_{t-} , that is

$$X_{t-} = \lim_{s \rightarrow t, s < t} X_s, \text{ for } t > 0,$$

and the jump size at time t by $\Delta X_t = X_t - X_{t-}$. It is known that the law of X_t is infinitely divisible with characteristic function of the form

$$\mathbb{E}[\exp(i\theta X_t)] = (\Phi(\theta))^t,$$

where Φ is the characteristic function of X_1 . Moreover, $\Psi = \log(\Phi)$ is called the characteristic exponent. Here $X_t, t \in [0, T]$ has a characteristic triplet (α, c^2, ν) . The characteristic exponent that is given by the following expression, named the Lévy-Khintchine formula

$$\Psi(u) = i\alpha u - \frac{1}{2}u^2 c^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathcal{J}_{\{|x| < 1\}}) \nu(dx),$$

where $\alpha \in \mathbb{R}, c \geq 0$, and the measure ν on \mathbb{R} satisfies $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty} (1 \wedge x^2) \nu(dx) < \infty$. Hence the characteristic triplet (α, c^2, ν) defines the law of X_t . The measure ν on \mathbb{R} is called the Lévy measure of X . Its value for any A in the Borel sigma algebra, is defined by

$$\nu(A) = \mathbb{E}(\#\{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A\}).$$

3. The model

Let $\{X_t, t \leq T\}$ be a Lévy process satisfying the properties we mentioned in Section 2. Our market consists of a risk-free asset (bank account process) denoted by B_t and a risky asset denoted by S_t for $t \in [0, T]$. We assume that the returns of the risk-free and the

risky assets have the following dynamics, respectively:

$$(3.1) \quad \frac{dB_t}{B_t} = r dt,$$

$$(3.2) \quad \frac{dS_t}{S_{t-}} = b dt + dX_t,$$

with the initial conditions $B_0 = 1$ and $S_0 \in \mathbb{R}^+$. Here r is the constant risk-free interest rate and b is the drift term. The solution to Equation (3.2) with initial condition S_0 is called the stochastic exponential of the process $X_t + bt$ and is given by

$$\varepsilon(X_t + bt) = e^{X_t + bt - \frac{1}{2}[X^c, X^c]_t} \prod_{s \leq t, \Delta X_s \neq 0} (1 + \Delta X_s) e^{-\Delta X_s},$$

where $[X^c, X^c]_t$ is the quadratic variation of the continuous part of the Lévy process.

By Lévy-Khintchine formula it can be showed that X has the following decomposition:

$$X_t = cW_t + L_t,$$

where $W = (W_t)_{\{0 \leq t \leq T\}}$ is a Brownian motion and $L = (L_t)_{\{0 \leq t \leq T\}}$ is a pure jump process which is independent of W . By the Lévy-Itô decomposition, L can be decomposed as follows:

$$(3.3) \quad L_t = \alpha t + \int_{(0,t] \times \{|x| < 1\}} x(N(ds, dx) - t\nu(dx)) + \int_{(0,t] \times \{|x| \geq 1\}} xN(ds, dx),$$

where $N(dt, dx)$ is a Poisson random measure on $(0, \infty) \times \mathbb{R} - \{0\}$ with intensity $dt \times \nu$. In the remainder of this study, we will assume that the Lévy measure satisfies, for some $\epsilon > 0$ and $\lambda > 0$,

$$\int_{(-\epsilon, \epsilon)^c} \exp(\lambda |x|) \nu(dx) < \infty.$$

As $g(x) = \exp(\lambda |x|)$ is sub multiplicative, we deduce that L has a g -moment for every $t \in [0, T]$, or equivalently the exponential moments $\mathbb{E}(g(x))$ exist by [16, Theorem 25.3]. Hence, this implies that

$$\int_{-\infty}^{\infty} |x|^i \nu(dx) < \infty, \quad i \geq 2.$$

Since every Lévy process is a semi martingale, by the semi martingale decomposition theorem L can be written as a sum of a martingale and a finite variation process as follows:

$$\begin{aligned} L_t &= M_t + at \\ M_t &= \int_{(0,t]} \int_{\mathbb{R}} x \bar{N}(ds, dx), \end{aligned}$$

where $\bar{N}(dt, dx)$ is the compensated Poisson random measure defined as $\bar{N}(dt, dx) = N(dt, dx) - dt\nu(dx)$. In the end, we have the following model for the returns of the risky asset:

$$\frac{dS_t}{S_{t-}} = (a + b)dt + cdW_t + dM_t.$$

By applying the Itô formula for càdlàg semi martingales [7], the stock price process has the following solution:

$$(3.4) \quad S_t = S_0 \exp \left(cW_t + M_t + \left(a + b - \frac{c^2}{2} \right) t \right) \prod_{0 < s \leq t} (1 + \Delta M_s) \exp(-\Delta M_s).$$

4. Equivalent Martingale measures

In this section, we investigate the structure preserving \mathbb{P} -equivalent measures which enable us to stay within an analytically tractable family of models. Under the structure-preserving \mathbb{P} -equivalent measures, X remains a Lévy process and the discounted stock price $\tilde{S} = \{\tilde{S}_t = e^{-rt}S_t, 0 \leq t \leq T\}$ is a martingale. Since we are working on a market with finite horizon T , locally equivalence will be regarded as the equivalence. The following theorem [16, Theorem 33.1. and 33.2] plays a very important role in characterizing the triplets of X under the equivalent probability measure.

