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IDEAL THEORY OF BE-ALGEBRAS BASED ON $\mathcal{N}$-STRUCTURES

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Abstract

Using $\mathcal{N}$-structures, the notion of an $\mathcal{N}$-ideal in a BE-algebra is introduced. Characterizations of an $\mathcal{N}$-ideal are discussed. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-ideal are provided. To obtain a more general form of an $\mathcal{N}$-ideal, a point $\mathcal{N}$-structure which is (conditionally) employed in an $\mathcal{N}$-structure is proposed. Using these notions, the concept of an ([e], [e] ∨ [c])-ideal is introduced, and related properties are investigated. Characterizations of ([e], [e] ∨ [c])-ideals are discussed.

Keywords: (Transitive, Self distributive) BE-algebra, Ideal, $\mathcal{N}$-ideal, ([e], [e]∨[c])-ideal.

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1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A.

So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$.

Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool.

To attain such an object, Jun *et al.* [4] introduced a new function which is called a negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. In 1966, Imai and Iséki [2] and Iséki [3] introduced two classes of abstract algebras:

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BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

As a generalization of a BCK-algebra, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They considered several descriptions of ideals in BE-algebras.

In this paper, we introduce the notion of an $\mathcal{N}$-ideal of BE-algebras, and investigate several characterizations of $\mathcal{N}$-ideals. To obtain a more general form of an $\mathcal{N}$-ideal, we propose a definition of a point $\mathcal{N}$-structure which is (conditionally) employed in an $\mathcal{N}$-structure. Using these notions, we introduce the concept of ([e], [e]$\vee$[c])-ideals, and investigate related properties. We provide characterizations of ([e], [e]$\vee$[c])-ideals.

We know that uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which have been pointed out in [6]. The long distance aim of this paper is to provide a new mathematical tool for dealing with uncertainties.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold (see [5]):

(a1) $(\forall x \in X)\ (x * x = 1)$,
(a2) $(\forall x \in X)\ (x * 1 = 1)$,
(a3) $(\forall x \in X)\ (1 * x = x)$,
(a4) $(\forall x, y, z \in X)\ (x * (y * z) = y * (x * z))$.

A relation "$\le$" on a BE-algebra X is defined by

$$(\forall x, y \in X)\,(x \le y \iff x * y = 1).$$

A BE-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

(a5) $(\forall x, y, z \in X)\ (y * z \le (x * y) * (x * z))$.

A BE-algebra $(X; *, 1)$ is said to be *self distributive* (see [5]) if it satisfies:

(a6) $(\forall x, y, z \in X)\ (x * (y * z) = (x * y) * (x * z))$.

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

A nonempty subset I of a BE-algebra X is called an *ideal* of X (see [1]) if it satisfies:

(a7) $(\forall x \in X)\ (\forall a \in I)(x * a \in I)$,
(a8) $(\forall x \in X)\ (\forall a, b \in I)\ (a * (b * x)) * x \in I)$.

Denote by $\mathcal{I}(X)$ the set of all ideals of X.

2.1. Lemma. [7] *A nonempty subset I of X is an ideal of X if and only if it satisfies:*

(1) $1 \in I$,
(2) $(\forall x, z \in X)\ (\forall y \in I)\ (x * (y * z) \in I \implies x * z \in I)$.

3. $\mathcal{N}$-ideals of BE-algebras

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$-*function* on X). By an $\mathcal{N}$-*structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

In what follows, let X denote a BE-algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0]$, the nonempty set

$$C(f;t) := \{x \in X \mid f(x) \leq t\}$$

is called a *closed* (f,t)-*cut* of (X, f).

3.1. Definition. By an $\mathcal{N}$-*ideal* of X we mean an $\mathcal{N}$-structure (X, f) which satisfies the following assertion:

$$(3.1) \qquad (\forall t \in [-1,0]) \ \big(C(f;t) \in \mathcal{I}(X) \cup \{\emptyset\}\big).$$

3.2. Example. Let $X = \{1, a, b, c, d, 0\}$ be a set with a multiplication table given by Table 1.

Table 1. Multiplication table

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X, *, 1)$ is a BE-algebra (see [5]).

(1) Consider an $\mathcal{N}$-structure (X, f) in which f is defined by

$$f(x) := \begin{cases} -0.6 & \text{if } x \in \{1, a, b\}, \\ -0.2 & \text{if } x \in \{c, d, 0\}. \end{cases}$$

Then

$$C(f;t) = \begin{cases} \text{X} & \text{if } t \in [-0.2, 0], \\ \{1, a, b\} & \text{if } t \in [-0.6, -0.2), \\ \emptyset & \text{if } t \in [-1, -0.6). \end{cases}$$

Note that $\{1, a, b\}$ and X are ideals of X, and so (X, f) is an $\mathcal{N}$-ideal of X.

(2) Consider an $\mathcal{N}$-structure (X, g) in which g is defined by

$$g(x) := \begin{cases} -0.9 & \text{if } x \in \{1, a\}, \\ -0.3 & \text{if } x \in \{b, c, d, 0\}. \end{cases}$$

Then

$$C(g;t) = \begin{cases} \text{X} & \text{if } t \in [-0.3, 0], \\ \{1, a\} & \text{if } t \in [-0.9, -0.3), \\ \emptyset & \text{if } t \in [-1, -0.9). \end{cases}$$

Note that $\{1, a\}$ is not an ideal of X since

$$(a * (a * b)) * b = (a * a) * b = 1 * b = b \notin \{1, a\}.$$

Hence (X, g) is not an $\mathcal{N}$-ideal of X.

