Hacettepe Journal of Mathematics and Statistics
Volume 41 (4) (2012), 449 – 465

CORRESPONDENCE BETWEEN FUZZY h-IDEALS OF A Γ-HEMIRING AND FUZZY h-IDEALS OF ITS OPERATOR HEMIRINGS

Debabrata Mandal*

Received 29 : 06 : 2010 : Accepted 14 : 09 : 2011

Abstract

In this paper we define a correspondence between the set of all fuzzy h-ideals of a Γ-hemiring S and the set of all fuzzy h-ideals of the operator hemirings of that Γ-hemiring. We deduce that the lattice of all fuzzy h-ideals of a Γ-hemiring is isomorphic to the lattice of all fuzzy h-ideals of the operator hemirings of that Γ-hemiring. Finally, the cartesian product of corresponding fuzzy h-ideals is defined and a characterization is obtained.

Keywords: Γ-hemiring, Cartesian product, Fuzzy h-ideal, Operator hemiring.

2000 AMS Classification: 16 Y 60, 16 Y 99, 03 E 72.

1. Introduction

There are many concepts of universal algebra generalizing an associative ring $(R, +, \cdot)$. Some of them - in particular semirings - have been found very useful for solving problems in different areas of applied mathematics and the information sciences, since the structure of a semiring provides an algebraic framework for modeling and studying the key factors in these applied areas. Ideals of semirings play a central role in the structure theory and are useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings.

To solve this problem, Henriksen [5] defined a more restricted class of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings was given by Iizuka [6], which are called h-ideals. LaTorre [9], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemirings and to amend the gap between ring ideals and semiring ideals.

The theory of Γ-semirings was introduced by Rao[12]. The theory of Γ-semirings has been enriched by the introduction of operator semirings of a Γ-semiring by Dutta and

*Department of Mathematics, Raja Peary Mohan College, Uttarpara, Hooghly-712258, India.
E-mail: debumndl@gmail.com

Sardar [3]. To make operator semirings effective in the study of Γ-semirings, Dutta *et al.* [3] established a correspondence between the ideals of a Γ-semiring S and the ideals of the operator semirings of S.

The concept of fuzzy set was introduced by Zadeh[16] and has been applied to many branches of mathematics. Jun and Lee[7] applied the concept to the theory of Γ-rings. The theory of fuzzy h-ideals in hemiring has been studied by many authors, for example [2, 8, 10, 11, 17]. As a continuation of this Sardar *et al.*[13] studied those properties in Γ-hemirings in terms of fuzzy h-ideals. Recently Ma *et al.* [10] investigated some properties of fuzzy h-ideals in Γ-hemirings. In this paper we establish various correspondences between the fuzzy h-ideals of a Γ-hemiring S and the fuzzy h-ideals of the operator hemirings of S.

2. Preliminaries

A *hemiring* (respectively *semiring*) [4] is a nonempty set S on which operations of addition and multiplication have been defined such that $(S,+)$ is a commutative monoid with identity 0_S, $(S,\cdot)$ is a semigroup (respectively monoid with identity 1_S), multiplication distributes over addition from either side, $1_S \neq 0$ and $0_S s = 0_S = s0_S$ for all $s \in S$.

Let S and Γ be two additive commutative semigroups with zero. According to [13], S is called a Γ-*hemiring* if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ by $(a,\alpha,b) \mapsto a\alpha b$ satisfying the following conditions:

(1) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(2) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(3) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$,
(5) $0_S\alpha a = 0_S = a\alpha 0_S$,
(6) $a0_\Gamma b = 0_S = b0_\Gamma a$,

for all $a,b,c \in S$ and for all $\alpha,\beta \in \Gamma$.

For simplicity we write 0 instead of 0_S and 0_Γ.

Let S be the set of all $m \times n$ matrices over $\mathbf{Z}_\mathbf{0}^-$ (the set of all non-positive integers) and Γ the set of all $n \times m$ matrices over $\mathbf{Z}_\mathbf{0}^-$. Then S forms a Γ-hemiring with the usual addition and multiplication of matrices.

Now, we recall the following definitions from [3].

Let S be a Γ-hemiring and F the free additive commutative semigroup generated by $S \times \Gamma$. We define a relation ρ on F as follows:

$$\sum_{i=1}^{m}(x_i,\alpha_i)\ \rho\ \sum_{j=1}^{n}(y_j,\beta_j) \text{ if and only if } \sum_{i=1}^{m} x_i\alpha_i a = \sum_{j=1}^{n} y_j\beta_j a,$$

for all $a \in S$ $(m,n \in \mathbf{Z}^+)$. Then ρ is a congruence relation on F. We denote the congruence class containing $\sum_{i=1}^{m}(x_i,\alpha_i)$ by $\sum_{i=1}^{m}[x_i,\alpha_i]$. Then F/ρ is an additive commutative semigroup. Now, F/ρ forms a hemiring with the multiplication defined by

$$\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)\left(\sum_{j=1}^{n}[y_j,\beta_j]\right) = \sum_{i,j}[x_i\alpha_i y_j,\beta_j].$$

We denote this hemiring by L and call it the *left operator hemiring* of the Γ-hemiring S. Dually we define the *right operator hemiring* R of the Γ-hemiring S.

Let S be a Γ-hemiring, L the left operator hemiring and R the right one. If there exists an element $\sum_{i=1}^{m}[e_i,\delta_i]\in L$ $\Big($resp. $\sum_{j=1}^{n}[\gamma_j,f_j]\in R\Big)$ such that $\sum_{i=1}^{m}e_i\delta_i a=a$ $\Big($respectively, $\sum_{j=1}^{n}a\gamma_j f_j=a\Big)$ for all $a\in S$, then S is said to have the *left unity* $\sum_{i=1}^{m}[e_i,\delta_i]$ $\Big($respectively, the *right unity* $\sum_{j=1}^{n}[\gamma_j,f_j]\Big)$.

Throughout this paper, unless otherwise mentioned, for different elements of L (respectively, R) we take the same index say 'i' whose range is finite, that is from 1 to n, for some positive integer n.

Let S be a Γ-hemiring, L the left operator hemiring and R the right one. If there exists an element $[e,\delta]\in L$ (respectively, $[\gamma,f]\in R$) such that $e\delta a=a$ (respectively, $a\gamma f=a$) for all $a\in S$, then S is said to have the *strong left unity* $[e,\delta]$ (respectively, strong right unity $[\gamma,f]$) [12].

Let S be a Γ-hemiring, L the left operator hemiring and R the right one. Let $P\subseteq L$ ($\subseteq R$). According to [3], we define

$$P^{+}=\{a\in S:[a,\Gamma]\subseteq P\}\ \text{(respectively,}\ P^{*}=\{a\in S:[\Gamma,a]\subseteq P\})$$

and for $Q\subseteq S$,

$$Q^{+'}=\left\{\sum_{i=1}^{m}[x_i,\alpha_i]\in L:\left(\sum_{i=1}^{m}([x_i,\alpha_i]\right)S\subseteq Q\right\},$$

where $\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)S$ denotes the set of all finite sums $\sum_{i,k}x_i\alpha_i s_k,\ s_k\in S$ and

$$Q^{*'}=\left\{\sum_{i=1}^{m}[\alpha_i,x_i]\in R:S\left(\sum_{i=1}^{m}([\alpha_i,x_i]\right)\subseteq Q\right\},$$

where $S\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)$ denotes the set of all finite sums $\sum_{i,k}s_k\alpha_i x_i,\ s_k\in S$.

A *fuzzy subset* μ of a non-empty set S is a function $\mu:S\longrightarrow[0,1]$. Let μ be a non-empty fuzzy subset of a Γ-hemiring S (i.e., $\mu(x)\neq 0$ for some $x\in S$). Then μ is called a *fuzzy left ideal* (respectively, *fuzzy right ideal*) of S [13] if

(1) $\mu(x+y)\geq\min\{\mu(x),\mu(y)\}$,
(2) $\mu(x\gamma y)\geq\mu(y)$ (respectively, $\mu(x\gamma y)\geq\mu(x)$),

for all $x,y\in S$ and $\gamma\in\Gamma$. A *fuzzy ideal* of a Γ-hemiring S is a non-empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S. Note that if μ is a fuzzy left or right ideal of a Γ-hemiring S, then $\mu(0)\geq\mu(x)$ for all $x\in S$.

A left ideal A of a Γ-hemiring S is called a *left h-ideal* if for any $x,z\in S$ and $a,b\in A$,

$$x+a+z=b+z\implies x\in A.$$

A *right h-ideal* is defined analogously. A fuzzy left ideal μ of a Γ-hemiring S is called a *fuzzy left h-ideal* if for all $a,b,x,z\in S$,

$$x+a+z=b+z\implies\mu(x)\geq\min\{\mu(a),\mu(b)\}.$$

A *fuzzy right h-ideal* is defined similarly. By a *fuzzy h-ideal* μ, we mean that μ is both a fuzzy left and a fuzzy right h-ideal.

