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ASYMPTOTIC EQUIVALENCE OF DOUBLE SEQUENCES

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Abstract

The goal of this paper is to present a four-dimensional matrix characterization of asymptotic equivalence of double sequences. This will be accomplished with the following notion of asymptotic equivalence of double sequences. Two double sequences are asymptotic equivalent if and only if $P - \lim_{k,l} \frac{x_{k,l}}{y_{k,l}} = 1$, where x and y are selected judicially. Using this notion necessary and sufficient conditions on the entries of a four-dimensional matrix are given to ensure that the transformation will preserve asymptotic equivalence.

Keywords: Divergent double sequences, Subsequences of a double sequences, Pringsheim limit point, P-convergent, P-divergent, RH-regular.

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1. Introduction

In 2002, Patterson [7] provided an answer to the following question : what type of four dimensional matrix transformation will preserve asymptotic equivalence under transformations $[\mu(x)]$?

In this paper, we present a regularity type of characterization of asymptotic equivalence of double sequences by using four-dimensional matrix transformations. To accomplish this we established the following theorem:

Suppose A is a non-negative four-dimensional matrix. Then

$$x \overset{P''}{\sim} y \text{ implies } \mu(Ax) \overset{P''}{\sim} \mu(Ay)$$

for any double sequences $x, y \in P''_\delta$ for some $\delta > 0$ if and only if A satisfies the following three conditions:

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(1) $\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\right)$ is a bounded double sequence dominated by some constant B;

(2) (a) For any order pair of $i, j = 1, 2, 3, \ldots$,

$$\text{P–}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}}{\sup_{k,l\geq m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,r,s}} = 0;$$

(b)

$$\text{P–}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{i=0}^{\infty} a_{k,l,i,j}}{\sup_{k,l\geq m,n} \sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = 0 \text{ for fixed } j = 0, 1, 2, \ldots;$$

(c)

$$\text{P–}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{j=0}^{\infty} a_{k,l,i,j}}{\sup_{k,l\geq m,n} \sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = 0 \text{ for fixed } i = 0, 1, 2, \ldots;$$

(3) for any infinite sequences $i_1 < i_2 < i_3 < \cdots$ and $j_1 < j_2 < j_3 < \cdots$,

$$\text{P–}\limsup_{m,n} \frac{\sup_{k,l>m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}}{\sup_{k,l\geq m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,r,s}} = \lambda.$$

2. Definitions, notations and preliminary results

2.1. Definition. (A. Pringsheim, [10]) A double sequence $x = [x_{k,l}]$ has **Pringsheim limit** L (denoted by P–lim $x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an x is describe more briefly as "P-convergent".

2.2. Definition. The double sequence y is a **double subsequence of** x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $x_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - \end{array}$$

We let

$$\text{P}_\delta = \{\text{ The set of all real double number sequences such that } x_{k,l} \geq \delta > 0 \text{ for all } k \text{ and } l\,\},$$

$$P_0 = \{\text{ The set of all nonnegative sequences which have at most a finite number of columns and/or rows with zero entries}\,\}.$$

2.3. Definition. (Patterson, [7]) Two nonnegative double sequences (x) and (y) are said to be **asymptotically equivalent** if

$$\text{P} - \lim_{k,l\to\infty} \frac{x_{k,l}}{y_{k,l}} = 1,$$

(denoted by $x \overset{P''}{\sim} y$).

$$d_A = \{x_{k,l} : \text{P–}\lim_{m,m\to\infty} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \text{ exists}\}.$$

In a manner similar to Pobyvanets, Patterson presented the following multidimensional Theorem in [6].