3.3. Theorem. *For an $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) (X, f) *is an* $\mathcal{N}$-*ideal of* X.

(2) (X, f) *satisfies the following two conditions:*

(2.1) $(\forall x, y \in X)\ \big(f(x * y) \leq f(y)\big)$,
(2.2) $(\forall x, y, z \in X)\ \big(f((x * (y * z)) * z) \leq \max\{f(x), f(y)\}\big)$.

Proof. Assume that (X, f) satisfies the two conditions (2.1) and (2.2). Let $t \in [-1, 0]$ be such that $C(f; t) \neq \emptyset$. Let $x \in X$ and $a \in C(f; t)$. Then $f(a) \leq t$, and so $f(x * a) \leq f(a) \leq t$ by (2.1). Thus $x * a \in C(f; t)$. Let $x \in X$ and $a, b \in C(f; t)$. Then $f(a) \leq t$ and $f(b) \leq t$. It follows from (2.2) that

$$f((a * (b * x)) * x) \leq \max\{f(a), f(b)\} \leq t$$

so that $(a * (b * x)) * x \in C(f; t)$. Hence $C(f; t)$ is an ideal of X, and therefore (X, f) is an $\mathcal{N}$-ideal of X.

Conversely, assume that (X, f) is an $\mathcal{N}$-ideal of X. If $f(a * b) > t_b := f(b)$ for some $a, b \in X$ and $t_b \in [-1, 0]$, then $b \in C(f; t_b)$ but $a * b \notin C(f; t_b)$. This is a contradiction, and so (2.1) is valid. Suppose that (2.2) is not valid. Then there exist $a, b, c \in X$ such that $f((a * (b * c)) * c) > \max\{f(a), f(b)\}$. Taking $t := \max\{f(a), f(b)\}$ implies that $a, b \in C(f; t)$ and $(a * (b * c)) * c \notin C(f; t)$. This is impossible, and thus (2.2) is true. □

3.4. Proposition. *Every $\mathcal{N}$-ideal (X, f) satisfies the following inequalities:*

(1) $(\forall x \in X)\ (f(1) \leq f(x))$,
(2) $(\forall x, y \in X)\ (f((x * y) * y) \leq f(x))$.

Proof. (1) Using (a1) and (2.1) in Theorem 3.3, we have $f(1) = f(x * x) \leq f(x)$ for all $x \in X$.

(2) Taking $y = 1$ and $z = y$ in Theorem 3.3(2.2) and using (a3) and (1), we get

$$f((x * y) * y) = f((x * (1 * y)) * y) \leq \max\{f(x), f(1)\} = f(x)$$

for all $x, y \in X$. □

3.5. Corollary. *Every $\mathcal{N}$-ideal (X, f) is order reversing.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f(y) = f(1 * y) = f((x * y) * y) \leq f(x)$$

by (a3) and Proposition 3.4(2). Hence (X, f) is order reversing. □

3.6. Proposition. *An $\mathcal{N}$-structure (X, f) satisfying the first condition of Proposition 3.4 and*

(3.2) $\qquad (\forall x, y, z \in X)\ \big(f(x * z) \leq \max\{f(x * (y * z)), f(y)\}\big)$

is order reversing.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f(y) = f(1 * y) \leq \max\{f(1 * (x * y)), f(x)\} = \max\{f(1 * 1), f(x)\} = f(x)$$

by (a1), (a3), (3.2) and Proposition 3.4(1). Therefore (X, f) is order reversing. □

3.7. Theorem. *For any $\mathcal{N}$-structure (X, f) in a transitive BE-algebra X, the following are equivalent:*

(1) *(X, f) is an $\mathcal{N}$-ideal of X.*
(2) *(X, f) satisfies two conditions Proposition 3.4(1) and (3.2).*

Proof. Assume that (X, f) is an N-ideal of X. It is sufficient to show that (X, f) satisfies (3.2). Since X is transitive, we have

$$(y * z) * z \leq (x * (y * z)) * (x * z), \tag{3.3}$$

i.e., $((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$ for all $x, y, z \in X$. It follows from (a3), (2.2) in Theorem 3.3 and Proposition 3.4(2) that

$$\begin{aligned} f(x * z) &= f(1 * (x * z)) \\ &= f((((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z)) \\ &\leq \max\{f((y * z) * z), f(x * (y * z))\} \\ &\leq \max\{f(x * (y * z)), f(y)\}. \end{aligned}$$

Hence (X, f) satisfies (3.2).