For example, let S be the additive commutative semigroup of all non-positive integers and Γ the additive commutative semigroup of all non-positive even integers. Then S is a Γ-hemiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of S, defined as follows

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.7 & \text{if } x \text{ is even,} \\ 0.1 & \text{if } x \text{is odd,} \end{cases}$$

The fuzzy subset μ of S is both a fuzzy ideal and a fuzzy h-ideal of S.

Let S be a Γ-hemiring and μ_1, μ_2 two fuzzy subsets of S . Then the sum $\mu_1 \oplus \mu_2$ is defined as follows:

$$(\mu_1 \oplus \mu_2)(x) = \begin{cases} \sup_{x=u+v}\{\min\{\mu_1(u), \mu_2(v)\} : u, v \in S\}, \\ \quad 0 \qquad \text{if for all } u, v \in S,\ u + v \neq x. \end{cases}$$

Let μ and θ be two fuzzy subsets of a Γ-hemiring S. We define the *generalized h-product* of μ and θ by

$$\mu \circ_h \theta(x) = \begin{cases} \sup\big\{\min_i\{\min\{\mu(a_i), \mu(c_i), \theta(b_i), \theta(d_i)\}\} \\ \qquad\qquad\qquad : x + \sum_{i=1}^n a_i\gamma_i b_i + z = \sum_{i=1}^n c_i\delta_i d_i + z\big\} \\ \ 0 \ \text{ if } x \text{ cannot be expressed as above,} \end{cases}$$

where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \dots, n$.

Ma *et al.* [10] also defined a simple h-product by

$$\mu\Gamma_h\theta(x) = \begin{cases} \sup\{\min\{\mu(a), \mu(c), \theta(b), \theta(d)\} : x + a\gamma b + z = c\delta d + z\}, \\ \ 0 \ \text{ if } x \text{ cannot be expressed as above,} \end{cases}$$

where $x, z, a, , c, d \in S$ and $\gamma, \delta \in \Gamma$.

We now recall the following two definitions from [10]

A fuzzy left(right) h-ideal ζ of a $\Gamma-$hemiring S is said to be *prime* if ζ is a non-constant function and for any two fuzzy left (right) h-ideals μ and ν of S, $\mu\Gamma_h\nu \subseteq \zeta$ implies $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Similarly we can define a *semiprime* fuzzy h-ideal.

A fuzzy subset μ of a Γ-hemiring S is called a *fuzzy h-bi-ideal* if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have

(1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$,
(3) $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$,
(4) $x + a + z = b + z \implies \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

A fuzzy subset μ of a Γ-hemiring S is called a *fuzzy h-quasi-ideal* if for all $x, y, z, a, b \in S$ we have

(1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $(\mu \circ_h \chi_S) \cap (\chi_S \circ_h \mu) \subseteq \mu$,
(3) $x + a + z = b + z \implies \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

For more preliminaries on semirings (hemirings) and Γ-semirings we refer to [4] and[3], respectively. Also, for more results on fuzzy h-ideals in Γ-hemirings we refer to [13].

Throughout this paper, unless otherwise mentioned, S denotes a Γ-hemiring with left unity and right unity and $FLh\text{-}I(S)$, $FRh\text{-}I(S)$ and $Fh\text{-}I(S)$ denote respectively the set of all fuzzy left h-ideals, the set of all fuzzy right h-ideals and the set of all fuzzy h-ideals of the Γ-hemiring S. The meaning of $FLh\text{-}I(L)$, $FLh\text{-}I(R)$, $FRh\text{-}I(L)$, $FRh\text{-}I(R)$,

Fh-$I(L)$, Fh-$I(R)$, where L and R are respectively the left operator and right operator hemirings of the Γ-hemiring S are defined similarly.

Also, we assume that $\mu(0) = 1$ for a fuzzy left h-ideal (respectively, fuzzy right h-ideal, fuzzy h-ideal) μ of a Γ-hemiring S. Similarly, we assume that $\mu(0_L) = 1$ (respectively, $\mu(0_R) = 1$) for a fuzzy left h-ideal (respectively, fuzzy right h-ideal, fuzzy h-ideal) μ of the left operator hemiring (respectively, right operator hemiring R) of a Γ-hemiring S.

3. Corresponding fuzzy h-ideals

Throughout this section S denotes a Γ-hemiring, R denotes the right operator hemiring and L denotes the left operator hemiring of the Γ-hemiring S.

3.1. Definition. Let μ be a fuzzy subset of L. We define a fuzzy subset μ^+ of S by

$$\mu^+(x) = \inf_{\gamma\in\Gamma} \mu([x,\gamma]) \text{ where } x \in S.$$

If σ is a fuzzy subset of S, we define a fuzzy subset $\sigma^{+'}$ of L by

$$\sigma^{+'}\Big(\sum_i [x_i, \alpha_i]\Big) = \inf_{s\in S} \sigma\Big(\sum_i x_i\alpha_i s\Big) \text{ where } \sum_i [x_i, \alpha_i] \in L.$$

3.2. Definition. If δ be a fuzzy subset of R, we define a fuzzy subset δ^* of S by

$$\delta^*(x) = \inf_{\gamma\in\Gamma} \delta([\gamma, x]) \text{ where } x \in S.$$

If η be a fuzzy subset of S, we define a fuzzy subset $\eta{*}'$ of R by

$$\eta^{*'}\Big(\sum_i [\alpha_i, x_i]\Big) = \inf_{s\in S} \eta\Big(\sum_i s\alpha_i x_i) \text{ where } \sum_i [\alpha_i, x_i] \in R.$$

3.3. Lemma. *If $\{\mu_i : i \in I\}$ is a collection of fuzzy subsets of L then* $\bigcap_{i\in I} \mu_i^+ = \Big(\bigcap_{i\in I} \mu_i\Big)^+$. □

3.4. Proposition. *If $\mu \in Fh$-$I(L)$ then $\mu^+ \in Fh$-$I(S)$.*

Proof. Let $\mu \in Fh$-$I(L)$. Then $\mu(0_L) = 1$.

Now $\mu^+(0_S) = \inf_{\gamma\in\Gamma}\{\mu([0_S, \gamma])\} = 1$ [Since for all $\gamma \in \Gamma$, $[0_S, \gamma]$ is the zero element of L]. So, μ^+ is non empty and $\mu^+(0_S) = 1$.

Let $x, y \in S$ and $\alpha \in \Gamma$. Now

$$\begin{aligned}\mu^+(x + y) &= \inf_{\gamma\in\Gamma}\{\mu([x + y, \gamma])\} = \inf_{\gamma\in\Gamma}\{\mu([x, \gamma] + [y, \gamma])\} \\ &\geq \inf_{\gamma\in\Gamma}\{\min\{\mu([x, \gamma]), \mu([y, \gamma])\}\} = \min\{\inf_{\gamma\in\Gamma}\{\mu([x, \gamma])\}, \inf_{\gamma\in\Gamma}\{\mu([y, \gamma])\}\} \\ &= \min\{\mu^+(x), \mu^+(y)\}.\end{aligned}$$

Therefore $\mu^+(x + y) \geq \min\{\mu^+(x), \mu^+(y)\}$. Again

$$\mu^+(x\alpha y) = \inf_{\gamma\in\Gamma}\{\mu([x\alpha y, \gamma])\} \geq \inf_{\gamma\in\Gamma}\{\mu([x, \alpha][y, \gamma])\} \geq \inf_{\gamma\in\Gamma}\mu[y, \gamma] = \mu^+(y)$$

and

$$\begin{aligned}\mu^+(x\alpha y) &= \inf_{\gamma\in\Gamma}\{\mu([x\alpha y, \gamma])\} = \inf_{\gamma\in\Gamma}\{\mu([x, \alpha][y, \gamma])\} \geq \mu([x, \alpha]) \\ &\geq \inf_{\delta\in\Gamma}\{\mu([x, \delta])\} = \mu^+(x).\end{aligned}$$

Hence μ^+ is a fuzzy ideal of S.

Now for an h-ideal, suppose $x+a+z=b+z$, where $x,a,b,z\in S$. Therefore

$$\begin{aligned}\mu^+(x) &= \inf_{\gamma\in\Gamma}\{\mu([x,\gamma])\} \geq \inf_{\gamma\in\Gamma}\min\{\mu([a,\gamma]),\mu([b,\gamma])\}\\ &= \min\Big\{\inf_{\gamma\in\Gamma}\{\mu([a,\gamma])\},\inf_{\gamma\in\Gamma}\{\mu([b,\gamma])\}\Big\} = \min\{\mu^+(a),\mu^+(b)\}.\end{aligned}$$

Hence μ^+ is a fuzzy h-ideal of S. □

3.5. Proposition. *If $\sigma \in Fh\text{-}I(S)$ [resp. $FRh\text{-}I(S), FLh\text{-}I(S)$] then $\sigma^{+'} \in Fh\text{-}I(L)$ [resp. $FRh\text{-}I(L), FLh\text{-}I(L)$].*

Proof. Let $\sigma \in Fh\text{-}I(S)$. Then $\sigma(0_S)=1$.

Now $\sigma^{+'}([0_S,\gamma]) = \inf_{s\in S}[\sigma(0_S\gamma s)] = \inf_{s\in S}[\sigma(0_S)] = 1$, $\forall\,\gamma\in\Gamma$.