4.1. Theorem. *Let X be a Lévy process with Lévy triplet (α, c^2, ν) on the filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Then the following conditions are equivalent:*

- (i) *There exists a probability measure \mathbb{Q} , locally equivalent to \mathbb{P} for any $t \geq 0$, such that X is a \mathbb{Q} -Lévy process with triplet $(\tilde{\alpha}, \tilde{c}^2, \tilde{\nu})$.*
- (ii) *All of the following conditions hold:*
 - (a) *$\tilde{\nu}(dx) = H(x)\nu(dx)$ where $H : \mathbb{R} \rightarrow (0, \infty)$ is a Borel function satisfying the following:*

$$\int_{-\infty}^{\infty} (1 - \sqrt{H(x)})^2 \nu(dx) < \infty,$$

- (b) *$\tilde{\alpha} = \alpha + \int_{-\infty}^{\infty} x \mathcal{J}_{\{|x| \leq 1\}}(H(x) - 1)\nu(dx) + Gc$ for some $G \in \mathbb{R}$,*
- (c) *$\tilde{c} = c$.*

From now on, let us assume that a structure preserving the equivalent martingale measure \mathbb{Q} is given so that X remains a Lévy process. We know that under this measure the discounted asset price process is a martingale and the process $\tilde{X} = \{X_t + (b-r)t, 0 \leq t \leq T\}$ is a Lévy process with the Lévy measure $\tilde{\nu}$. Hence, we get the dynamics of a risky asset under measure \mathbb{Q} as follows:

$$\frac{dS_t}{S_{t-}} = r dt + d\tilde{X}_t.$$

Solving the above SDE, the discounted asset price process under \mathbb{Q} denoted by \tilde{S} is found in the following form: (see [9])

$$\begin{aligned} \tilde{S}_t = S_0 \exp & \left(c\tilde{W}_t + \tilde{M}_t + \left(a + b - r + cG - \frac{c^2}{2} \right) t \right) \\ & \times \exp \left(t \int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx) \right) \prod_{0 < s \leq t} (1 + \Delta\tilde{M}_s) \exp(-\Delta\tilde{M}_s). \end{aligned}$$

Here, \tilde{W} is the Brownian motion under structure preserving measure \mathbb{Q} and

$$(4.1) \quad \tilde{M}_t = \int_{(0,t]} \int_{\mathbb{R}} x \overline{N}_{\mathbb{Q}}(ds, dx),$$

where $\overline{N}_{\mathbb{Q}}(dt, dx)$ is the compensated Poisson random measure under \mathbb{Q} , defined as $\overline{N}_{\mathbb{Q}}(dt, dx) = N_{\mathbb{Q}}(dt, dx) - dt\tilde{\nu}(dx)$ and $N_{\mathbb{Q}}(dt, dx)$ is the Poisson random measure under \mathbb{Q} .

The necessary and sufficient condition for \tilde{S} to be a \mathbb{Q} -martingale is the existence of G and $H(x)$ with the condition $\int_{-\infty}^{\infty} (1 - \sqrt{H(x)})^2 \nu(dx) < \infty$ such that

$$(4.2) \quad cG + a + b - r + \int_{-\infty}^{\infty} x(H(x) - 1)\nu(dx) = 0.$$

4.1. The power-jump processes. Consider the process

$$(4.3) \quad \tilde{X}_t^{(i)} = \sum_{0 < s \leq t} (\Delta \tilde{X}_s)^i, \quad i \geq 2,$$

where $\Delta \tilde{X}_s = \tilde{X}_s - \tilde{X}_{s-}$ is just the jump size at time s under measure \mathbb{Q} . Set $\tilde{X}_t^{(1)} = \tilde{X}_t$. Notice that, $\Delta \tilde{X}_t = \Delta \tilde{L}_t$ since the two processes jump at the same point. We denote the compensated power jump process of order $i \geq 1$ as follows:

$$(4.4) \quad \tilde{Y}_t^{(i)} := \tilde{X}_t^{(i)} - \mathbb{E}_{\mathbb{Q}}[\tilde{X}_t^{(i)}],$$

where

$$\mathbb{E}_{\mathbb{Q}}[\tilde{X}_t^{(i)}] = \mathbb{E}_{\mathbb{Q}} \left[\sum_{0 < s \leq t} (\Delta \tilde{X}_s)^i \right] = t \int_{-\infty}^{\infty} x^i \tilde{\nu}(dx) = m_i t.$$

Here $\tilde{Y}^{(i)}$ is a normal martingale specifically called a Teugel martingale, which are firstly introduced in [15].

We are looking for a set of pairwise strongly orthogonal martingales $\{\tilde{T}^{(i)}, i \geq 1\}$ such that $\tilde{T}^{(i)}$ is a linear combination of $\tilde{Y}^{(j)}$ for $j = 1, 2, \dots, i$.

4.2. Theorem. Let $\tilde{Y}^{(i)}, i \geq 1$ be a Teugel martingale under measure \mathbb{Q} described above. Set

$$\tilde{T}^{(i)} = a_{i,i} \tilde{Y}^{(i)} + a_{i,i-1} \tilde{Y}^{(i-1)} + \dots + a_{i,1} \tilde{Y}^{(1)}, \quad i \geq 1.$$