Conversely, suppose that (X, f) satisfies the two conditions Proposition 3.4(1) and (3.2). Using (a1), (a2), (3.2) and Proposition 3.4(1), we have

$$\begin{aligned} f(x * y) &\leq \max\{f(x * (y * y)), f(y)\} \\ &= \max\{f(x * 1), f(y)\} \\ &= \max\{f(1), f(y)\} = f(y) \end{aligned}$$

and

$$\begin{aligned} f((x * y) * y) &\leq \max\{f((x * y) * (x * y)), f(x)\} \\ &= \max\{f(1), f(x)\} = f(x) \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Since f is order reversing by Proposition 3.6, it follows from (3.3) that $f((y * z) * z) \geq f((x * (y * z)) * (x * z))$ so from (3.2) and (3.4) that

$$\begin{aligned} f((x * (y * z)) * z) &\leq \max\{f(((x * (y * z)) * (x * z)), f(x)\} \\ &\leq \max\{f((y * z) * z), f(x)\} \leq \max\{f(x), f(y)\} \end{aligned}$$

for all $x, y, z \in X$. Hence (X, f) is an N-ideal of X. □

3.8. Corollary. *For any N-structure (X, f) in a self distributive BE-algebra X, the following are equivalent:*

(1) *(X, f) is an N-ideal of X.*
(2) *(X, f) satisfies the two conditions Proposition 3.4(1) and (3.2).*

Proof. Straightforward. □

For any $a, b \in X$, the set

$$A(a, b) := \{x \in X \mid a * (b * x) = 1\}$$

is called the *upper set* of a and b (see [5]). Clearly, $1, a, b \in A(a, b)$ for all $a, b \in X$ (see [5]). Note that $A(a, b)$ is not an ideal of X in general (see [1]).

For every $a, b \in X$, let (X, f_a^b) be an N-structure in which f_a^b is given by

$$f_a^b(x) := \begin{cases} \alpha & \text{if } x \in A(a, b), \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [-1, 0]$ with $\alpha < \beta$.

The following example shows that there exist $a, b \in X$ such that f_a^b is not an N-ideal of X.

3.9. Example. Consider the BE-algebra which is described in Example 3.2. Then (X, f_1^a) is not an $\mathcal{N}$-ideal of X since

$$\begin{aligned} f_1^a((a*(a*b))*b) &= f_1^a((a*a)*b) = f_1^a(1*b) = f_1^a(b) \\ &= \beta < \alpha = f_1^a(a) = \max\{f_1^a(a), f_1^a(a)\}. \end{aligned}$$

We provide a condition for an $\mathcal{N}$-structure (X, f_a^b) to be an $\mathcal{N}$-ideal of X.

3.10. Theorem. *If X is self-distributive, then an $\mathcal{N}$-structure (X, f_a^b) is an $\mathcal{N}$-ideal of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. For every $x, y \in X$, if $y \notin A(a,b)$, then $f_a^b(y) = \beta \geq f_a^b(x*y)$. If $y \in A(a,b)$, then

$$\begin{aligned} a*(b*(x*y)) &= a*((b*x)*(b*y)) \\ &= (a*(b*x))*(a*(b*y)) \\ &= (a*(b*x))*1 = 1, \end{aligned}$$

i.e., $x*y \in A(a,b)$. Hence $f_a^b(x*y) = \alpha = f_a^b(y)$. Therefore $f_a^b(y) \geq f_a^b(x*y)$ for all $x, y \in X$. Now let $x, y, z \in X$. If $x \notin A(a,b)$ or $y \notin A(a,b)$, then $f_a^b(x) = \beta$ or $f_a^b(y) = \beta$. Thus

$$f_a^b((x*(y*z))*z) \leq \beta = \max\{f_a^b(x), f_a^b(y)\}.$$

Suppose that $x, y \in A(a,b)$. Then $a*(b*x) = 1$ and $a*(b*y) = 1$. Hence

$$\begin{aligned} a*(b*((x*(y*z))*z)) &= a*((b*(x*(y*z)))*(b*z)) \\ &= (a*(b*(x*(y*z))))*(a*(b*z)) \\ &= ((a*(b*x))*(a*(b*(y*z))))*(a*(b*z)) \\ &= (1*(a*(b*(y*z))))*(a*(b*z)) \\ &= (a*(b*(y*z)))*(a*(b*z)) \\ &= ((a*(b*y))*(a*(b*z)))*(a*(b*z)) \\ &= (1*(a*(b*z)))*(a*(b*z)) \\ &= (a*(b*z))*(a*(b*z)) \\ &= 1, \end{aligned}$$

i.e., $(x*(y*z))*z \in A(a,b)$, and so

$$f_a^b((x*(y*z))*z) = \alpha = \max\{f_a^b(x), f_a^b(y)\}.$$

Therefore $f_a^b((x*(y*z))*z) \leq \max\{f_a^b(x), f_a^b(y)\}$ for all $x, y, z \in X$. Consequently, f_a^b is an $\mathcal{N}$-ideal of X for all $a, b \in X$. □

3.11. Lemma. *Every $\mathcal{N}$-ideal (X, f) satisfies the following inequality.*

(3.5) $\quad (\forall x, y \in X)\ \big(f(y) \leq \max\{f(x), f(x*y)\}\big).$

Proof. Using (a1), (a3) and (2.2) in Theorem 3.3, we have

$$f(y) = f(1*y) = f((x*y)*(x*y))*y) \leq \max\{f(x), f(x*y)\}$$

for all $x, y \in X$. □

3.12. Theorem. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of X if and only if (X, f) satisfies two conditions:*

(1) $(\forall x \in X)\ \big(f(1) \leq f(x)\big)$,
(2) $(\forall x, y, z \in X)\ \big(f(x*z) \leq \max\{f(x*(y*z)), f(y)\}\big)$.