Therefore $\sigma^{+'}$ is non empty and $\sigma^{+'}(0_L)=1$ as $[0_S,\gamma]$ is the zero element of L.

Let $\sum_i [x_i,\alpha_i]$, $\sum_j [y_j,\beta_j] \in L$. Then

$$\begin{aligned}&\sigma^{+'}\Big(\sum_i[x_i,\alpha_i]+\sum_j[y_j,\beta_j]\Big)\\ &\quad= \inf_{s\in S}\Big\{\sigma\Big(\sum_i x_i\alpha_i s+\sum_j y_j\beta_j s\Big)\Big\}\\ &\quad\geq \inf_{s\in S}\Big\{\min\Big\{\sigma\Big(\sum_i x_i\alpha_i s\Big),\sigma\Big(\sum_j y_j\beta_j s\Big)\Big\}\Big\}\\ &\quad= \min\Big\{\inf_{s\in S}\Big\{\sigma\Big(\sum_i x_i\alpha_i s\Big)\Big\},\inf_{s\in S}\Big\{\sigma\Big(\sum_j y_j\beta_j s)\Big\}\Big\}\\ &\quad= \min\Big\{\sigma^{+'}\Big(\sum_i[x_i,\alpha_i]\Big),\sigma^{+'}\Big(\sum_j[y_j,\beta_j]\Big)\Big\}.\end{aligned}$$

Again

$$\begin{aligned}&\sigma^{+'}\Big(\sum_i[x_i,\alpha_i]\sum_j[y_j,\beta_j]\Big)\\ &\quad= \sigma^{+'}\Big(\sum_{i,j}[x_i\alpha_i y_j,\beta_j]\Big) = \inf_{s\in S}\sigma\Big(\sum_{i,j}x_i\alpha_i y_j\beta_j s\Big)\\ &\quad\geq \inf_{s\in S}\Big\{\min\Big\{\sigma\Big(\sum_i x_i\alpha_i y_1\Big),\sigma\Big(\sum_i x_i\alpha_i y_2\Big),\sigma\Big(\sum_i x_i\alpha_i y_3\Big),\ldots\Big\}\Big\}\\ &\quad\geq \min\Big\{\sigma\Big(\sum_i x_i\alpha_i y_1\Big),\sigma\Big(\sum_i x_i\alpha_i y_2\Big),\sigma\Big(\sum_i x_i\alpha_i y_3\Big),\ldots\Big\}\\ &\quad\geq \inf_{s\in S}\Big\{\sigma\Big(\sum_i(x_i\alpha_i s)\Big)\Big\} = \sigma^{+'}\Big(\sum_i[x_i,\alpha_i]\Big).\end{aligned}$$

Similarly we can show that $\sigma^{+'}\Big(\sum_j[y_j,\beta_j]\sum_i[x_i,\alpha_i]\Big) \geq \sigma^{+'}\Big(\sum_j[y_j,\beta_j]\Big)$.

Thus $\sigma^{+'}$ is a fuzzy ideal of L.

Now for an h-ideal suppose

$$\sum_i[x_i,e_i]+\sum_i[a_i,\alpha_i]+\sum_i[z_i,\delta_i] = \sum_i[b_i,\beta_i]+\sum_i[z_i,\delta_i]$$

where $\sum_i [x_i, e_i]$, $\sum_i [a_i, \alpha_i]$, $\sum_i [z_i, \delta_i]$, $\sum_i [b_i, \beta_i] \in L$. Then

$$\begin{aligned}
\sigma^{+'}\Big(\sum_i [x_i, e_i]\Big) &= \inf_{s \in S}\Big\{\sigma\Big(\sum_i x_i e_i s\Big)\Big\} \\
&\geq \inf_{s \in S}\Big\{\min\Big\{\sigma\Big(\sum_i a_i \alpha_i s\Big), \sigma\Big(\sum_j b_i \beta_i s\Big)\Big\}\Big\} \\
&= \min\Big\{\inf_{s \in S}\Big\{\sigma\Big(\sum_i a_i \alpha_i s\Big)\Big\}, \inf_{s \in S}\Big\{\sigma\Big(\sum_j b_i \beta_i s\Big)\Big\}\Big\} \\
&= \min\Big\{\sigma^{+'}\Big(\sum_i [a_i, \alpha_i]\Big), \sigma^{+'}\Big(\sum_j [b_i, \beta_i]\Big)\Big\}
\end{aligned}$$

Therefore $\sigma^{+'}$ is a fuzzy h-ideal of L. □

Similarly we can prove the following propositions.

3.6. Proposition. *If $\delta \in Fh\text{-}I(R)$ [resp. $FRh\text{-}I(R), FLh\text{-}I(R)$] then $\delta^* \in Fh\text{-}I(S)$ [resp. $FRh\text{-}I(S), FLh\text{-}I(S)$].* □

3.7. Proposition. *If $\eta \in Fh\text{-}I(S)$ [resp. $FRh\text{-}I(S), FLh\text{-}I(S)$] then $\eta^{*'} \in Fh\text{-}I(R)$ [resp. $FRh\text{-}I(R), FLh\text{-}I(R)$].* □

3.8. Theorem. *The lattices of all fuzzy h-ideals of S and L are isomorphic via the inclusion preserving bijection $\sigma \mapsto \sigma^{+'}$, where $\sigma \in Fh\text{-}I(S)$ and $\sigma^{+'} \in Fh\text{-}I(L)$.*

Proof. First we shall show that $(\sigma^{+'})^+ = \sigma$, where $\sigma \in Fh\text{-}I(S)$. Let $x \in S$. Then

$$\begin{aligned}
\big((\sigma^{+'})^+\big)(x) &= \inf_{\gamma \in \Gamma}\{\sigma^{+'}([x, \gamma])\} = \inf_{\gamma \in \Gamma}\big\{\inf_{s \in S}\{\sigma(x\gamma s)\}\big\} \\
&\geq \inf_{\gamma \in \Gamma}\big\{\inf_{s \in S}\{\sigma(x)\}\big\} = \sigma(x).
\end{aligned}$$

So $\sigma \subseteq \big(\sigma^{+'}\big)^+$.

Let $\sum_i [\gamma_i, f_i]$ be the right unity of S. Then $\sum_i x\gamma_i f_i = x$ for all $x \in S$. Now

$$\begin{aligned}
\sigma(x) &= \sigma(\sum_i x\gamma_i f_i) \geq \min\{\sigma(x\gamma_1 f_1), \sigma(x\gamma_2 f_2), \dots\} \\
&\geq \inf_{\gamma \in \Gamma}\big\{\inf_{s \in S}\{\sigma(x\gamma s)\}\big\} = \big(\sigma^{+'}\big)^+(x).
\end{aligned}$$

Therefore $\big(\sigma^{+'}\big)^+ \subseteq \sigma$ and hence $\big(\sigma^{+'}\big)^+ = \sigma$.

Next let $\mu \in Fh\text{-}I(L)$. Then

$$\begin{aligned}
\big((\mu^+)^{+'}\big)\Big(\sum_i [x_i, \alpha_i]\Big) &= \inf_{s \in S}\Big\{\mu^+\Big(\sum_i x_i \alpha_i s\Big)\Big\} = \inf_{s \in S}\Big\{\inf_{\gamma \in \Gamma}\Big\{\mu\Big(\sum_i [x_i \alpha_i s, \gamma]\Big)\Big\}\Big\} \\
&= \inf_{s \in S}\Big\{\inf_{\gamma \in \Gamma}\Big\{\mu(\sum_i [x_i, \alpha_i][s, \gamma])\Big\}\Big\} \\
&\geq \inf_{s \in S}\Big\{\inf_{\gamma \in \Gamma}\Big\{\mu\Big(\sum_i [x_i, \alpha_i]\Big)\Big\}\Big\} = \mu\Big(\sum_i [x_i, \alpha_i]\Big).
\end{aligned}$$

So $\mu \subseteq \big(\mu^+\big)^{+'}$.

Let $\sum_i [e_i, \delta_i]$ be the left unity of S. Then

$$\begin{aligned}\mu\Big(\sum_j [x_j, \alpha_j]\Big) &= \mu\Big(\sum_j [x_j, \alpha_j] \sum_i [e_i, \delta_i]\Big) \\ &\geq \min\Big\{\mu\Big(\sum_j [x_j, \alpha_j][e_1, \delta_1]\Big), \mu\Big(\sum_j [x_j, \alpha_j][e_2, \delta_2]\Big), \dots\Big\} \\ &\geq \inf_{s\in S}\Big\{\inf_{\gamma\in\Gamma}\Big\{\mu\Big(\sum_j [x_j, \alpha_j][s, \gamma]\Big)\Big\}\Big\} = (\mu^+)^{+'}\Big(\sum_j [x_j, \alpha_j]\Big)\end{aligned}$$

Therefore $(\mu^+)^{+'} \subseteq \mu$, and so $(\mu^+)^{+'} = \mu$. Thus the correspondence $\sigma \mapsto \sigma^{+'}$ is a bijection.