2.4. Theorem. *In order for a four dimensional matrix A to be asymptotically regular it is necessary and sufficient that for fixed integers l_0 or k_0,*

$$\sum_{k,l\in b_0} a_{m,n,k,l} \text{ is bounded for all } m \text{ and } n, \text{ and} \tag{2.1}$$

$$P-\lim_{m,n\to\infty}\frac{\sum_{k,l\in b_0} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} = 0, \tag{2.2}$$

where $b_0 = \{(p,q) : 1 \le p \le k_0 \text{ or } 1 \le q \le l_0\}$

3. Main results

3.1. Theorem. *Let A be a four-dimensional non-negative matrix. Suppose* $x \overset{P''}{\sim} y$ *and* $x, y \in P''_\delta$ *for some* $\delta > 0$ *and x is bounded. Then* $\mu_{m,n}(Ax) \overset{P''}{\sim} \mu_{m,n}(Ay)$ *if and only if*

(1) *for each* $i, j = 0, 1, 2, 3, \ldots$,

$$P\text{–}\lim_{m,n}\frac{a_{m,n,i,j}}{\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}} = 0;$$

(2)

$$P\text{–}\lim_{m,n}\frac{\sup_{k,l>m,n}\sum_{i=0}^{\infty} a_{k,l,i,j}}{\sup_{k,l\ge m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = 0 \text{ for fixed } j = 0, 1, 2, \ldots;$$

(3)

$$P\text{–}\lim_{m,n}\frac{\sup_{k,l>m,n}\sum_{j=0}^{\infty} a_{k,l,i,j}}{\sup_{k,l\ge m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = 0 \text{ for fixed } i = 0, 1, 2, \ldots;$$

Proof. If

$$\text{P–}\lim_{m,n}\frac{a_{m,n,i,j}}{\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}} = 0,\ \ i, j = 0, 1, 2, 3, \ldots$$

then we need to show that $\mu_{m,n}(Ax) \overset{P''}{\sim} \mu_{m,n}(Ay)$. Since $x \overset{P''}{\sim} y$ there exists a bounded P–null double sequence z such that $x_{k,l} = y_{k,l}(1 + z_{k,l})$. Thus for each (m,n) we have the following:

$$\begin{aligned}
\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} &= \frac{\sup_{p,q\ge m,n}(Ax)_{p,q}}{\sup_{p,q\ge}(Ay)_{p,q}}\\
&= \frac{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}x_{k,l}}{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}}\\
&= \frac{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}(y_{k,l}+z_{k,l}y_{k,l})}{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}}\\
&= 1 + \frac{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}|z_{k,l}|y_{k,l}}{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}}
\end{aligned}$$

$$= 1 + \frac{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\bar{M},\bar{N}} a_{p,q,k,l}|z_{k,l}|y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} + \frac{\sup_{p,q\geq m,n}\sum_{k=0,l=\bar{N}+1}^{\bar{M}-1,\infty} a_{p,q,k,l}|z_{k,l}|y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} + \frac{\sup_{p,q\geq m,n}\sum_{k=\bar{M}+1,l=0}^{\infty,\bar{N}-1} a_{p,q,k,l}|z_{k,l}|y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} + \frac{\sup_{p,q\geq m,n}\sum_{k=\bar{M}+1,l=\bar{N}+1}^{\infty,\infty} a_{p,q,k,l}|z_{k,l}|y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}}$$

for some positive integers $\bar{M}$ and $\bar{N}$. Since z is a bounded P–null double sequence, $\sup_{k,l}|z_{k,l}| < \infty$ and for any $\epsilon > 0$ then there exist $\bar{M}$ and $\bar{N}$ such that $k,l \geq \bar{M}, \bar{N}|z_{k,l}| < \epsilon$. Therefore,

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \leq 1 + \sup_{k,l}|z_{k,l}|\sum_{k,l=0,0}^{\bar{M},\bar{N}} \frac{\sup_{p,q\geq m,n} a_{p,q,k,l}y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} + \sup_{k,l}|z_{k,l}|\sum_{k=0,l=\bar{N}+1}^{\bar{M}-1,\infty} \frac{\sup_{p,q\geq m,n} a_{p,q,k,l}y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} + \sup_{k,l}|z_{k,l}|\sum_{k=\bar{M}+1,l=0}^{\infty,\bar{N}-1} \frac{\sup_{p,q\geq m,n} a_{p,q,k,l}y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} + \epsilon\frac{\sum_{k=\bar{M}+1,l=\bar{N}+1}^{\infty,\infty}\sup_{p,q\geq m,n} a_{p,q,k,l}y_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}y_{k,l}}.$$