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of X. Then (1) is valid by Proposition 3.4. Using Lemma 3.11 and (a4), we have

$$f(x * z) \leq \max\{f(y * (x * z)), f(y)\} = \max\{f(x * (y * z)), f(y)\}$$

for all $x, y, z \in X$.

Conversely, suppose that an $\mathcal{N}$-structure (X, f) satisfies conditions (1) and (2). Using (2), (a1), (a2) and (1), we get

$$\begin{aligned} f(x * a) &\leq \max\{f(x * (a * a)), f(a)\} = \max\{f(x * 1), f(a)\} \\ &= \max\{f(1), f(a)\} = f(a). \end{aligned}$$

Let $a, b, x, y \in X$ and take $x = a * x$, $y = a$ and $z = x$ in (2). Then

$$f((a * x) * x) \leq \max\{f((a * x) * (a * x)), f(a)\} = \max\{f(1), f(a)\} = f(a)$$

by using (a1) and (1), which implies that $f((a * (b * x)) * (b * x)) \leq f(a)$. It follows from (2) that

$$f((a * (b * x)) * x) \leq \max\{f((a * (b * x)) * (b * x)), f(b)\} \leq \max\{f(a), f(b)\}.$$

Using Theorem 3.3, we conclude that (X, f) is an $\mathcal{N}$-ideal of X. □

3.13. Theorem. *If (X, f) is an $\mathcal{N}$-ideal of X, then*

(3.6) $$(\forall a, b \in X)\ (\forall t \in [-1, 0])\ \big(a, b \in C(f; t) \implies A(a, b) \subseteq C(f; t)\big).$$

Proof. Let $a, b \in C(f; t)$ for any $t \in [-1, 0]$. Then $f(a) \leq t$ and $f(b) \leq t$. If $x \in A(a, b)$, then $a * (b * x) = 1$. Hence

$$f(x) = f(1 * x) = f((a * (b * x)) * x) \leq \max\{f(a), f(b)\} \leq t,$$

and so $x \in C(f; t)$. Therefore $A(a, b) \subseteq C(f; t)$. □

We now consider the converse of Theorem 3.13. Let $t \in [-1, 0]$ and (X, f) an $\mathcal{N}$-structure satisfying (3.6). Note that $1 \in A(a, b) \subseteq C(f; t)$ for all $a, b \in X$. Let $x, y, z \in X$ be such that $x * (y * z) \in C(f; t)$ and $y \in C(f; t)$. Using (a4) and (a1), we know that

$$(x * (y * z)) * (y * (x * z)) = (x * (y * z)) * (x * (y * z)) = 1.$$

Thus $x * z \in A(x * (y * z), y) \subseteq C(f; t)$, and so $C(f; t)$ is an ideal of X by Lemma 2.1. Therefore (X, f) is an $\mathcal{N}$-ideal of X. Hence we have the following theorem.

3.14. Theorem. *If an $\mathcal{N}$-structure* (X, f) *satisfies* (3.6)*, then* (X, f) *is an* $\mathcal{N}$*-ideal of* X. □

3.15. Corollary. *For any $\mathcal{N}$-ideal (X, f), we have*

(3.7) $$(\forall t \in [-1, 0])\ \Big(C(f; t) \neq \emptyset \implies C(f; t) = \bigcup_{a, b \in C(f; t)} A(a, b)\Big).$$

Proof. Assume that $C(f; t) \neq \emptyset$ for all $t \in [-1, 0]$. Since $1 \in C(f; t)$, we get

$$C(f; t) \subseteq \bigcup_{a \in C(f; t)} A(a, 1) \subseteq \bigcup_{a, b \in C(f; t)} A(a, b).$$

Now, let $x \in \bigcup_{a, b \in C(f; t)} A(a, b)$. Then there exist $u, v \in C(f; t)$ such that $x \in A(u, v) \subseteq C(f; t)$. Hence $\bigcup_{a, b \in C(f; t)} A(a, b) \subseteq C(f; t)$. This completes the proof. □

4. ([e], [e]∨[c])-ideals

Let (X, f) be an N-structure in which f is given by

$$f(y) = \begin{cases} 0 & \text{if } y \neq x, \\ \theta & \text{if } y = x, \end{cases}$$

where $\theta \in [-1, 0]$. In this case, f is denoted by $\frac{x}{\theta}$ and we call $(X, \frac{x}{\theta})$ a *point* N*-structure*.

We say that a point N-structure $(X, \frac{x}{\theta})$ is *employed* in an N-structure (X, f), denoted by $(X, \frac{x}{\theta})[\mathrm{e}](X, f)$ (or briefly $\frac{x}{\theta}[\mathrm{e}]f$), if $f(x) \leq \theta$. A point N-structure $(X, \frac{x}{\theta})$ is said to be *conditionally employed* in an N-structure (X, f), denoted by $(X, \frac{x}{\theta})[\mathrm{c}](X, f)$ (or briefly $\frac{x}{\theta}[\mathrm{c}]f$), if $f(x) + \theta + 1 < 0$.