Now let $\sigma_1, \sigma_2 \in Fh\text{-}I(S)$ be such that $\sigma_1 \subseteq \sigma_2$. Then

$$\begin{aligned}\sigma_1^{+'}\Big(\sum_i [x_i, \alpha_i]\Big) &= \inf_{s\in S}\Big\{\sigma_1\Big(\sum_i x_i\alpha_i s\Big)\Big\} \\ &\leq \inf_{s\in S}\Big\{\sigma_2\Big(\sum_i x_i\alpha_i s\Big)\Big\} = \sigma_2^{+'}\Big(\sum_i [x_i, \alpha_i]\Big)\end{aligned}$$

for all $\sum_i [x_i, \alpha_i] \in L$. Thus, $\sigma_1^{+'} \subseteq \sigma_2^{+'}$.

Similarly we can deduce that if $\mu_1 \subseteq \mu_2$ where $\mu_1, \mu_2 \in Fh\text{-}I(L)$ then $\mu_1^+ \subseteq \mu_2^+$.

We shall now show that $(\sigma_1 \oplus \sigma_2)^{+'} = \sigma_1^{+'} \oplus \sigma_2^{+'}$ and $(\sigma_1 \cap \sigma_2)^{+'} = \sigma_1^{+'} \cap \sigma_2^{+'}$.

Let $\sum_i [a_i, \alpha_i] \in L$. Then

$$\begin{aligned}&\big((\sigma_1 \oplus \sigma_2)^{+'}\big)\Big(\sum_i [a_i, \alpha_i]\Big) \\ &= \inf_{s\in S}(\sigma_1 \oplus \sigma_2)\Big(\sum_i a_i\alpha_i s\Big) \\ &= \inf_{s\in S}\Big\{\sup\Big\{\min\Big\{\sigma_1(\sum_k x_k\delta_k s), \sigma_2\Big(\sum_j y_j\beta_j s\Big)\Big\} : \\ &\qquad \sum_i a_i\alpha_i s = \sum_k x_k\delta_k s + \sum_j y_j\beta_j s\Big\}\Big\} \\ &= \sup\Big\{\min\Big\{\inf_{s\in S}\sigma_1\Big(\sum_k x_k\delta_k s), \inf_{s\in S}\sigma_2\Big(\sum_j y_j\beta_j s\Big)\Big\}\Big\} \\ &= \sup\Big\{\min\Big\{\sigma_1^{+'}\Big(\sum_k [x_k, \delta_k]\Big), \sigma_2^{+'}\Big(\sum_j [y_j, \beta_j]\Big)\Big\}\Big\} \\ &= \big(\sigma_1^{+'} \oplus \sigma_2^{+'}\big)\Big(\sum_i [a_i, \alpha_i]\Big)\end{aligned}$$

Thus $(\sigma_1 \oplus \sigma_2)^{+'} = \sigma_1^{+'} \oplus \sigma_2^{+'}$. Again

$$\begin{aligned}(\sigma_1 \cap \sigma_2)^{+'}\Big(\sum_i [a_i, \alpha_i]\Big) &= \inf_{s\in S}\Big\{(\sigma_1 \cap \sigma_2)\Big(\sum_i a_i\alpha_i s\Big)\Big\} \\ &= \inf_{s\in S}\Big\{\min\Big\{\sigma_1\Big(\sum_i a_i\alpha_i s\Big), \sigma_2\Big(\sum_i a_i\alpha_i s\Big)\Big\}\Big\} \\ &= \min\Big\{\inf_{s\in S}\sigma_1\Big(\sum_i a_i\alpha_i s\Big), \inf_{s\in S}\sigma_2\Big(\sum_i a_i\alpha_i s\Big)\Big\}\end{aligned}$$

$$= \min\Big\{\sigma_1^{+'}\Big(\sum_i [a_i,\alpha_i]\Big), \sigma_2^{+'}\Big(\sum_i [a_i,\alpha_i]\Big)\Big\}$$
$$= \big(\sigma_1^{+'} \cap \sigma_2^{+'}\big)\Big(\sum_i [a_i,\alpha_i]\Big).$$

So $(\sigma_1 \cap \sigma_2)^{+'} = \sigma_1^{+'} \cap \sigma_2^{+'}$. Hence the mapping $\sigma \mapsto \sigma^{+'}$ is a lattice isomorphism. □

Similarly we can obtain the following theorem.

3.9. Theorem. *The lattices of all fuzzy h-ideals of S and R are isomorphic via the inclusion preserving bijection $\sigma \mapsto \sigma^{*'}$, where $\sigma \in Fh\text{-}I(S)$ and $\sigma^{*'} \in Fh\text{-}I(R)$.* □

3.10. Corollary. *$FLh\text{-}I(L)$ [resp. $FRh\text{-}I(L)$, $FLh\text{-}I(R)$, $FRh\text{-}I(R)$] is a complete lattice.*

Proof. The corollary follows from the above theorems and the fact that $FLh\text{-}I(S)$ [resp. $FRh\text{-}I(S)$, $Fh\text{-}I(S)$] is a complete lattice [13]. □

By routine verification the following Lemmas can be obtained.

3.11. Lemma. *Let I be an h-ideal (left h-ideal, right h-ideal) of a Γ-hemiring S and let λ_I be the characteristic function of I. Then $\big(\lambda_I\big)^{+'} = \lambda_{\left(I^{+'}\right)}$.* □

3.12. Lemma. *Let I be an h-ideal (left h-ideal, right h-ideal) of the left operator hemiring L of a Γ-hemiring S and let λ_I be the characteristic function of I. Then $\big(\lambda_I\big)^{+} = \lambda_{\left(I^{+}\right)}$.* □

3.13. Lemma. *Let I be an h-ideal (left h-ideal, right h-ideal) of a Γ-hemiring S and let λ_I be the characteristic function of I. Then $\big(\lambda_I\big)^{*'} = \lambda_{\left(I^{*'}\right)}$.* □

3.14. Lemma. *Let I be an h-ideal (left h-ideal, right h-ideal) of the right operator hemiring R of a Γ-hemiring S and let λ_I be the characteristic function of I. Then $\big(\lambda_I\big)^{*} = \lambda_{\left(I^{*}\right)}$.* □

3.15. Theorem. *The lattices of all h-ideals of S and L are isomorphic via the mapping $I \mapsto I^{+'}$, where I denotes an h-ideal of S.*

Proof. First we shall show that the mapping $I \mapsto I^{+'}$ is one-one. Let I_1 and I_2 be two h-ideals of S such that $I_1 \neq I_2$. Then λ_{I_1} and λ_{I_2} are fuzzy h-ideals of S where λ_{I_1} and λ_{I_2} are the characteristic functions of I_1 and I_2 respectively. Evidently, $\lambda_{I_1} \neq \lambda_{I_2}$. Then by Theorem 3.8, $\lambda_{I_1}^{+'} \neq \lambda_{I_2}^{+'}$. Hence by Lemma 3.11, $\lambda_{I_1^{+'}} \neq \lambda_{I_2^{+'}}$ whence $I_1^{+'} \neq I_2^{+'}$. Consequently the mapping $I \mapsto I^{+'}$ is one-one.

Next let J be an h-ideal of L. Then λ_J is a fuzzy h-ideal of L. By Proposition 3.4 and Theorem 3.8, λ_J^{+} is a fuzzy h-ideal of S. Now by Lemma 3.12, $\lambda_J^{+} = \lambda_{J^+}$ and consequently, J^{+} is an h-ideal of S. Hence the mapping is onto.

Let I_1, I_2 be two h-ideals of S such that $I_1 \subseteq I_2$. Then $\lambda_{I_1} \subseteq \lambda_{I_2}$ and by Theorem 3.8, $\lambda_{I_1}^{+'} \subseteq \lambda_{I_2}^{+'}$ and by Lemma 3.11, $\lambda_{I_1^{+'}} \subseteq \lambda_{I_2^{+'}}$ and consequently $I_1^{+'} \subseteq I_2^{+'}$. Thus the mapping is inclusion preserving. Hence the theorem. □

Similarly we can prove the following theorem:

3.16. Theorem. *The lattices of all h-ideals of S and R are isomorphic via the mapping $I \mapsto I^{*'}$, where I denotes an h-ideal of S.* □

3.17. Proposition. *For any two fuzzy h-ideals μ and ν of S,*

$$(\mu \circ_h \nu)^{+'} = \big((\mu)^{+'} \circ_h (\nu)^{+'}\big).$$

Proof. Let

$$\sum_i [x_i, e_i],\ \Big(\sum_i [a_i, \alpha_i]\Big)_j,\ \sum_i [z_i, \eta_i],\ \Big(\sum_i [b_i, \beta_i]\Big)_j,\ \Big(\sum_i [c_i, \gamma_i]\Big)_j,\ \Big(\sum_i [d_i, \delta_i]\Big)_j \in L$$

be such that

$$\sum_i [x_i, e_i] + \sum_j \Big(\sum_i [a_i, \alpha_i]\Big)_j \Big(\sum_i [c_i, \gamma_i]\Big)_j + \sum_i [z_i, \eta_i] = \sum_j \Big(\sum_i [b_i, \beta_i]\Big)_j \Big(\sum_i [d_i, \delta_i]\Big)_j + \sum_i [z_i, \eta_i].$$