Then the coefficients in the Gram-Schmidt process are as follows:

$$(4.5) \quad a_{n,k} = - \sum_{i=k}^{n-1} \frac{\langle \tilde{Y}^n, \tilde{T}^{(i)} \rangle_2}{\langle \tilde{T}^{(i)}, \tilde{T}^{(i)} \rangle_2} a_{i,k},$$

for $k = 1, 2, \dots, n-1$ and $n = 2, 3, \dots$ with $a_{i,i} = 1$ for all $i \geq 1$. Here

$$\begin{aligned} \langle \tilde{Y}^n, \tilde{T}^{(i)} \rangle_2 &= \mathbb{E}([\tilde{Y}^{(n)} \tilde{T}^{(i)}]_1) \\ &= m_{i+n} + a_{i,i-1} m_{i+n-1} + \dots + a_{i,1} m_{n+1} + a_{i,1} \tilde{c}^2 J_{\{n=1\}} \end{aligned}$$

and

$$\langle \tilde{T}^{(i)}, \tilde{T}^{(i)} \rangle_2 = \mathbb{E}([\tilde{T}^{(i)} \tilde{T}^{(i)}]_1) = \sum_{l=1}^i \sum_{l'=1}^i a_{i,l} a_{i,l'} m_{l+l'} + a_{i,1}^2 \tilde{c}^2.$$

Proof. We give a sketch of the proof. Let

$$\tilde{T}^{(1)} = \tilde{Y}^{(1)}.$$

By the Gram-Schmidt process,

$$(4.6) \quad \tilde{T}^{(2)} = \tilde{Y}^{(2)} - \frac{\langle \tilde{Y}^{(2)}, \tilde{Y}^{(1)} \rangle_2}{\langle \tilde{Y}^{(1)}, \tilde{Y}^{(1)} \rangle_2} \tilde{Y}^{(1)}.$$

Here $\langle \tilde{Y}^{(1)}, \tilde{Y}^{(1)} \rangle_2 = m_2 + \tilde{c}^2$ and $\langle \tilde{Y}^{(2)}, \tilde{Y}^{(1)} \rangle_2 = m_3$. By induction, one can deduce the above formula. \square

The completeness of the market is proved in [9] using the Martingale Representation Property (MRP), obtained in [15]. MRP allows any square integrable \mathbb{Q} -martingale to be represented as an orthogonal sum of stochastic integrals with respect to the orthogonalized martingales $\tilde{T}^{(i)} = (\tilde{T}_t^{(i)})_{0 \leq t \leq T}$ and implies that in the market enlarged with the i -th power-jump assets, every square-integrable contingent claim can be replicated by a sequence of self-financing portfolios. The following theorem, which is proved in [9],

shows that the market enlarged with the i -th power-jump assets $H^{(i)} = \{H_t^{(i)}, t \geq 0\}$, is complete, where

$$H_t^{(i)} = e^{rt} \tilde{Y}_t^{(i)}$$

and $\tilde{Y}^{(i)}$ is the Teugel martingale defined in Equation (4.4).

4.3. Theorem. *The Lévy market model enlarged with the i -th power-jump assets is complete in the sense that any square integrable contingent claim X can be replicated, that is, the final values of the self-financing portfolios will converge to X in the $L^2(\mathbb{Q})$ sense. \square*

Since the market is complete by Theorem 4.3 and the completion is basically based on the drift term, we can take $H \equiv 1$ in Theorem 4.2. We conclude that

$$(4.7) \quad \tilde{\nu}(dx) = \nu(dx),$$

$$(4.8) \quad \tilde{\alpha} = \alpha + r - a - b,$$

$$(4.9) \quad \tilde{c} = c.$$

Hence, under the unique martingale measure \mathbb{Q} , one can define the Teugel martingales as follows:

$$(4.10) \quad \tilde{Y}_t^{(i)} := \tilde{X}_t^{(i)} - \mathbb{E}_{\mathbb{Q}}[\tilde{X}_t^{(i)}] = \tilde{X}_t^{(i)} - t \int_{-\infty}^{\infty} x^i \nu(dx).$$

In practice, once we make the necessary assumptions on the distribution of the jump sizes, we can calculate the Teugel martingales. Moreover, by Theorem 4.2, we get the orthogonalized family of Teugel martingales \tilde{T} . Note that, the completion technique applied in this paper results in an arbitrage-free market as shown in [9].

5. Pricing

In this section we discuss two predominant methods for pricing European options on assets driven by Lévy processes. We obtain the explicit prices by two methods which are the *Martingale pricing approach* and the *fast Fourier transform based characteristic formula*.

Throughout the pricing section, we assume that the jump size of the compound Poisson process has a particular distribution which will be clarified below. Ultimately we obtain the value at time t of a European call option with strike price K and payoff function $f(S_T)$ only depending on the stock price at maturity. Equation (4.1) can be restated as follows:

$$\tilde{M}_t = \int_{(0,t]} \int_{\mathbb{R}} x N_{\mathbb{Q}}(ds, dx) - \int_{(0,t]} \int_{\mathbb{R}} x ds \tilde{\nu}(dx),$$

where the first part, the compound Poisson process, can be represented as

$$\sum_{0 < s \leq t} \Delta \tilde{M}_t.$$

Moreover, by [7, Proposition 3.5], we can represent the compensator in the following form:

$$\int_{(0,t]} \int_{\mathbb{R}} x ds \tilde{\nu}(dx) = \int_{(0,t]} \int_{\mathbb{R}} x \tilde{\lambda} F(dx) dt = t \tilde{\lambda} \mathbb{E}_{\mathbb{Q}}(\Delta \tilde{M}_t),$$

where $\tilde{\lambda}$ is the intensity of the Poisson random measure, F is the jump size distribution and $\mathbb{E}_{\mathbb{Q}}(\Delta \tilde{M}_t)$ denotes the expected jump size under \mathbb{Q} .