The inequalities above yields the following:

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \leq 1 + \sup_{k,l}|z_{k,l}|\sup_{0\leq k\leq \bar{M};0\leq l\leq \bar{N}}(y_{k,l})\sum_{k,l=0,0}^{\bar{M},\bar{N}}\sup_{p,q\geq m,n}\left(\frac{a_{p,q,k,l}}{\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}}\right) + \sup_{k,l}|z_{k,l}|\sup_{0\leq k\leq \bar{M}-1;\bar{N}+1\leq l\leq\infty}(y_{k,l})\sum_{k=0,l=\bar{N}+1}^{\bar{M}-1,\infty}\sup_{p,q\geq m,n}\left(\frac{a_{p,q,k,l}}{\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}}\right) + \sup_{k,l}|z_{k,l}|\sup_{\bar{M}\leq k\leq\infty;0\leq l\leq \bar{N}-1}(y_{k,l})\sum_{k=\bar{M}+1,l=0}^{\infty,\bar{N}-1}\sup_{p,q\geq m,n}\left(\frac{a_{p,q,k,l}}{\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}}\right) + \epsilon\sum_{k=\bar{M}+1,l=\bar{N}+1}^{\infty,\infty}\sup_{p,q\geq m,n}\left(\frac{a_{p,q,k,l}}{\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}}\right).$$

According to the hypothesis there exist $M_1, N_1 \in \mathbf{N}$ such that if $p,q > M_1, N_1$ then

$$\frac{a_{p,q,k,l}}{\sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} < \frac{\epsilon}{\bar{M}\bar{N}3\sup_{k,l}|z_{k,l}|\sup_{0\leq k\leq \bar{M};0\leq l\leq \bar{N}}(y_{k,l})} \quad \text{for } 0\leq k\leq \bar{M};\ 0\leq l\leq \bar{N}, \tag{3.1}$$

$$\sum_{l=\bar{N}+1}^{\infty} \frac{a_{p,q,k,l}}{\sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} < \frac{\epsilon}{\bar{M}3\sup_{k,l}|z_{k,l}|\sup_{0\le k\le \bar{M}-1;\bar{N}+1\le l\le\infty}(y_{k,l})} \quad \text{for } 0 \le k \le \bar{M}-1, \tag{3.2}$$

$$\sum_{k=\bar{M}+1}^{\infty} \frac{a_{p,q,k,l}}{\sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} < \frac{\epsilon}{\bar{N}3\sup_{k,l}|z_{k,l}|\sup_{\bar{M}+1\le k\le\infty;0\le l\le \bar{N}-1}(y_{k,l})} \quad \text{for } 0 \le l \le \bar{N}-1. \tag{3.3}$$

Thus for $m, n > \bar{M}, \bar{N}$ we have the following:

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \le 1 + 2\epsilon.$$

Therefore

$$\text{P}-\lim_{m,n} \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \le 1.$$

Similarly we can also show that

$$\text{P}-\lim_{m,n} \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \ge 1.$$

Thus

$$\text{P}-\lim_{m,n} \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} = 1.$$

Next suppose $\mu_{m,n}(Ax) \overset{P''}{\sim} \mu_{m,n}(Ay)$ for $x \overset{P''}{\sim} y$ such that $x, y \in P''_\delta$; $\delta > 0$. If $x = y = 1$ for all (k,l) then $\mu_{m,n}(Ax) \overset{P''}{\sim} \mu_{m,n}(Ay)$, that is

$$\text{P}-\lim_{m,n} \frac{\sup_{p,q\ge m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}}{\sup_{p,q\ge m,n}\sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} = 1.$$