To say that $(X, \frac{x}{\theta})[\mathrm{e}] \vee [\mathrm{c}](X, f)$ (or briefly, $\frac{x}{\theta}[\mathrm{e}] \vee [\mathrm{c}]f$) we mean $(X, \frac{x}{\theta})[\mathrm{e}](X, f)$ or $(X, \frac{x}{\theta})[\mathrm{c}](X, f)$ (or briefly, $\frac{x}{\theta}[\mathrm{e}]f$ or $\frac{x}{\theta}[\mathrm{c}]f$).

To say that $\frac{x}{\theta}\, \overline{\alpha}\, f$, we mean $\frac{x}{\theta}\, \alpha\, f$ does not hold for $\alpha \in \{[\mathrm{e}], [\mathrm{c}], [\mathrm{e}]\vee[\mathrm{c}]\}$.

4.1. Definition. An N-structure (X, f) is called a $([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}])$*-ideal* of X if it satisfies:

(1) $\frac{y}{t}[\mathrm{e}]f \implies \frac{x*y}{t}[\mathrm{e}]\vee[\mathrm{c}]f$,

(2) $\frac{x}{t}[\mathrm{e}]f,\ \frac{y}{r}[\mathrm{e}]f \implies \frac{(x*(y*z))*z}{\max\{t,r\}}[\mathrm{e}]\vee[\mathrm{c}]f$

for all $x, y, z, \in X$ and $t, r \in [-1, 0]$.

4.2. Example. Consider the BE-algebra X described in Example 3.2. Let (X, f) be an N-structure in which f is given by

$$f = \begin{pmatrix} 1 & a & b & c & d & 0 \\ -0.5 & -0.9 & -0.6 & t & t & t \end{pmatrix}.$$

where $t \in (-0.5, 0]$. It is routine to check that f is a $([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}])$-ideal of X.

4.3. Theorem. *For any* N*-structure* (X, f)*, the following are equivalent:*

(1) (X, f) *is a* $([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}])$*-ideal of* X.

(2) (X, f) *satisfies the following inequalities:*

(2.1) $(\forall x, y \in X)\ \big(f(x*y) \leq \max\{f(y), -0.5\}\big)$,

(2.2) $(\forall x, y, z \in X)\ \big(f((x*(y*z))*z) \leq \max\{f(x), f(y), -0.5\}\big)$.

Proof. Let (X, f) be a $([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}])$-ideal of X. Assume that there exist $a, b \in X$ such that $f(a*b) > \max\{f(b), -0.5\}$. If we take $t_b := \max\{f(b), -0.5\}$, then $t_b \in [-0.5, 0]$, $\frac{b}{t_b}[\mathrm{e}]f$ and $\frac{a*b}{t_b}\,\overline{[\mathrm{e}]}\,f$. Also, $f(a*b) + t_b + 1 > 2t_b + 1 \geq 0$, and so $\frac{a*b}{t_b}\,\overline{[\mathrm{c}]}\,f$. This is a contradiction. Hence $f(x*y) \leq \max\{f(y), -0.5\}$ for all $x, y \in X$.

Suppose that $f((a*(b*c))*c) > \max\{f(a), f(b), -0.5\}$ for some $a, b, c \in X$. Take $t := \max\{f(a), f(b), -0.5\}$. Then $t \geq -0.5$, $\frac{a}{t}[\mathrm{e}]f$ and $\frac{b}{t}[\mathrm{e}]f$, but $\frac{(a*(b*c))*c}{t}\,\overline{[\mathrm{e}]}\,f$. Also, $f((a*(b*c))*c) + t + 1 > 2t + 1 \geq 0$, i.e., $\frac{(a*(b*c))*c}{t}\,\overline{[\mathrm{c}]}\,f$. This is a contradiction, and hence

$$f((x*(y*z))*z) \leq \max\{f(x), f(y), -0.5\}$$

for all $x, y, z \in X$.

Conversely, assume that (X, f) satisfies (2.1) and (2.2). Let $x, y \in X$ and $t \in [-1, 0]$ be such that $\frac{y}{t}[\mathrm{e}]f$. Then $f(y) \leq t$. Suppose that $\frac{x*y}{t}\,\overline{[\mathrm{e}]}\,f$, i.e., $f(x*y) > t$. If $f(y) > -0.5$, then

$$f(x*y) \leq \max\{f(y), -0.5\} = f(y) \leq t,$$

a contradiction. Hence $f(y) \leq -0.5$, which implies that

$$f(x*y) + t + 1 < 2f(x*y) + 1 \leq 2\max\{f(y), -0.5\} + 1 = 0,$$

i.e., $\frac{x*y}{t}[\mathrm{c}]f$. Thus $\frac{x*y}{t}[\mathrm{e}]\vee[\mathrm{c}]f$.