Since a compound Poisson process has almost surely a finite number of jumps in interval $(0, t]$, the summation is finite. Hence, we do not have convergence problems. If

$U = (U_i)_{i \geq 1}$ is the sequence of independent random variables, then the jump size process of \tilde{M}_t takes the following form:

$$\tilde{M}_t = \sum_{i=1}^{N(t)} U_i - \tilde{\lambda} \mathbb{E}_{\mathbb{Q}}(\Delta \tilde{M}_t)t.$$

Under these circumstances, the risk-neutral dynamics of the stock price process can be rewritten as:

$$\begin{aligned} S_t &= S_0 \exp \left(c\tilde{W}_t + \tilde{M}_t + \left(r - \frac{c^2}{2} \right) t \right) \prod_{0 < s \leq t} (1 + \Delta \tilde{M}_s) \exp(-\Delta \tilde{M}_s) \\ &= S_0 \exp \left(c\tilde{W}_t - \tilde{\lambda} \mathbb{E}_{\mathbb{Q}}(\Delta \tilde{M}_t)t + \left(r - \frac{c^2}{2} \right) t + \sum_{i=1}^{N(t)} \ln(1 + U_i) \right), \end{aligned}$$

where $U_i > -1$ for all $1 \leq i \leq N(t)$.

Henceforth we assume that jumps in the log-price process have a Gaussian distribution: $\ln(1 + U) \sim N(m, \varrho^2)$ as in the Merton Jump-Diffusion (JD) model [14]. This allows us to obtain the probability of the intensity process of the jumps and find the price of European options using the density function.

Hence, the stock price process under the measure \mathbb{Q} becomes as follows:

$$(5.1) \quad S_t = S_0 \exp \left(c\tilde{W}_t + \left(r - \tilde{\lambda} \left(\exp \left(m + \frac{\varrho^2}{2} \right) - 1 \right) - \frac{c^2}{2} \right) t + \sum_{i=1}^{N(t)} \ln(1 + U_i) \right).$$

5.1. Martingale pricing approach. The Martingale pricing approach is based on the fact that the discounted value of a contingent claim is a martingale under measure \mathbb{Q} . In the Black-Scholes market model, it is rather easy to obtain the option price by this approach. Hence it is useful to price in extended market models based on the Black-Scholes settings. In this subsection, we obtain the price of a European call option under the Merton JD model, in terms of the Black-Scholes model price.

5.1. Theorem. *The price of a European call option on an underlying asset satisfying Equation (5.1) has the following form:*

$$(5.2) \quad C(t, S_t) = \sum_{n=0}^{+\infty} \frac{(\tilde{\lambda}\tau)^n \exp(-\tilde{\lambda}\tau)}{n!} C_{BS}(\tau, S_n; c_n),$$

where $\tau = T - t$ is the time to maturity, C_{BS} is the Black-Scholes model price, $c_n = \sqrt{\frac{c^2\tau + n\varrho^2}{\tau}}$, $S_n = S_t \exp \left(nm + \frac{n\varrho^2}{2\tau} - \tilde{\lambda}(\exp(m + \frac{\varrho^2}{2}) - 1)\tau \right)$ and m, ϱ^2 are the mean and the variance of the jump size distribution.

Proof. The value of a contingent claim X with a payoff function $f(S_T)$ at time $t \in [0, T]$ is defined as:

$$(5.3) \quad C(t, S_t) = \exp(-r(T-t)) \mathbb{E}_{\mathbb{Q}}(f(S_T) | \mathcal{F}_t).$$

The proof can be completed by substituting Equation (5.1) in Equation (5.3) and doing some straightforward calculations. \square

Note that, the formula we derived in Equation (5.2) is the same as the one derived from solving the partial differential equation (PDE) based on the first and second derivatives of the option price. In other words, the PDE approach and the Martingale approach give

the same result. The corresponding derivatives used in the PDE approach in our setting are given as follows :

$$D_2^n C(t, x) = \mathbb{E}_{\mathbb{Q}} \left(\exp(n\tilde{M}_{T-t}) \prod_{0 < s \leq T-t} (1 + \Delta\tilde{M}_s)^n \exp(-n\Delta\tilde{M}_s) \right. \\ \left. \times D_2^n C_{BS} \left(t, x \exp(n\tilde{M}_{T-t}) \prod_{0 < s \leq T-t} (1 + \Delta\tilde{M}_s) \exp(-\Delta\tilde{M}_s) \right) \right),$$

where \tilde{M} is the process defined in Equation (4.1) and $D_2^n C_{BS}(\cdot, \cdot)$ is the n -th order derivative of the Black-Scholes model price, C_{BS} , with respect to the second component, for $n = 1, 2$. Note that the necessary Black-Scholes price derivatives $D_2^n(C_{BS})$, $n = 1, 2$ are very simple to derive. For instance for a European call the first two derivatives, namely *Delta* and *Gamma*, are given in terms of the cumulative probability distribution function $N(x)$ and the density function $n(x)$ of a standard Normal random variable.

5.2. Characteristic formula approach via the fast Fourier transform. In this section we obtain the price of a European call option by using the fast Fourier transform (FFT) method, following the steps presented in the paper of Carr *et. al.* [5]. This method has significant advantages compared to the classical Martingale approach. First of all, when the risk neutral density is unknown in the models (as is often the case), we can find the price by using the Fourier transform of S_t , which can be calculated by the Lévy-Khintchine representation [7]. Secondly, the algorithms used for the inversion of the Fourier transform are fast and optimized. Also the method enables us to price options with different strikes in a single calculation.

In this subsection, we consider a European call option written on S_T with payoff function $f(S_T)$ at maturity time T and strike price K . Note that the properties of this contingent claim are the same as the properties of the one introduced in the previous subsection.