Therefore there exists $M > 0$ such that $\{\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}\}$ is bounded by M. If $\text{P}-\lim_{p,q} \frac{a_{p,q,k,l}}{\sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} \ne 0$ for some (p,q) then there exists $\lambda > 0$ and index sequences $m_1 < m_2 < m_3 < \cdots ; n_1 < n_2 < n_3 < \cdots$ such that

$$\frac{a_{m_p,n_q,k,l}}{\sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}} \ge \lambda,$$

where $k, l = 0, 1, 2, \ldots$ and $\lambda > 0$. Let $t > 0$ and

$$y_{k,l} = 1 \text{ for } k, l = 0, 1, 2, \ldots$$

and

$$x_{k,l} = \begin{cases} 1, & \text{if } k \ne m_k;\ l \ne n_l, \\ 1+t, & \text{if } k = m_k;\ l = n_l. \end{cases}$$

Since $x \overset{P''}{\sim} y$ and $x, y \in P''_1$ we have the following

$$\begin{aligned} \text{P}-\lim_{k,l} & \frac{\sup_{p,q\ge k,l}\sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j}x_{i,j}}{\sup_{p,q\ge k,l}\sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j}y_{i,j}} \\ &= \text{P}-\lim_{k,l} \frac{\sup_{p,q\ge k,l}\sum_{i,j=0,0}^{\infty,\infty}(a_{m_p,n_q,i,j}) + t a_{m_p,n_q,k,l}}{\sup_{p,q\ge k,l}\sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j}y_{i,j}} \\ &\ge \text{P}-\lim_{k,l} \frac{\sup_{p,q\ge k,l}\sum_{i,j=0,0}^{\infty,\infty}(a_{m_p,n_q,i,j}) + t\lambda\sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j}}{\sup_{p,q\ge k,l}\sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j}y_{i,j}} \\ &= 1 + t\lambda. \end{aligned}$$

If $t = \frac{1}{\lambda}$ then

$$\text{P-}\lim_{k,l} \frac{\mu_{m_k,n_l}(Ax)}{\mu_{m_k,n_l}(Ay)} \geq 2.$$

Thus $\mu_{m,n}(Ax) \overset{P''}{\not\sim} \mu_{m,n}(Ay)$. This establishes condition (1).

Now let us establish condition (2). Similar to the proof of condition (1) there exist M positive such that $\{\sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}\}$ is bounded by M. Now suppose

$$\text{P-}\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{k=0}^{\infty} a_{m_p,n_q,k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0}^{\infty,\infty} a_{m_p,n_q,k,l}} \neq 0 \text{ for fixed } l.$$

Then there exists $\lambda > 0$ and double index sequences $\{m_p\}$ and $\{n_q\}$ such that

$$\frac{\sup_{p,q\geq m,n} \sum_{k=0}^{\infty} a_{m_p,n_q,k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0}^{\infty,\infty} a_{m_p,n_q,k,l}} \geq \lambda$$

where $p, q = 0, 1, 2, \ldots$, $\lambda > 0$. Let $t > 0$ and define x and y as follows:

$y_{k,l} = 1$ for all $k, l = 0, 1, 2, \ldots$ and

$$x_{k,l} = \begin{cases} 1, & \text{if } l \neq l_0;\ k = 0, 1, 2, \ldots, \\ 1 + t, & \text{if } l = l_0;\ k = 0, 1, 2, \ldots. \end{cases}$$

Observe that $x \overset{P''}{\sim} y$ and $x, y \in P_1''$, thus we have the following:

$$\begin{aligned} &\text{P-}\lim_{k,l} \frac{\sup_{p,q\geq k,l} \sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j} x_{i,j}}{\sup_{p,q\geq k,l} \sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j} y_{i,j}} \\ &= \text{P-}\lim_{k,l} \frac{\sup_{p,q\geq k,l} \sum_{i,j=0,0}^{\infty,\infty} (a_{m_p,n_q,i,j}) + t \sum_{k=0}^{\infty} a_{m_p,n_q,k,l_0}}{\sup_{p,q\geq k,l} \sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j} y_{i,j}} \\ &\geq \text{P-}\lim_{k,l} \frac{\sup_{p,q\geq k,l} \sum_{i,j=0,0}^{\infty,\infty} (a_{m_p,n_q,i,j}) + t\lambda \sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j}}{\sup_{p,q\geq k,l} \sum_{i,j=0,0}^{\infty,\infty} a_{m_p,n_q,i,j} y_{i,j}} \\ &= 1 + t\lambda. \end{aligned}$$