Then

$$\begin{aligned}
&\big((\mu)^{+'} \circ_h (\nu)^{+'}\big)\Big(\sum_i [x_i, e_i]\Big) \\
&= \sup\Big\{\min_j\Big\{(\mu)^{+'}\Big(\Big(\sum_i [a_i, \alpha_i]\Big)_j\Big),\ (\nu)^{+'}\Big(\Big(\sum_i [c_i, \gamma_i]\Big)_j\Big),\ (\mu)^{+'}\Big(\Big(\sum_i [b_i, \beta_i]\Big)_j\Big), \\
&\qquad (\nu)^{+'}\Big(\Big(\sum_i [d_i, \delta_i]\Big)_j\Big)\Big\}\ :\ \sum_i [x_i, e_i] + \sum_j \Big(\sum_i [a_i, \alpha_i]\Big)_j \Big(\sum_i [c_i, \gamma_i]\Big)_j \\
&\qquad + \sum_i [z_i, \eta_i] = \sum_j \Big(\sum_i [b_i, \beta_i]\Big)_j \Big(\sum_i [d_i, \delta_i]\Big)_j + \sum_i [z_i, \eta_i]\Big\} \\
&= \sup\Big\{\min_j\Big\{\inf_{s\in S}(\mu)\Big(\Big(\sum_i a_i\alpha_i s\Big)_j\Big),\ \inf_{s\in S}(\nu)\Big(\Big(\sum_i c_i\gamma_i s\Big)_j\Big), \\
&\qquad \inf_{s\in S}(\mu)\Big(\Big(\sum_i b_i\beta_i s\Big)_j\Big),\ \inf_{s\in S}(\nu)\Big(\Big(\sum_i d_i\delta_i s\Big)_j\Big)\Big\}\Big\} \\
&= \inf_{s\in S}\Big\{\sup\Big\{\min_j\Big\{(\mu)\Big(\Big(\sum_i a_i\alpha_i s\Big)_j\Big),\ (\nu)\Big(\Big(\sum_i c_i\gamma_i s\Big)_j\Big),\ (\mu)\Big(\Big(\sum_i b_i\beta_i s\Big)_j\Big), \\
&\qquad (\nu)\Big(\sum_i (d_i\delta_i s)_j\Big)\Big\}\ :\ \sum_i x_i e_i s + \sum_j \Big(\sum_i a_i\alpha_i s\Big)_j \gamma_j \Big(\sum_i c_i\gamma_i s\Big)_j \\
&\qquad + \sum_i z_i\eta_i s = \sum_j \Big(\sum_i b_i\beta_i s\Big)_j \delta_j \Big(\sum_i d_i, \delta_i s\Big)_j + \sum_i z_i\eta_i s\Big\}\Big\} \\
&= \inf_{s\in S}(\mu \circ_h \nu)\Big(\sum_i x_i e_i s\Big) \\
&= (\mu \circ_h \nu)^{+'}\Big(\sum_i [x_i, e_i]\Big).
\end{aligned}$$

□

3.18. Remark. Similarly we can show that for any two fuzzy h-ideals μ and ν of S, $(\mu\Gamma_h\nu)^{+'} = \mu^{+'}\Gamma_h\nu^{+'}$.

3.19. Proposition. *If ζ is a prime [semiprime] fuzzy h-ideal of S then $\zeta^{+'}$(resp. $\zeta^{*'}$) is a prime [semiprime] fuzzy h-ideal of L (resp. R).*

Proof. Suppose ζ is a prime fuzzy h-ideal of S and let $\mu^{+'}$, $\nu^{+'}$ be fuzzy h-ideals of L such that $\mu^{+'}\Gamma_h\nu^{+'} \subseteq \zeta^{+'}$. Then from Remark 3.18 we have $(\mu\Gamma_h\nu)^{+'} \subseteq \zeta^{+'}$ which implies $(\mu\Gamma_h\nu) \subseteq \zeta$, that is $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$ [since ζ is a prime fuzzy h-ideal of S]. Hence $\mu^{+'} \subseteq \zeta^{+'}$ or $\nu^{+'} \subseteq \zeta^{+'}$. Therefore $\zeta^{+'}$ is a prime fuzzy h-ideal of L.

Similarly we can prove the result for R.

Now for semiprime fuzzy h-ideal, the proof is very similar. □

3.20. Proposition. *If ζ is a prime [semiprime] fuzzy h-ideal of L (resp. R), then ζ^{+} (resp. ζ^{*}) is a prime [semiprime] fuzzy h-ideal of S.*

Proof. The proof follows from a routine verification. □

3.21. Proposition. *If μ is a fuzzy h-bi-ideal of S. Then $\mu^{+'}$ (resp. $\mu^{*'}$) is a fuzzy h-bi-ideal of L (resp. R).*

Proof. Suppose μ is a fuzzy h-bi-ideal of S and $\sum_i[x_i,\alpha_i], \sum_i[y_i,\beta_i], \sum_i[z_i,\gamma_i] \in L$. Then from Proposition 3.5 we have

$$\mu^{+'}\Big(\Big(\sum_i[x_i,\alpha_i]\Big)+\Big(\sum_i[y_i,\beta_i]\Big)\Big) \geq \min\Big\{\mu^{+'}\Big(\sum_i[x_i,\alpha_i]\Big), \mu^{+'}\Big(\sum_i[y_i,\beta_i]\Big)\Big\}$$

and

$$\mu^{+'}\Big(\Big(\sum_i[x_i,\alpha_i]\Big)\Big(\sum_i[y_i,\beta_i]\Big)\Big) \geq \min\Big\{\mu^{+'}\Big(\sum_i[x_i,\alpha_i]\Big), \mu^{+'}\Big(\sum_i[y_i,\beta_i]\Big)\Big\}.$$

Now suppose $\sum_i[x_i,e_i], \sum_i[a_i,\alpha_i], \sum_i[z_i,\delta_i], \sum_i[b_i,\beta_i] \in L$ are such that

$$\sum_i[x_i,e_i]+\sum_i[a_i,\alpha_i]+\sum_i[z_i,\delta_i]=\sum_i[b_i,\beta_i]+\sum_i[z_i,\delta_i].$$

Since $\mu^{+'}$ is a fuzzy h-ideal of L,

$$\mu^{+'}\Big(\sum_i[x_i,e_i]\Big) \geq \min\Big\{\mu^{+'}\Big(\sum_i[a_i,\alpha_i]\Big),\ \mu^{+'}\Big(\sum_j[b_i,\beta_i]\Big)\Big\}.$$

Now

$$\begin{aligned}\mu^{+'}\Big(\Big(\sum_i[x_i,\alpha_i]\Big)\Big(\sum_i[y_i,\beta_i]\Big)\Big(\sum_i[z_i,\gamma_i]\Big)\Big) &= \mu^{+'}\Big(\sum_i[x_i,\alpha_i]\sum_i[y_i\beta_i z_i,\gamma_i]\Big)\\ &\geq \mu^{+'}\Big(\sum_i[x_i,\alpha_i]\Big).\end{aligned}$$

Similarly,

$$\begin{aligned}\mu^{+'}\Big(\Big(\sum_i[x_i,\alpha_i]\Big)\Big(\sum_i[y_i,\beta_i]\Big)\Big(\sum_i[z_i,\gamma_i]\Big)\Big) &= \mu^{+'}\Big(\sum_i[x_i,\alpha_i y_i\beta_i]\sum_i[z_i,\gamma_i]\Big)\\ &\geq \mu^{+'}\Big(\sum_i[z_i,\gamma_i]\Big).\end{aligned}$$

Therefore we have,

$$\begin{aligned}&\mu^{+'}\Big(\Big(\sum_i[x_i,\alpha_i]\Big)\Big(\sum_i[y_i,\beta_i]\Big)\Big(\sum_i[z_i,\gamma_i]\Big)\Big)\\ &\qquad\geq \min\Big\{\mu^{+'}\Big(\sum_i[x_i,\alpha_i]\Big),\ \mu^{+'}\Big(\sum_i[z_i,\gamma_i]\Big)\Big\}\end{aligned}$$

Hence $\mu^{+'}$ is a fuzzy h-bi-ideal of L.