5.2. Theorem. *The price of a European call option obtained by the characteristic formula approach and the fast Fourier transform is*

$$(5.4) \quad C(k_v) = \frac{e^{-\alpha k_v}}{\pi} \sum_{j=1}^{N-1} \left(\frac{e^{i(u_j - i(1+\alpha))(r - \frac{c^2}{2})t - \frac{1}{2}(u_j - i(1+\alpha))^2 c^2 t}}{\alpha^2 + \alpha - u_j^2 + i(2\alpha + 1)u_j} \right. \\ \left. - \frac{\tilde{\lambda}t \left(e^{m + \frac{\rho^2}{2}} - e^{i(u_j - i(1+\alpha))m - \frac{1}{2}(u_j - i(1+\alpha))^2 \rho^2} \right)}{\alpha^2 + \alpha - u_j^2 + i(2\alpha + 1)u_j} \right) \\ \times \frac{e^{-rT + i b u_j - i \frac{2\pi}{N}(j-1)(v-1)}}{\alpha^2 + \alpha - u_j^2 + i(2\alpha + 1)u_j} \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1}),$$

where k_v are the segmented log-strike prices (the log-strikes that are equally spaced around the log-spot price for $v = 1, \dots, N$), α is the dampening factor which is a parameter used to modify the call price so that it becomes square integrable for a range of its values and m, ρ^2 are the mean and the variance of the jump size distribution.

Proof. The characteristic function of the log-price process $s_T = \ln(S_T)$ can be found explicitly as follows:

$$(5.5) \quad \phi_T(u) = \exp \left(iu \left(r - \frac{c^2}{2} \right) t - \frac{1}{2} u^2 c^2 t - iu \tilde{\lambda} t \left(\exp \left(m + \frac{\rho^2}{2} \right) - \exp \left(ium - \frac{1}{2} u^2 \rho^2 \right) \right) \right).$$

If k denotes the log of the strike price K , then $C_T(k)$ is the desired value of a T maturity call option with strike $\exp(k)$. Let the risk-neutral density of the log-price be $q_T(s)$. The

characteristic function of $s_T = \ln(S_T)$ is defined by:

$$\phi_T(u) = \int_{-\infty}^{+\infty} e^{ius} q_T(s) ds.$$

The initial call price $C_T(k)$ is related to the risk-neutral density by the following steps:

$$(5.6) \quad \begin{aligned} C_T(k) &= \mathbb{E}_{\mathbb{Q}} \left(e^{-rT} (S_T - K)_+ | \mathcal{F}_0 \right) \\ &= \int_k^{\infty} e^{-rT} (e^s - e^k) q_T(s) ds. \end{aligned}$$

Since $C_T(k)$ does not decay on the negative log-strike axis, it is not integrable. To overcome this problem, we use a dampening coefficient α as in [6] and define a new square integrable modified call price $\bar{C}_T(k)$ as follows:

$$\bar{C}_T(k) = \exp(\alpha k) C_T(k)$$

for $\alpha > 0$.

For more details on the selection of the dampening coefficient α such that $\bar{C}_T(k)$ does have a well-defined Fourier transform, one can see [5].

Considering the Fourier transform of $\bar{C}_T(k)$ defined by:

$$\psi_T(u) = \int_{-\infty}^{+\infty} e^{iuk} \bar{C}_T(k) dk,$$

we have the following direct Fourier transform by the inversion formula in [17]

$$(5.7) \quad C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{iuk} \psi_T(u) du.$$

By Equation (5.6) and Fubini's Theorem, the analytical relation is obtained as follows:

$$(5.8) \quad \psi_T(u) = \frac{e^{-rT} \phi_T(u - i(1 + \alpha))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u},$$

where $-\infty < k < s < +\infty$ and $\phi_T(u)$ is the characteristic function of $s_T = \ln(S_T)$. Substituting Equation (5.8) into Equation (5.7), the call option price is obtained as:

$$(5.9) \quad C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{-iuk} \frac{e^{-rT} \phi_T(u - i(1 + \alpha))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u} du.$$

Afterwards, we perform the required integration using FFT and obtain an approximation of $C_T(k)$. FFT is an efficient algorithm for computing the sum:

$$(5.10) \quad \omega(k) = \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j)$$

for $k = 1, \dots, N$.

Using the Trapezoidal rule for the integral in Equation (5.7), an approximation of $C_T(k)$ is obtained as follows:

$$(5.11) \quad C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N-1} e^{-iu_j k} \psi_T(u_j) \eta,$$

where η is the width of the segmented intervals, $u_j = \eta(j - 1)$. Employing a regular spacing size of $\hat{\lambda}$ between two consecutive log-strikes, values of k are determined as:

$$k_v = -b + \hat{\lambda}(v - 1) + s_0,$$

for $v = 1, \dots, N$ where $b = \frac{N\hat{\lambda}}{2}$. However, when we consider $S_0 = 1$, we obtain $s_0 = 0$ and this implies

$$k_v = -b + \hat{\lambda}(v - 1),$$

for $v = 1, \dots, N$. Applying Simpson's rule, the weighting call price can be written in the following way:

$$(5.12) \quad C_T(k_v) = \frac{e^{-\alpha k_v}}{\pi} \sum_{j=1}^{N-1} e^{-i\frac{2\pi}{N}(j-1)(v-1)} e^{ibu_j} \psi_T(u_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1}),$$

where δ_n is the Kronecker delta function.