Thus as above we can take $t = \frac{1}{\lambda}$ and obtain a contradiction. Condition (3) can be established in a similar manner and the proof is omitted. This completes the proof. □

3.2. Theorem. *Suppose A is a non-negative four-dimensional matrix. Then $\mu(x) \overset{P''}{\sim} \mu(y)$ implies $\mu(Ax) \overset{P''}{\sim} \mu(Ay)$ for any bounded double sequences $x, y \in P_\delta''$ for some $\delta > 0$ if and only if A satisfies the following three conditions*

(1) $\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\right)$ *is a bounded double sequence dominated by some constant B;*

(2) (a) *For any order pair of $i, j = 1, 2, 3, \ldots$,*

$$P\text{-}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}}{\sup_{k,l\geq m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,r,s}} = 0;$$

(b)

$$P\text{-}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{i=0}^{\infty} a_{k,l,i,j}}{\sup_{k,l\geq m,n} \sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = 0 \textit{ for fixed } j = 0, 1, 2, \ldots;$$

(c)

$$P\text{-}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{j=0}^{\infty} a_{k,l,i,j}}{\sup_{k,l\geq m,n} \sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = 0 \textit{ for fixed } i = 0, 1, 2, \ldots;$$

(3) *for any infinite sequences* $i_1 < i_2 < i_3 < \cdots$ *and* $j_1 < j_2 < j_3 < \cdots$

$$P\text{–}\lim_{m,n} \frac{\sup_{k,l>m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}}{\sup_{k,l\geq m,n} \sum_{r,s=0,0}^{\infty,\infty} a_{k,l,r,s}} = 1.$$

Proof. Let x and y be bounded double sequences in P''_δ for some $\delta > 0$, $\mu(x) \overset{P''}{\sim} \mu(y)$ implies $\mu(Ax) \overset{P''}{\sim} \mu(Ay)$. Let $x = y = 1$ for all k and l. Then x and y are bounded double sequences in P''_1 and $\mu(x) \overset{P''}{\sim} \mu(y)$ and $\mu(Ax) \overset{P''}{\sim} \mu(Ay)$. However $\mu_{m,n}(Ax) = \sup_{k,l\geq m,n} \sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}$. Thus $(\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j})_{k,l=0,0}^{\infty,\infty}$ is bounded. This proves (1).

To establish (2a) we consider p and q such that

$$\text{P–}\limsup_{m,n} \frac{\sup_{k,l>m,n} a_{k,l,p,q}}{\sup_{k,l\geq m,n} \sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}} = \lambda$$

for some $\lambda > 0$. Let $t > 0$ and let

$$y_{k,l} = 1 \text{ for } k, l = 0, 1, 2, \ldots$$

and

$$x_{k,l} = \begin{cases} 1, & \text{if } m \neq p;\ n \neq q, \\ 1 + t & \text{if } m = p;\ n = q. \end{cases}$$

Then $x, y \in P_1$, both x and y are bounded and $\mu(x) \overset{P''}{\sim} \mu(y)$ and thus $\mu(Ax) \overset{P''}{\sim} \mu(Ay)$. However,

$$\begin{aligned}
&\text{P–}\limsup_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} x_{k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} y_{k,l}} \\
&= \text{P–}\limsup_{m,n} \frac{\sup_{p,q\geq m,n}(t a_{p,q,i,j} + \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l})}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} y_{k,l}} \\
&\geq \text{P–}\limsup_{m,n} \frac{\sup_{p,q\geq m,n} t a_{p,q,i,j}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} y_{k,l}} - 1 \\
&= t\lambda - 1.
\end{aligned}$$

We now choose $t = \frac{3}{\lambda}$. This implies

$$\text{P–}\limsup_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} x_{k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} y_{k,l}} \geq 3 - 1 = 2.$$

This is a contradiction to $\mu(Ax) \overset{P''}{\sim} \mu(Ay)$, thus (2) holds.