Let $x, y, z \in X$ and $t, r \in [-1, 0]$ be such that $\frac{x}{t}[\mathrm{e}]f$ and $\frac{y}{r}[\mathrm{e}]f$. Then $f(x) \leq t$ and $f(y) \leq r$. Suppose that $\frac{(x*(y*z))*z}{\max\{t,r\}}\,\overline{[\mathrm{e}]}\,f$, i.e., $f((x*(y*z))*z) > \max\{t, r\}$. If $\max\{f(x), f(y)\} > -0.5$, then

$$f((x*(y*z))*z) \leq \max\{f(x), f(y), -0.5\} = \max\{f(x), f(y)\} \leq \max\{t, r\}.$$

This is impossible, and so $\max\{f(x), f(y)\} \leq -0.5$. It follows that

$$\begin{aligned} f((x*(y*z))*z) + \max\{t, r\} + 1 &< 2f((x*(y*z))*z) + 1 \\ &\leq 2\max\{f(x), f(y), -0.5\} + 1 = 0 \end{aligned}$$

so that $\frac{(x*(y*z))*z}{\max\{t,r\}}[\mathrm{c}]f$. Hence $\frac{(x*(y*z))*z}{\max\{t,r\}}[\mathrm{e}]\vee[\mathrm{c}]f$, and therefore (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X. □

4.4. Theorem. *For any $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) (X, f) *is a* $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$*-ideal of* X.

(2) $(\forall t \in [-0.5, 0])\ \big(C(f; t) \in \mathcal{I}(X) \cup \{\emptyset\}\big)$.

Proof. Assume that (X, f) is a ([e], [e]$\vee$[c])-ideal of $X$and let $t \in [-0.5, 0]$ be such that $C(f; t) \neq \emptyset$. Using (2.1) in Theorem 4.3, we have

$$f(x*y) \leq \max\{f(y), -0.5\}$$

for any $y \in C(f; t)$. It follows that $f(x*y) \leq \max\{t, -0.5\} = t$ so that $x*y \in C(f; t)$. Let $x \in X$ and $a, b \in C(f; t)$. Then $f(a) \leq t$ and $f(b) \leq t$. Using (2.2) in Theorem 4.3, we get

$$f((a*(b*x))*x) \leq \max\{f(a), f(b), -0.5\} \leq \max\{t, -0.5\} = t.$$

Thus $(a*(b*x))*x \in C(f; t)$. Therefore $C(f; t)$ is an ideal of X.

Conversely assume that (2) is valid. If there exist $a, b \in X$ such that $f(a*b) > \max\{f(b), -0.5\}$, then $f(a*b) > t_b \geq \max\{f(b), -0.5\}$ for some $t_b \in [-0.5, 0]$. Thus $b \in C(f; t_b)$ and $a*b \notin C(f; t_b)$, which is a contradiction. Thus $f(x*y) \leq \max\{f(y), -0.5\}$ for all $x, y \in X$.

Suppose that there are $a, b, c \in X$ such that $f((a*(b*c))*c) > \max\{f(a), f(b), -0.5\}$. If we take $t := \max\{f(a), f(b), -0.5\}$, then $t \in [-0.5, 0]$, $a \in C(f; t)$ and $b \in C(f; t)$, but $(a*(b*c))*c \notin C(f; t)$. This is a contradiction. Thus

$$f((x*(y*z))*z) \leq \max\{f(x), f(y), -0.5\}$$

for all $x, y, z \in X$. Using Theorem 4.3, we conclude that (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X. □

4.5. Theorem. *Every* $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$*-ideal* (X, f) *of* X *satisfies the following inequalities:*

(1) $(\forall x \in X)\ (f(1) \leq \max\{f(x), -0.5\})$,

(2) $(\forall x, y \in X)\ (f((x*y)*y) \leq \max\{f(x), -0.5\})$.

Proof. (1) Using (a1) and Theorem 4.3(2.1), we have

$$f(1) = f(x*x) \leq \max\{f(x), -0.5\}$$

for all $x \in X$.

(2) If we put $y = 1$ and $z = y$ in Theorem 4.3(2.2), then

$$f((x*y)*y) = f((x*(1*y))*y) \leq \max\{f(x), f(1), -0.5\} = \max\{f(x), -0.5\}$$

for all $x, y \in X$ by using (a3) and (1). □

4.6. Corollary. *Every* ([e], [e]∨[c])*-ideal* (X, f) *satisfies the following implication.*

$$(\forall x, y \in X)\ (x \leq y \implies f(y) \leq \max\{f(x), -0.5\}).$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f(y) = f(1 * y) = f((x * y) * y) \leq \max\{f(x), -0.5\}.$$

This completes the proof. □

4.7. Proposition. *Let* (X, f) *be an* $\mathcal{N}$*-structure such that*

(1) $(\forall x \in X)\ (f(1) \leq \max\{f(x), -0.5\})$,
(2) $(\forall x, y, z \in X)\ (f(x * z) \leq \max\{f(x * (y * z)), f(y), -0.5\})$.

Then the following implication is valid.

$$(\forall x, y \in X)\ (x \leq y \implies f(y) \leq \max\{f(x), -0.5\}).$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and thus

$$\begin{aligned} f(y) = f(1 * y) &\leq \max\{f(1 * (x * y)), f(x), -0.5\} \\ &= \max\{f(1 * 1), f(x), -0.5\} \\ &= \max\{f(1), f(x), -0.5\} \\ &= \max\{f(x), -0.5\}. \end{aligned}$$

This completes the proof. □

4.8. Theorem. *Let* X *be a transitive BE-algebra. Then an* $\mathcal{N}$*-structure* (X, f) *is a* ([e], [e]∨[c])*-ideal of* X *if and only if it satisfies:*

(1) $(\forall x \in X)\ (f(1) \leq \max\{f(x), -0.5\})$,
(2) $(\forall x, y, z \in X)\ (f(x * z) \leq \max\{f(x * (y * z)), f(y), -0.5\})$.