Similarly we can prove the result for R. □

3.22. Proposition. *If μ is a fuzzy h-bi-ideal of L (resp. R), then μ^{+} (resp. μ^{*}) is also a fuzzy h-bi-ideal of S.* □

3.23. Proposition. *If μ is a fuzzy h-quasi-ideal of S then $\mu^{+'}$ (resp. $\mu^{*'}$) is a fuzzy h-quasi-ideal of L (resp. R).*

Proof. Suppose μ is a fuzzy h-quasi-ideal of S and $\sum_i [x_i, \alpha_i]$, $\sum_i [y_i, \beta_i]$, $\sum_i [z_i, \gamma_i] \in L$. Then by Proposition 3.5 we obtain

$$\mu^{+'}\Big(\sum_i [x_i, \alpha_i] + \sum_i [y_i, \beta_i]\Big) \geq \min\Big\{\mu^{+'}\Big(\sum_i [x_i, \alpha_i]\Big), \mu^{+'}\Big(\sum_i [y_i, \beta_i]\Big)\Big\},$$

and if

$$\sum_i [x_i, e_i] + \sum_i [a_i, \alpha_i] + \sum_i [z_i, \delta_i] = \sum_i [b_i, \beta_i] + \sum_i [z_i, \delta_i]$$

for $\sum_i [x_i, e_i], \sum_i [a_i, \alpha_i], \sum_i [z_i, \delta_i], \sum_i [b_i, \beta_i] \in L$, then

$$\mu^{+'}\Big(\sum_i [x_i, e_i]\Big) \geq \min\Big[\mu^{+'}\Big(\sum_i [a_i, \alpha_i]\Big), \mu^{+'}\Big(\sum_j [b_i, \beta_i]\Big)\Big].$$

Let χ_S be the characteristic function of S. Then by using Proposition 3.17 and Theorem 3.8 we deduce that

$$\begin{aligned}(\mu^{+'} \circ_h \chi_S^{+'}) \cap (\chi_S^{+'} \circ_h \mu^{+'}) &= (\mu \circ_h \chi_S)^{+'} \cap (\chi_S \circ_h \mu)^{+'} \\ &= ((\mu \circ_h \chi_S) \cap (\chi_S \circ_h \mu))^{+'} \subseteq \mu^{+'}\end{aligned}$$

[since μ is a fuzzy h-quasi ideal]. Hence $\mu^{+'}$ is an h-quasi-ideal of L.

Similarly we can prove the result for R. □

3.24. Proposition. *If μ is a fuzzy h-quasi-ideal of L (resp. R), then μ^{+} (resp. μ^{*}) is also a fuzzy h-quasi-ideal of S.* □

4. Cartesian product of corresponding fuzzy *h*-ideals

Let $\{S_i\}_{i \in I}$ be a family of Γ-hemirings. Now if we define addition (+) and multiplication on the Cartesian product $\Pi_{i \in I} S_i$ as follows:

$$(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I} \text{ and } (x_i)_{i \in I} \alpha (y_i)_{i \in I} = (x_i \alpha y_i)_{i \in I}$$

for all $(x_i)_{i \in I}, (y_i)_{i \in I} \in \Pi_{i \in I} S_i$ and for all $\alpha \in \Gamma$, then $\Pi_{i \in I} S_i$ becomes a Γ-hemiring.

4.1. Definition. [1] Let μ and σ be two fuzzy subsets of a set X. Then the *Cartesian product of μ and σ* is defined by $(\mu \times \sigma)(x, y) = \min\{\mu(x), \sigma(y)\}\ \forall\, x, y \in X$.

4.2. Definition. Let $\mu \times \sigma$ be the Cartesian product of two fuzzy subsets μ and σ of R. Then the corresponding cartesian product $(\mu \times \sigma)^*$ of $S \times S$ is defined by

$$(\mu \times \sigma)^*(x, y) = \inf_{\alpha, \beta \in \Gamma} (\mu \times \sigma)([\alpha, x], [\beta, y]) \text{ where } x, y \in S.$$

4.3. Definition. Let $\mu \times \sigma$ be the Cartesian product of two fuzzy subsets μ and σ of S. Then the corresponding cartesian product $(\mu \times \sigma)^{*'}$ of $R \times R$ is defined by

$$(\mu \times \sigma)^{*'}\Big(\sum_{i=1}^{n} [\alpha_i, x_i], \sum_{j=1}^{m} [\beta_j, y_j]\Big) = \inf_{s_i, s_j \in S} (\mu \times \sigma)\Big(\sum_{i=1}^{n} s_i \alpha_i x_i, \sum_{j=1}^{m} s_j \beta_j y_j\Big),$$

where $\sum_{i=1}^{n} [\alpha_i, x_i]$, $\sum_{j=1}^{m} [\beta_j, y_j] \in R$.

4.4. Definition. Let $\mu \times \sigma$ be the cartesian product of two fuzzy subsets μ and σ of L. Then the corresponding *Cartesian product* $(\mu \times \sigma)^{+}$ *of* $S \times S$ is defined by

$$(\mu \times \sigma)^{+}(x, y) = \inf_{\alpha,\beta\in\Gamma} (\mu \times \sigma)([x, \alpha], [y, \beta]),$$

where $x, y \in S$.

4.5. Definition. Let $\mu \times \sigma$ be the Cartesian product of two fuzzy subsets μ and σ of S. Then the corresponding *Cartesian product* $(\mu \times \sigma)^{+'}$ *of* $L \times L$ is defined by

$$(\mu \times \sigma)^{+'}\Big(\sum_{i=1}^{n}[x_i, \alpha_i], \sum_{j=1}^{m}[y_j, \beta_j]\Big) = \inf_{s_i, s_j \in S} (\mu \times \sigma)\Big(\sum_{i=1}^{n} x_i\alpha_i s_i, \sum_{j=1}^{m} y_j\beta_j s_j\Big)$$

where $\sum_{i=1}^{n}[x_i, \alpha_i],\ \sum_{j=1}^{m}[y_j, \beta_j] \in L$.

4.6. Proposition. *Let μ, μ', ν, ν' are four fuzzy h-ideals of S. Then*

$$\big(\mu \times \mu'\big)\Gamma_h\big(\nu \times \nu'\big) = \big(\mu\Gamma_h\nu\big) \times \big(\mu'\Gamma_h\nu'\big).$$

Proof. Let $(x, y) \in S\times S$ be such that $(x, y)+(a, c)\gamma(a', c')+(z, z') = (b, d)\delta(b', d')+(z, z')$, where $a,\ c,\ a',\ c',\ z,\ z',\ b,\ d,\ b',\ d' \in S$ and $\gamma,\ \delta \in \Gamma$. Then $(x, y) + (a\gamma a', c\gamma c') + (z, z') = (b\delta b', d\delta d') + (z, z')$, which implies that $(x + a\gamma a' + z, y + c\gamma c' + z') = (b\delta b' + z, d\delta d' + z')$. Now

$$\begin{aligned}
&(\mu \times \mu')\Gamma_h(\nu \times \nu')(x, y)\\
&= \sup\{\min\{(\mu \times \mu')(a, c), (\mu \times \mu')(b, d), (\nu \times \nu')(a', c'), (\nu \times \nu')(b', d')\} \ :\\
&\qquad\qquad (x, y) + (a, c)\gamma(a', c') + (z, z') = (b, d)\delta(b', d') + (z, z')\}\\
&= \sup\{\min\{\min\{\mu(a), \mu'(c)\}, \min\{\mu(b), \mu'(d)\}, \min\{\nu(a'), \nu'(c')\},\\
&\qquad\qquad \min\{\nu(b'), \nu'(d')\}\}\}\\
&= \min\{\sup\{\min\{\mu(a), \mu(b), \nu(a'), \nu(b')\} \ : \ x + a\gamma a' + z = b\delta b' + z\},\\
&\qquad\qquad \sup\{\min\{\mu'(c), \mu'(d), \nu'(c'), \nu'(d')\} \ : \ y + c\gamma c' + z' = d\delta d' + z'\}\}\\
&= \min\big\{\big(\mu\Gamma_h\nu\big)(x), \big(\mu'\Gamma_h\nu'\big)(y)\big\}\\
&= \big(\big(\mu\Gamma_h\nu\big) \times \big(\mu'\Gamma_h\nu'\big)\big)(x, y)
\end{aligned}$$

Hence the proof. □

In this section we prove the results for the Γ-hemiring S and its right operator hemiring R. Similar results hold for the Γ-hemiring S and its left operator hemiring L.