After expressing these relations in the Lévy market model described in the previous section and using the characteristic function of the log-price process in Equation (5.5), we rewrite the following equation:

$$\begin{aligned} \phi_T(u - i(1 + \alpha)) &= \exp \left\{ i(u - i(1 + \alpha)) \left(r - \frac{c^2}{2} \right) t - \frac{1}{2} (u - i(1 + \alpha))^2 c^2 t \right\} \\ &\quad \times \exp \left\{ -\tilde{\lambda} t \left(e^{(m + \frac{\sigma^2}{2})} - e^{(i(u - i(1 + \alpha))m - \frac{1}{2}(u - i(1 + \alpha))^2 \sigma^2)} \right) \right\}. \end{aligned}$$

Substituting the above equality into the Equation (5.8), we obtain the Fourier transform of $\bar{C}_T(k)$, ψ_T , as follows:

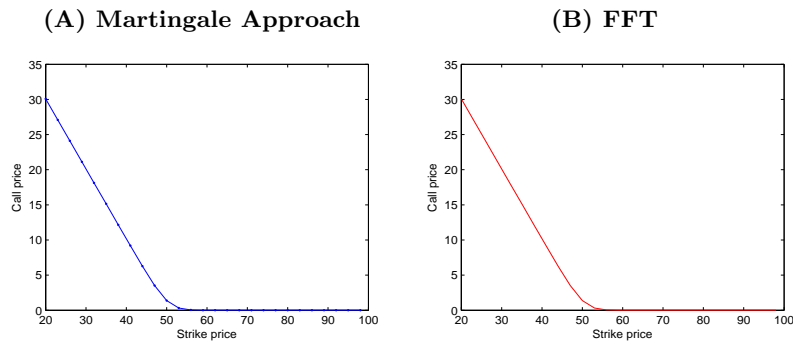
$$(5.13) \quad \begin{aligned} \psi_T(u) &= \frac{e^{i(u - i(1 + \alpha))(r - \frac{c^2}{2})t - \frac{1}{2}(u - i(1 + \alpha))^2 c^2 t - \tilde{\lambda} t (e^{m + \frac{\sigma^2}{2}} - e^{i(u - i(1 + \alpha))m - \frac{1}{2}(u - i(1 + \alpha))^2 \sigma^2}) - rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u} \end{aligned}$$

Finally, substitution of Equation (5.13) into Equation (5.12) yields the desired call option price. □

In the next subsection, we will compare the prices computed by the two approaches in this section. We obtain the price values for fixed parameters and estimated parameters using market option price data by computing the results via computer software.

5.3. Comparison of Pricing Methods. For the sake of the simplicity, we use the same parameters and variables for both formulas derived in Theorem 5.1 and in Theorem 5.2. In both calculations, we fix $S_0 = 50$, $c = 0.02$, $r = 0.05$, $T = \frac{20}{252}$, $\lambda = 1$, $m = -0.1$, $\sigma = 0.01$. As shown in Figure ??, the outcomes of both methods are almost the same. In fact, it is the expected result since Equation (5.4) is just a series representation of the martingale price formula.

Figure 1. Call prices generated by the two approaches



6. Application

In this section we illustrate the theory of the previous parts and calibrate the JD model to option data set on Standard&Poors 500 index options. The calibration consists of an estimation of the unknown parameters of our model which reproduce almost perfectly the market option prices. In the calibration of Lévy driven models, qualitative features of the model must be considered such as infinite/finite activity. Moreover, testing the data concerned for the presence of jumps will provide accuracy in the model choice. Several studies have investigated the presence of a jump component looking at the S&P 500 option data and it is shown in [7] that in addition to a continuous diffusion component, the jump component exists but may not be present every single day in the sample. It is shown in [8] that Lévy processes produce better implied volatility smile for a single maturity, but when calibrating several maturities at the same time, the calibration of Lévy processes becomes much less precise. This difficulty of calibrating an exponential-Lévy model to option prices of several different maturities arises due to independence and stationarity properties of the increments.

Data consists of S&P 500 index call option prices observed in the market on 20 August 2010 at 2.42 pm and at that time the S&P 500 index quote was 1075.63. Options are selected with 21 different strike prices ranging from 925 to 1500 with the maturities; $T_1 = 21$ days, $T_2 = 51$ days, $T_3 = 82$ days, $T_4 = 203$ days, $T_5 = 295$ days, $T_6 = 478$ days, $T_6 = 478$ days, $T_7 = 662$ days, $T_8 = 845$ days.

Maturities of the selected options are at least 20 days as prices of short maturity options are very close to the intrinsic value. Also, deep in the money and deep out of the money options are eliminated because of unrealistic implied volatilities and difficulties in implied volatility calculation.

6.1. Parameter estimation. The parameter estimation procedure is carried out by a sampling algorithm that minimizes numerically the sum of squares of the difference between market and JD model prices. The vector of unknown parameters θ are hence determined by minimizing the following expression

$$(6.1) \quad \sum_{i=1}^N (C_{market}^i - C_{model}^{i,\theta})^2.$$

The algorithm used in this process is the DIRECT optimization algorithm developed by Jones *et. al.* in [10], which is a method for finding the global minimum of a multivariate function subject to simple bounds. It was created to solve difficult global optimization problems with bound constraints and a real-valued objective function.

The minimization function concerned is non-convex, so a gradient-based method may not succeed in locating the global minimum, which in turn reduces the quality of the calibration algorithm. Therefore, we employ the DIRECT sampling algorithm which does not require any knowledge of the objective function's gradient. Instead, it samples points in the domain, and uses the information it has obtained to decide where to search next. The algorithm will globally converge to the minimal value of the objective function in Equation (6.1).

DIRECT's sample points are centers of hyperrectangles and it initiates its search by sampling the objective function at the center of Ω . The entire domain is treated as the first hyperrectangle, which DIRECT identifies as potentially optimal and divides. In the division process, it determines hyperrectangles that has the most potential to contain unsampled points and in the sampling stage it samples f at the centers of the newly-created hyperrectangles.