Proof. Assume that (X, f) is a ([e], [e] ∨ [c])-ideal of X. The first result follows from Theorem 4.5. Since X is transitive,

$$((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$$

for all $x, y, z \in X$. Using (a3) and Theorems 4.3(2.2) and 4.5(2), we have

$$\begin{aligned} f(x * z) &= f(1 * (x * z)) \\ &= f(((y * z) * z) * ((x * (y * z)) * (x * z)) * (x * z)) \\ &\leq \max\{f((y * z) * z), f(x * (y * z)), -0.5\} \\ &\leq \max\{f(x * (y * z)), f(y), -0.5\} \end{aligned}$$

for all $x, y, z \in X$.

Conversely, suppose that (X, f) satisfies (1) and (2). Using (2), (a1), (a2) and (1), we get

$$\begin{aligned} f(x * y) &\leq \max\{f(x * (y * y)), f(y), -0.5\} \\ &= \max\{f(x * 1), f(y), -0.5\} \\ &= \max\{f(1), f(y), -0.5\} \\ &= \max\{f(y), -0.5\} \end{aligned}$$

and

$$\begin{aligned} f((x * y) * y) &\leq \max\{f((x * y) * (x * y)), f(x), -0.5\} \\ &= \max\{f(1), f(x), -0.5\} \qquad (4.1) \\ &= \max\{f(x), -0.5\} \end{aligned}$$

for all $x, y \in X$. Now, since $(y * z) * z \le (x * (y * z)) * (x * z)$ for all $x, y, z \in X$, it follows from Proposition 4.7 that

$$f((x * (y * z)) * (x * z)) \le \max\{f((y * z) * z), -0.5\}$$

so from (2) and (4.1) that

$$\begin{aligned} f((x * (y * z)) * z) &\le \max\{f((x * (y * z)) * (x * z)), f(x), -0.5\} \\ &\le \max\{f((y * z) * z), f(x), -0.5\} \\ &\le \max\{f(x), f(y), -0.5\} \end{aligned}$$

for all $x, y, z \in X$. Using Theorem 4.3, we conclude that (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X. □

4.9. Theorem. *Let X be a transitive BE-algebra. If (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X such that $f(1) > -0.5$, then (X, f) is an $\mathcal{N}$-ideal of X.*

Proof. Assume that (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X such that $f(1) > -0.5$. Then $f(x) > -0.5$ and so $f(x) \ge f(1) > -0.5$ for all $x \in X$ by Theorem 4.8(1). It follows from Theorem 4.8(2) that

$$\begin{aligned} f(x * z) &\le \max\{f(x * (y * z)), f(y), -0.5\} \\ &= \max\{f(x * (y * z)), f(y)\} \end{aligned}$$

for all $x, y, z \in X$. Using Theorem 3.7, we conclude that (X, f) is an $\mathcal{N}$-ideal of X. □

4.10. Theorem. *If (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of a transitive BE-algebra X, then*

$$\big(\forall\, t \in [-1, -0.5)\big)\ \big(Q(f; t) \in \mathcal{I}(X) \cup \{\emptyset\}\big),$$

where $Q(f; t) := \{x \in X \mid \frac{x}{t}[\mathrm{c}]f\}$.

Proof. Suppose that $Q(f; t) \ne \emptyset$ for all $t \in [-1, -0.5)$. Then there exists $x \in Q(f; t)$, and so $\frac{x}{t}[\mathrm{c}]f$, i.e., $f(x) + t + 1 < 0$. Using Theorem 4.8(1), we have

$$\begin{aligned} f(1) &\le \max\{f(x), -0.5\} \\ &= \begin{cases} -0.5 & \text{if } f(x) \le -0.5, \\ f(x) & \text{if } f(x) > -0.5 \end{cases} \\ &< -1 - t, \end{aligned}$$

which implies that $1 \in Q(f; t)$. Let $x, y, z \in X$ be such that $x * (y * z) \in Q(f; t)$ and $y \in Q(f; t)$. Then $\frac{x*(y*z)}{t}[\mathrm{c}]f$ and $\frac{y}{t}[\mathrm{c}]f$, i.e., $f(x*(y*z))+t+1 < 0$ and $f(y)+t+1 < 0$. Using Theorem 4.8(2), we get

$$f(x * z) \le \max\{f(x * (y * z)), f(y), -0.5\}.$$

Thus, if $\max\{f(x * (y * z)), f(y)\} > -0.5$, then

$$f(x * z) \le \max\{f(x * (y * z)), f(y)\} < -1 - t.$$

If $\max\{f(x*(y*z)), f(y)\} \le -0.5$, then $f(x*z) \le -0.5 < -1-t$. It follows that $\frac{x*z}{t}[\mathrm{c}]f$, i.e., $x * z \in Q(f; t)$. Using Lemma 2.1, we know that $Q(f; t)$ is an ideal of X. □

4.11. Theorem. *Let X be a transitive BE-algebra. Then an $\mathcal{N}$-structure (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X if and only if the following assertion is valid:*

(4.2) $\big(\forall t \in [-1, 0]\big)\ \big([f]_t \in \mathcal{I}(X) \cup \{\emptyset\}\big),$

where $[f]_t := C(f; t) \cup \{x \in X \mid f(x) + t + 1 \le 0\}$.