4.7. Proposition. *Let μ and σ be two fuzzy subsets of R (L) [the right (left) operator hemiring of the Γ-hemiring S]. Then $(\mu \times \sigma)^{*} = \mu^{*} \times \sigma^{*}$ (resp. $(\mu \times \sigma)^{+} = \mu^{+} \times \sigma^{+}$).*

Proof. Let $x, y \in S$. Then

$$\begin{aligned}
(\mu \times \sigma)^{*}(x, y) &= \inf_{\alpha,\beta\in\Gamma} (\mu \times \sigma)([\alpha, x], [\beta, y]) = \inf_{\alpha,\beta\in\Gamma} \min\{\mu([\alpha, x]), \sigma([\beta, y])\}\\
&= \min\big\{\inf_{\alpha\in\Gamma} \mu([\alpha, x]), \inf_{\beta\in\Gamma} \sigma([\beta, y])\big\} = \min\big\{\mu^{*}(x), \sigma^{*}(y)\big\}\\
&= \big(\mu^{*} \times \sigma^{*}\big)(x, y).
\end{aligned}$$

Consequently, $(\mu \times \sigma)^{*} = \mu^{*} \times \sigma^{*}$. Similarly, we can show that $(\mu \times \sigma)^{+} = \mu^{+} \times \sigma^{+}$. □

4.8. Proposition. *Let μ and σ be two fuzzy subsets of S. Then*

$$(\mu \times \sigma)^{*'} = \mu^{*'} \times \sigma^{*'} ((\mu \times \sigma)^{+'} = \mu^{+'} \times \sigma^{+'}).$$

Proof. Let $\sum_{i=1}^{n}[\alpha_i, x_i], \sum_{j=1}^{m}[\beta_j, y_j] \in R$. Then

$$\begin{aligned}(\mu\times\sigma)^{*'}\Big(\sum_{i=1}^{n}[\alpha_i,x_i],\sum_{j=1}^{m}[\beta_j,y_j]\Big) &= \inf_{s_i,s_j\in S}(\mu\times\sigma)\Big(\sum_{i=1}^{n}s_i\alpha_i x_i,\sum_{j=1}^{m}s_j\beta_j y_j\Big)\\ &= \inf_{s_i,s_j\in S}\min\Big\{\mu\Big(\sum_{i=1}^{n}s_i\alpha_i x_i\Big),\sum_{j=1}^{m}\sigma(s_j\beta_j y_j)\Big\}\\ &= \min\Big\{\inf_{s_i\in S}\mu\Big(\sum_{i=1}^{n}s_i\alpha_i x_i\Big),\inf_{s_j\in S}\sigma\sum_{j=1}^{m}(s_j\beta_j y_j)\Big\}\\ &= \min\Big\{\mu^{*'}\Big(\sum_{i=1}^{n}[\alpha_i,x_i]\Big),\sigma^{*'}\Big(\sum_{j=1}^{m}[\beta_j,y_j]\Big)\Big\}\\ &= (\mu^{*'}\times\sigma^{*'})\Big(\sum_{i=1}^{n}[\alpha_i,x_i],\sum_{j=1}^{m}[\beta_j,y_j]\Big).\end{aligned}$$

Consequently, $(\mu\times\sigma)^{*'} = \mu^{*'}\times\sigma^{*'}$. Similarly we can show that $(\mu\times\sigma)^{+'} = \mu^{+'}\times\sigma^{+'}$. □

4.9. Proposition. *Suppose μ and σ are two fuzzy h-ideals of R (resp. L). Then $\mu^*\times\sigma^*$ (resp. $\mu^+\times\sigma^+$) is a fuzzy h-ideal of $S\times S$.*

Proof. Since μ, σ are fuzzy h-ideals of R, by Proposition 3.6, μ^*, σ^* are fuzzy h-ideals of S and so from [13, Theorem 35] we have that $\mu^*\times\sigma^*$ is a fuzzy h-ideal of $S\times S$.

In similar way we can prove the result for L. □

4.10. Proposition. *Let μ and σ be two prime (semiprime) fuzzy h-ideals of R (L). Then $\mu^*\times\sigma^*$ (resp. $\mu^+\times\sigma^+$) is a prime (semiprime) fuzzy h-ideal of $S\times S$.*

Proof. Since μ and σ are two prime fuzzy h-ideals of R, by Proposition 4.9 we see that $\mu^*\times\sigma^*$ is a fuzzy h-ideal of $S\times S$. To show that $\mu^*\times\sigma^*$ is prime, suppose $\theta,\theta',\eta,\eta'\in Fh\text{-}I(S)$ are such that $(\theta\times\theta')\Gamma_h(\eta\times\eta')\subseteq\mu^*\times\sigma^*$ Then from Proposition 4.6 we obtain $(\theta\Gamma_h\eta)\times(\theta'\Gamma_h\eta')\subseteq\mu^*\times\sigma^*$. Therefore $(\theta\Gamma_h\eta)\subseteq\mu^*$ and $(\theta'\Gamma_h\eta')\subseteq\sigma^*$. Hence $\theta\subseteq\mu^*$ or $\eta\subseteq\mu^*$ and $\theta'\subseteq\sigma^*$ or $\eta'\subseteq\sigma^*$, that is $\theta\times\theta'\subseteq\mu^*\times\sigma^*$ or $\eta\times\eta'\subseteq\mu^*\times\sigma^*$. So, $\mu^*\times\sigma^*$ is a prime fuzzy h-ideal of $S\times S$.

Similarly we can prove the result for semiprime fuzzy h-ideals and the left operator hemiring L. □

By suitable modification of the above argument we obtain the following result.

4.11. Proposition. *Let μ and σ be two fuzzy h-ideals (prime fuzzy h-ideals, semiprime fuzzy h-ideals) of S. Then $\mu^{*'}\times\sigma^{*'}$ [resp. $\mu^{+'}\times\sigma^{+'}$] is a fuzzy h-ideal (resp. prime fuzzy h-ideal, semiprime fuzzy h-ideal) of $R\times R$ [resp. $L\times L$].* □

4.12. Theorem. *Let S be a Γ-hemiring with unities and let R be its right operator hemiring. Then there exists an inclusion preserving bijection $\mu\times\sigma\longmapsto\mu^{*'}\times\sigma^{*'}$ between the set of all Cartesian products of fuzzy h-ideals (prime fuzzy h-ideals, semiprime fuzzy h-ideals) of S and the set of all Cartesian products of fuzzy h-ideals (prime fuzzy h-ideals, semiprime fuzzy h-ideals) of R, where μ and σ are fuzzy h-ideals (prime fuzzy h-ideals, semiprime fuzzy h-ideals) of S.*

Proof. Let μ and σ are fuzzy h-ideals of S and $x, y \in S$. Then

$$\begin{aligned}
\big(\mu^{*'} \times \sigma^{*'}\big)^*(x,y) &= \inf_{\alpha,\beta\in\Gamma}\big(\mu^{*'} \times \sigma^{*'}\big)([\alpha,x],\, [\beta,y]) \\
&= \inf_{\alpha,\beta\in\Gamma} \min\big\{\mu^{*'}([\alpha,x]),\, \sigma^{*'}([\beta,y])\big\} \\
&= \min\big\{\inf_{\alpha\in\Gamma}\mu^{*'}([\alpha,x]),\, \inf_{\beta\in\Gamma}\sigma^{*'}([\beta,y])\big\} \\
&= \min\big\{\inf_{\alpha\in\Gamma}\inf_{s_1\in S}\mu(s_1\alpha x),\, \inf_{\beta\in\Gamma}\inf_{s_2\in S}\sigma(s_2\beta y)\big\} \\
&\geq \min\{\mu(x), \sigma(y)\}(\text{ since } \mu \text{ and } \sigma \text{ are fuzzy ideals}) \\
&= (\mu \times \sigma)(x,y).
\end{aligned}$$

Therefore $\mu \times \sigma \subseteq \big(\mu^{*'} \times \sigma^{*'}\big)^*$. Let $[e,\delta]$ be the strong left unity of S. Then $e\delta x = x$ and $e\delta y = y$ for all $x, y \in S$. Now,

$$\begin{aligned}
(\mu \times \sigma)(x,y) &= \min\{\mu(x), \sigma(y)\} = \min\{\mu(e\delta x),\, \sigma(e\delta y)\} \\
&\geq \min\{\inf_{\alpha\in\Gamma}\inf_{s_1\in S}\mu(s_1\alpha x),\, \inf_{\beta\in\Gamma}\inf_{s_2\in S}\sigma(s_2\beta y)\} \\
&= \min\{\inf_{\alpha\in\Gamma}\mu^{*'}([\alpha,x]),\, \inf_{\beta\in\Gamma}\sigma^{*'}([\beta,\, y])\} \\
&= \min\big\{(\mu^{*'})^*(x),\, (\sigma^{*'})^*(y)\big\} \\
&= \big(\big(\mu^{*'}\big)^* \times \big(\sigma^{*'}\big)^*\big)(x,y) \\
&= \big(\mu^{*'} \times \sigma^{*'}\big)^*(x,y).
\end{aligned}$$

So $\mu \times \sigma \supseteq (\mu^{*'} \times \sigma^{*'})^*$. Hence $\mu \times \sigma = (\mu^{*'} \times \sigma^{*'})^*$.