Let $a, b \in \mathbb{R}$, $\Omega = \{x \in \mathbb{R}^N : a_i \leq x_i \leq b_i\}$ and $f : \Omega \rightarrow \mathbb{R}$ be Lipschitz continuous with constant α . The algorithm attempts to find a x_{opt} such that

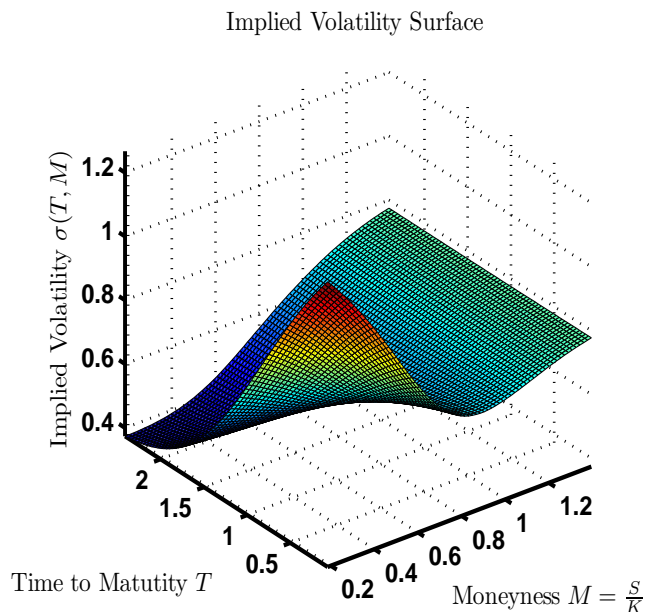
$$f_{opt} = f(x_{opt}) \leq f^* + \varepsilon,$$

where ε is a given small positive constant and f^* is the current best function value.

The DIRECT algorithm converges globally to the minimal value of the objective function. However, this global convergence may occur after a large and exhaustive search over the domain. The algorithm is advantageous because of the balanced effort it gives to local and global searches, and the few parameters it requires to run. The algorithm has been shown to be very competitive with other Lipschitzian optimization algorithms in its class, see [10] for further details.

6.2. Results. Merton’s JD model generates volatility smiles by adding discontinuous jumps to diffusion dynamics. By choosing the parameters of the jump process appropriately, the JD model can generate a multitude of volatility smiles and skews. In addition to calculating a volatility smile, a volatility term structure can be calculated as a function of maturity for a fixed strike price. Combining the volatility smile and the volatility term structure, we can generate a volatility surface, one dimension for maturity and the other for the strike price.

Figure 2. Implied volatility surface generated by the Jump-Diffusion process. ($r = 0.05$, $\tilde{\lambda} = 0.01$, $c = 0.15$, mean jump size $m = -0.05$, jump size standard deviation $\varrho = 0.4$)



In Figure 2 we plot the implied volatility surface of the JD model for fixed parameters. Observing the figure, it can be seen the jump part of the process dynamics drives the skew in the short time to expiration limit which is short dated and dies with time to maturity. Therefore while failing for longer maturities, the model explains a significant part of the volatility smile/skew observed in the market.

On the other hand, stochastic volatility models cannot generate enough skew for very short expirations but in some degree fit for longer expirations. So, a combination of stochastic volatility and jumps in one model can display a relatively good performance in volatility smile modeling.

Estimated parameter values for separate and simultaneous calibration of JD and are given in Tables ?? and ??, respectively.

Table 1. Separately calibrated JD model parameters for different times to maturity

Maturity	c	$\tilde{\lambda}$	m	ϱ
21 days	0.2550	0.0261	0.0750	0.1289
51 days	0.2006	0.1550	-0.2167	0.2600
82 days	0.2006	0.1550	-0.0500	0.2600
203 days	0.1824	0.2517	-0.1056	0.2067
295 days	0.1824	0.1550	-0.0056	0.2067
478 days	0.1824	0.1872	-0.1611	0.3133
662 days	0.1461	0.1550	-0.2667	0.1000
845 days	0.1461	0.2194	0.1167	0.4733

Table 2. Simultaneously calibrated JD model parameters for all times to maturity

Maturity	c	$\tilde{\lambda}$	m	ϱ
All Maturities	0.2369	0.2517	-0.1015	0.42

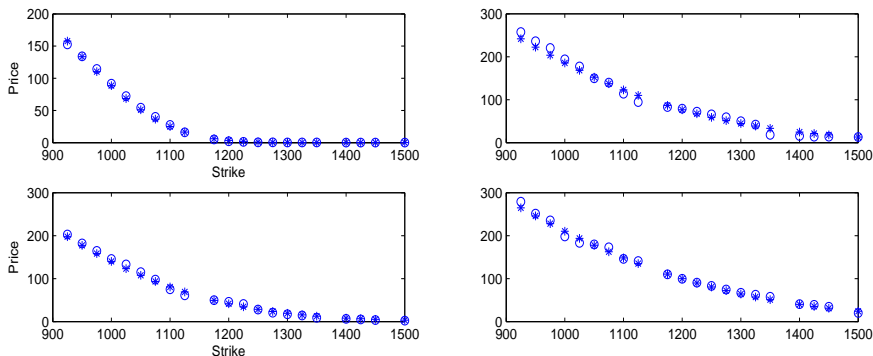
In Table 1, we observe different parameter values for the S&P500 index for given strike price and different time to maturities. As we can see from the table, the qualitative behavior varies through the maturities. Volatility is seen to decline as time to maturity increases, which is the case for at the money options. Since we select our data set among the money options, indeed this phenomenon is expected. Moreover, the range of the estimates points out that the true volatility of the asset for the corresponding period lies between 0.1461 and 0.255. Moreover, the jump intensity, denoted by $\tilde{\lambda}$, oscillates around 0.2 which indicates a very low average. This finding supports our model's assumption about jumps being rare events. The expected jump size, m , seems to have a random behavior which shows the consistency with the random jump size assumption and the random jump direction. Finally, we see that ϱ , which is the standard deviation of the jump sizes, follows a random pattern. This can be interpreted as the occurrence of independent shocks in the market.