Proof. Assume that (X, f) is a $\big([\mathrm{e}], [\mathrm{e}] \vee [\mathrm{c}]\big)$-ideal of $X$and let $t \in [-1, 0]$ be such that $[f]_t \neq \emptyset$. Then there exists $x \in [f]_t$, and so $f(x) \leq t$ or $f(x)+t+1 \leq 0$. If $f(x) \leq t$, then

$$f(1) \leq \max\{f(x), -0.5\} \leq \max\{t, -0.5\} = \begin{cases} t & \text{if } t > -0.5, \\ -0.5 \leq -1-t & \text{if } t \leq -0.5 \end{cases}$$

by Theorem 4.8(1). Hence $1 \in [f]_t$.

If $f(x)+t+1 \leq 0$, then

$$f(1) \leq \max\{f(x), -0.5\} \leq \max\{-1-t, -0.5\} = \begin{cases} -1-t & \text{if } t < -0.5, \\ -0.5 \leq t & \text{if } t \geq -0.5 \end{cases}$$

and so $1 \in [f]_t$. Let $x, a, y \in X$ be such that $a \in [f]_t$ and $x*(a*y) \in [f]_t$. Then $f(a) \leq t$ or $f(a)+t+1 \leq 0$, and $f(x*(a*y)) \leq t$ or $f(x*(a*y))+t+1 \leq 0$. Thus we can consider the following four cases:

(i) $f(a) \leq t$ and $f(x*(a*y)) \leq t$,
(ii) $f(a) \leq t$ and $f(x*(a*y))+t+1 \leq 0$,
(iii) $f(a)+t+1 \leq 0$ and $f(x*(a*y)) \leq t$,
(iv) $f(a)+t+1 \leq 0$ and $f(x*(a*y))+t+1 \leq 0$.

For the first case, Theorem 4.8(2) implies that

$$f(x*y) \leq \max\{f(x*(a*y)), f(a), -0.5\} \leq \max\{t, -0.5\} = \begin{cases} -0.5 & \text{if } t < -0.5, \\ t & \text{if } t \geq -0.5 \end{cases}$$

so that $x*y \in C(f;t)$ or $f(x*y)+t \leq -0.5 + -0.5 = -1$. Thus $x*y \in [f]_t$. The second case implies that

$$\begin{aligned} f(x*y) &\leq \max\{f(x*(a*y)), f(a), -0.5\} \\ &\leq \max\{-1-t, t, -0.5\} \\ &= \begin{cases} -1-t & \text{if } t < -0.5, \\ t & \text{if } t \geq -0.5. \end{cases} \end{aligned}$$

Thus $x*y \in [f]_t$. For case (iii), the proof is similar to case (ii). The final case implies that

$$\begin{aligned} f(x*y) &\leq \max\{f(x*(a*y)), f(a), -0.5\} \\ &\leq \max\{-1-t, -0.5\} \\ &= \begin{cases} -1-t & \text{if } t < -0.5, \\ -0.5 & \text{if } t \geq -0.5 \end{cases} \end{aligned}$$

so that $x*y \in [f]_t$. Consequently, $[f]_t$ is an ideal of X by Lemma 2.1.

Conversely, let (X, f) be an $\mathcal{N}$-structure satisfying (4.2). If there exists $a \in X$ such that $f(1) > \max\{f(a), -0.5\}$, then $f(1) > t_a \geq \max\{f(a), -0.5\}$ for some $t_a \in [-0.5, 0)$. It follows that $a \in C(f;t_a) \subseteq [f]_{t_a}$ but $1 \notin C(f;t_a)$. Also, $f(1)+t_a+1 > 2t_a+1 \geq 0$. Hence $1 \notin [f]_{t_a}$, which is a contradiction. Therefore $f(1) \leq \max\{f(x), -0.5\}$ for all $x \in X$. Suppose that

$$f(x*y) > \max\{f(x*(a*y)), f(a), -0.5\} \tag{4.3}$$

for some $x, a, y \in X$. Taking $t := \max\{f(x*(a*y)), f(a), -0.5\}$ implies that $t \in [-0.5, 0)$, $a \in C(f;t) \subseteq [f]_t$ and $x*(a*y) \in C(f;t) \subseteq [f]_t$. Since $[f]_t$ is an ideal of X, we have $x*y \in [f]_t$ and so $f(x*y) \leq t$ or $f(x*y)+t+1 \leq 0$. The inequality (4.3) induces $x*y \notin C(f;t)$ and $f(x*y)+t+1 > 2t+1 \geq 0$. Thus $x*y \notin [f]_t$. This is a contradiction. Hence $f(x*y) \leq \max\{f(x*(a*y)), f(a), -0.5\}$ for all $x, a, y \in X$. Using Theorem 4.8 we conclude that (X, f) is a $\big([\mathrm{e}], [\mathrm{e}]\vee[\mathrm{c}]\big)$-ideal of X. □

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