Now let μ and σ be two fuzzy h-ideals of R. Then

$$\begin{aligned}
&\big(\mu^* \times \sigma^*\big)^{*'}\Big(\sum_{i=1}^{n}[\alpha_i,x_i],\, \sum_{j=1}^{m}[\beta_j,y_j]\Big) \\
&= \inf_{s_i,s_j\in S}\big(\mu^* \times \sigma^*\big)\Big(\sum_{i=1}^{n}s_i\alpha_i x_i,\, \sum_{j=1}^{m}s_j\beta_j y_j\Big) \\
&= \inf_{s_i,s_j\in S}\min\Big\{\mu^*\Big(\sum_{i=1}^{n}s_i\alpha_i x_i),\, \sigma^*\Big(\sum_{j=1}^{m}s_j\beta_j y_j\Big)\Big\} \\
&= \min\Big\{\inf_{s_i\in S}\mu^*\Big(\sum_{i=1}^{n}s_i\alpha_i x_i),\, \inf_{s_j\in S}\sigma^*\Big(\sum_{j=1}^{m}s_j\beta_j y_j\Big)\Big\} \\
&= \min\Big\{\inf_{s_i\in S}\inf_{\gamma\in\Gamma}\mu\Big(\Big[\gamma,\, \sum_{i=1}^{n}s_i\alpha_i x_i\Big]\Big),\, \inf_{s_j\in S}\inf_{\delta\in\Gamma}\sigma\Big(\Big[\delta,\, \sum_{j=1}^{m}s_j\beta_j y_j\Big]\Big)\Big\} \\
&= \min\Big\{\inf_{s_i\in S}\inf_{\gamma\in\Gamma}\mu\Big(\sum_{i=1}^{n}[\gamma,s_i]\sum_{i=1}^{n}[\alpha_i,x_i]\Big),\, \inf_{s_j\in S}\inf_{\delta\in\Gamma}\sigma\Big(\sum_{j=1}^{m}[\delta,s_j]\sum_{j=1}^{m}[\beta_j,y_j]\Big)\Big\} \\
&\geq \min\Big\{\mu\Big(\sum_{i=1}^{n}[\alpha_i,x_j]),\, \sigma\Big(\sum_{j=1}^{m}[\beta_j,y_j]\Big)\Big\} \\
&= (\mu \times \sigma)\Big(\sum_{i=1}^{n}[\alpha_i,x_i],\, \sum_{j=1}^{m}[\beta_j,y_j]\Big).
\end{aligned}$$

Thus we obtain $(\mu^* \times \sigma^*)^{*'} \supseteq \mu \times \sigma$. Let $\sum_{k=1}^{p}[\gamma_k, f_k]$ be the right unity of S. Then

$$\begin{aligned}
&(\mu \times \sigma)\Big(\sum_{i=1}^{n}[\alpha_i, x_i], \sum_{j=1}^{m}[\beta_j, y_j]\Big) \\
&\quad = \min\Big\{\mu\Big(\sum_{i=1}^{n}[\alpha_i, x_i]\Big), \sigma\Big(\sum_{j=1}^{m}[\beta_j, y_j]\Big)\Big\} \\
&\quad = \min\Big\{\mu\Big(\sum_{i=1}^{n}[\alpha_i, x_i]\sum_{k=1}^{p}[\gamma_k, f_k]\Big), \sigma\Big(\sum_{j=1}^{m}[\beta_j, y_j]\sum_{k=1}^{p}[\gamma_k, f_k]\Big)\Big\} \\
&\quad \geq \min\Big\{\inf_{s_i\in S}\inf_{\gamma\in\Gamma}\mu\Big(\sum_{i=1}^{n}[\alpha_i, x_i]\sum_{i=1}^{n}[\gamma, s_i]\Big), \inf_{s_j\in S}\inf_{\delta\in\Gamma}\sigma\Big(\sum_{j=1}^{m}[\beta_j, y_j]\sum_{j=1}^{m}[\delta, s_j]\Big)\Big\} \\
&\quad \geq \min\Big\{(\mu^*)^{*'}\Big(\sum_{i=1}^{n}[\alpha_i, x_i]\Big), (\sigma^*)^{*'}\Big(\sum_{j=1}^{m}[\beta_j, y_j]\Big)\Big\} \\
&\quad = \big((\mu^*)^{*'} \times (\sigma^*)^{*'}\big)\Big(\sum_{i=1}^{n}[\alpha_i, x_i], \sum_{j=1}^{m}[\beta_j, y_j]\Big) \\
&\quad = \big(\mu^* \times \sigma^*\big)^{*'}\Big(\sum_{i=1}^{n}[\alpha_i, x_i], \sum_{j=1}^{n}[\beta_j, y_j]\Big).
\end{aligned}$$

Hence $\big(\mu^* \times \sigma^*\big)^{*'} \subseteq \mu \times \sigma$. Consequently, $\big(\mu^* \times \sigma^*\big)^{*'} = \mu \times \sigma$. Thus we see that the correspondence $\mu \times \sigma \longmapsto \mu^{*'} \times \sigma^{*'}$ is a bijection.

Now let $\mu_1, \mu_2, \sigma_1, \sigma_2$ be fuzzy h-ideals of S such that $\mu_1 \times \sigma_1 \subseteq \mu_2 \times \sigma_2$. Then:

$$\begin{aligned}
&\big(\mu_1^{*'} \times \sigma_1^{*'}\big)\Big(\sum_{i=1}^{n}[\alpha_i, x_i], \sum_{j=1}^{m}[\beta_j, y_j]\Big) \\
&\quad = \inf_{s_i, s_j\in S}\big(\mu_1 \times \sigma_1\big)\Big(\sum_{i=1}^{n}s_i\alpha_i x_i, \sum_{j=1}^{m}s_j\beta_j y_j\Big) \\
&\quad = \inf_{s_i, s_j\in S}\min\Big\{\mu_1\Big(\sum_{i=1}^{n}s_i\alpha_i x_i\Big), \sigma_1\sum_{j=1}^{m}\Big(s_j\beta_j y_j\Big)\Big\} \\
&\quad \leq \inf_{s_i, s_j\in S}\min\Big\{\mu_2\Big(\sum_{i=1}^{n}s_i\alpha_i x_i\Big), \sigma_2\Big(\sum_{j=1}^{m}s_j\beta_j y_j\Big)\Big\} \\
&\quad = \min\Big\{\inf_{s_i\in S}\mu_2\Big(\sum_{i=1}^{n}s_i\alpha_i x_i\Big), \inf_{s_j\in S}\sigma_2\Big(\sum_{j=1}^{m}s_j\beta_j y_j\Big)\Big\} \\
&\quad = \min\Big\{\mu_2^{*'}\Big(\sum_{i=1}^{n}[\alpha_i, x_i]\Big), \sigma_2^{*'}\Big(\sum_{j=1}^{m}[\beta_j, y_j]\Big)\Big\} \\
&\quad = \big(\mu_2^{*'} \times \sigma_2^{*'}\big)\Big(\sum_{i=1}^{n}[\alpha_i, x_i], \sum_{j=1}^{m}[\beta_j, y_j]\Big).
\end{aligned}$$

Hence $\mu_1^{*'} \times \sigma_1^{*'} \subseteq \mu_2^{*'} \times \sigma_2^{*'}$. Therefore, $\mu \times \sigma \longmapsto \mu^{*'} \times \sigma^{*'}$ is an inclusion preserving bijection.

Similarly, we can prove the results for prime fuzzy h-ideals and semiprime fuzzy h-ideals. □

Acknowledgement

The author is grateful to Dr. S. K. Sardar of Jadavpur University for suggesting this problem, and for his continuous guidance in writing this paper. The author is also thankful to the learned referee for his valuable comments to improve the paper.

References

[1] Bhattacharya, P. and Mukherjee, N. P. *Fuzzy relations and fuzzy groups*, Information Sciences **36**, 267–282, 1985.

[2] Dudek, W. A., Shabir, M. and Anjum, R. *Characterizations of hemirings by their h-ideals*, Comput. Math. Appl. **59**, 3167–3179, 2010.

[3] Dutta, T. K. and Sardar, S. K. *On the operator semirings of a Γ-semiring*, Southeast Asian Bull. Math. **26**, 203–213, 2002.

[4] Golan, J. S. *Semirings and their Applications* (Kluwer Academic Publishers, Netherlands, 1999).

[5] Henriksen, M. *Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices **6**, 321, 1958.

[6] Iizuka, K. *On the Jacobson radical of semiring*, Tohoku Math. J. **11** (2), 409–421, 1959.

[7] Jun, Y. B. and Lee, C. Y. *Fuzzy Γ-rings*, Pusom Kyongnam Math. J. **8** (2), 63–170, 1992.

[8] Jun, Y. B., Öztürk, M. A. and Song, S. Z. *On Fuzzy h-ideals in hemiring*, Information Sciences **162**, 211–226, 2004.

[9] La Torre, D. R. *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen **12**, 219–226, 1965.

[10] Ma, X. and Zhan, J. *Fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemirings*, Neural Comput and Applications **19**, 477–485, 2010.

[11] Ma, X. and Zhan, J. *Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings*, Information Sciences **179**, 1249–1268, 2009.

[12] Rao, M. M. K. *Γ-semirings-1*, Southeast Asian Bull. Math. **19**, 49–54, 1995.

[13] Sardar, S. K. and Mandal, D. *Fuzzy h-ideals in Γ-hemiring*, Int. J. Pure. Appl. Math. **56** (3), 439–450, 2009.

[14] Sardar, S. K. and Mandal, D. *Prime (semiprime) fuzzy h-ideals in Γ-hemiring*, Int. J. Comput. Math. Sci. **5** (3), 160–164, 2011.

[15] Sardar, S. K. and Mandal, D. *On fuzzy h-ideals in h-regular Γ-hemirings and h-duo Γ-hemirings*, Gen. Math. Notes. **2** (1), 64–85, 2011.

[16] Zadeh, L. A. *Fuzzy sets*, Information and Control **8**, 338–353, 1965.

[17] Zhan, J. and Dudek, W. A. *Fuzzy h-ideals of hemirings*, Information Sciences **177**, 876–886, 2007.