The calibration procedure is carried out both simultaneously for all maturities and separately for each maturity in the S&P 500 option data sample. In the next figures we plot the market and model prices which are calculated with the parameters found after calibration for all the strikes and maturities.

Figure 3. Calibration of JD model to market data separately for each maturity

(A) Top: maturity 51 days
Bottom: maturity 203 days

(B) Top: maturity 662 days
Bottom: maturity 845 days



As seen in Figure 3, the calibration for each individual maturity gives quite good results, despite the fact that the options with different maturities correspond to the same underlying and the same trading day, the parameter values for each maturity are different, as seen from Table 1.

Figure 4. Calibration of the jump-diffusion model simultaneously to 8 maturities. (circles are market prices, stars are model prices)

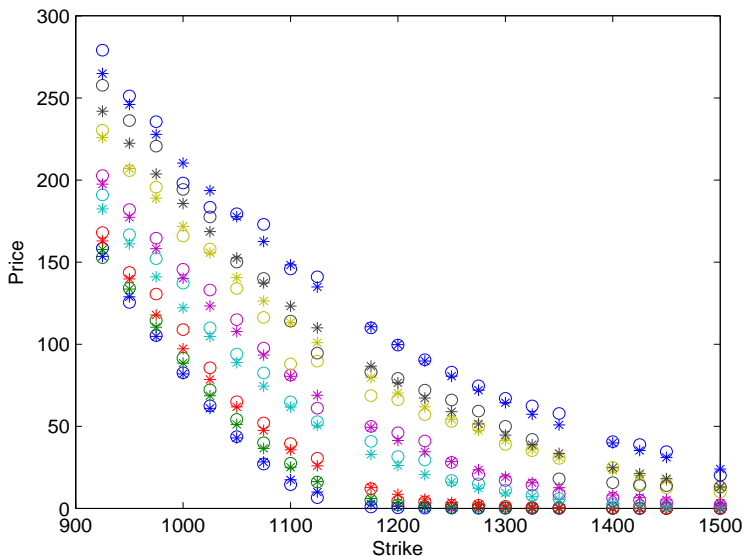


Figure 4 presents the result of the simultaneous calibration of the model to 8 different maturities ranging from 1 month to 3 years. As can be seen from the Figure, the calibration error is much higher than in Figures 3A and 3B.

This problem arises due to the fact that log-price process, which is a Lévy process, has independent and stationary increments. So the law of the entire process is completely determined by its law at any given time t . If we have calibrated the model parameters for a single maturity T , this fixes completely the risk-neutral stock price distribution for all other maturities.

To adequately calibrate a jump-diffusion model to options of different maturities at the same time, the model must have a sufficient number of degrees of freedom to generate different term structures.

7. Conclusion

In this paper, a geometric Lévy market model is considered in three parts. In the first part, the market setup is examined. Since generally these markets are incomplete, for completion the market is enlarged by a series of artificial assets called power-jump assets which are related to the power-jump processes of the underlying Lévy process as in [9]. These assets are linked to options on the stock and contracts on realized variance that are traded in OTC markets and thus can be traded for volatility expectation purposes. Next, it is shown by using the martingale representation property that the enlarged market is complete. Then the equivalent martingale measure conditions are given and the market is shown to be arbitrage-free. Finally, the explicit hedging portfolios for contingent claims whose payoff function depends on the prices of the stock are derived.

In the second part, we obtained prices for European call options by using two pricing methods under a specified jump-size distribution for the jump sizes. The methods were the Martingale pricing approach and the characteristic formula approach via the fast Fourier transform. Moreover, we made a comparison of the performances and speeds of these methods. We found that the FFT produces very small pricing errors and the results generated by the methods are nearly identical.

In the third part, we performed calibration of the specified jump-diffusion model on S&P 500 call option data. The optimization algorithm used is discussed and the parameter estimation results are presented. We found that for longer maturities the jump intensity tends to increase meanwhile the mean jump size decreases for extensive holding period. The calibration of the jump-diffusion model is executed both simultaneously for all maturities and separately for each maturity in the S&P 500 call option data sample. We came to the conclusion that the calibration for each individual maturity gives better results than the simultaneous calibration. The reason for this outcome is the fact that the log-price process, which is a Lévy process, has independent and stationary increments. So the law of the entire process is determined by its law at any given time t . Therefore, when we calibrate the model parameters for a single maturity T , this fixes completely the risk-neutral stock price distribution for all other maturities.

Note that, it is also possible to do the calibration of the parameters of the asset price directly using stock prices and then use those estimations as inputs for the corresponding option price. Or instead, one can estimate the parameters directly from the option price model like in our setting. In fact, in theory both estimations should yield the same option price assuming that option prices reflect underlying properties. But when we make a calibration for different maturities we observe different parameter values for the stock price. Hence for this case, calibration through option prices could reveal other embedded impacts of the price properties. For the simultaneous calibration, however, the parameters from both data sets yield the same option price. Because of this fact, in this work, we did not use any stock price data and work with one compact option data set. As an accuracy test, the estimations obtained using the stock prices could be

inserted into the option price formula as inputs (like historical volatility in the BS model) and the difference between the estimations made through option prices and stock prices could be analyzed.

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