Let us now establish (3). For any given infinite index sequences $i_1 < i_2 < i_3 < \cdots$ and $j_1 < j_2 < j_3 < \cdots$ we define x and y by

$$y_{k,l} = 2 \text{ for all } k, l$$

and

$$x_{k,l} = \begin{cases} 2, & \text{if } k = i_r;\ l = j_s, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that x and y are bounded and in P_1'', and $\mu(x) \overset{P''}{\sim} \mu(y)$. Thus $\mu(Ax) \overset{P''}{\sim} \mu(Ay)$. Therefore we have the following

$$\begin{aligned} 1 &= \text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} x_{k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} y_{k,l}} \\ &= \text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} (\sum_{(k,l)\in\Lambda} a_{p,q,k,l} x_{k,l} + \sum_{(k,l)\notin\Lambda} a_{p,q,k,l} x_{k,l})}{2\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l} y_{k,l}}, \end{aligned}$$

where Λ denotes the collection of ordered pairs (i_r, j_s) for $r, s = 1, 2, 3, \ldots$. Hence

$$\begin{aligned} 1 &= \text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} (2\sum_{(k,l)\in\Lambda} a_{p,q,k,l} + \sum_{(k,l)\notin\Lambda} a_{p,q,k,l})}{2\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} \\ &= \text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} (\sum_{(k,l)\in\Lambda} a_{p,q,k,l} + \sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l})}{2\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} \\ &\leq \text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{(k,l)\in\Lambda} a_{p,q,k,l} + \sup_{p,q\geq m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{p,q,k,l}}{2\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} \\ &= \text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{(k,l)\in\Lambda} a_{p,q,k,l}}{2\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} + \frac{1}{2}. \end{aligned}$$

Therefore

$$1 \leq \frac{1}{2}\text{P} - \lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{(k,l)\in\Lambda} a_{p,q,k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} + \frac{1}{2}.$$

This implies

$$\text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{(k,l)\in\Lambda} a_{p,q,k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} \geq 1.$$

It is clear that a similar argument will establish the following

$$\text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{(k,l)\in\Lambda} a_{p,q,k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} \leq 1.$$

Thus

$$\text{P}-\lim_{m,n} \frac{\sup_{p,q\geq m,n} \sum_{(k,l)\in\Lambda} a_{p,q,k,l}}{\sup_{p,q\geq m,n} \sum_{k,l=,0,0}^{\infty,\infty} a_{p,q,k,l}} = 1.$$

Now let us establish the converse of this theorem. We are granted conditions (1), (2), and (3). In addition suppose x and y are bounded by some $M > 0$; $x, y \in P_\delta''$ for some $\delta > 0$, and $\mu(x) \overset{P''}{\sim} \mu(y)$. For any $\epsilon > 0$ and since x and y are bounded there exist $M_1, N_1 \in \mathbf{N}$ such that for $i, j \geq M, N$ the following holds:

$$y_{i,j} \leq \text{P}-\lim_{k,l} \sup_{i,j\geq k,l} y_{i,j} + \epsilon,$$

and also there exist infinite index sequences $i_1 < i_2 < i_3 < \cdots$ and $j_1 < j_2 < j_3 < \cdots$ such that

$$x_{i_r,j_s} \geq \text{P}-\lim_{k,l} \sup_{i,j\geq k,l} x_{i,j} - \epsilon \text{ for } r, s = 1, 2, 3, \ldots.$$

Now,

$$
\begin{aligned}
&\text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s} x_{i_r,j_s}}{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} \\
&\geq \text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,li_r,j_s} x_{i_r,j_s}}{\bar{\Delta}+\sup_{k,l\geq m,n}\sum_{i,j=M_1+1,N_1+1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} \\
&\geq \text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}-\epsilon\right)}{\bar{\Delta}+\sup_{k,l\geq m,n}\sum_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim_{k,l}\sup_{i,j\geq k,l} y_{i,j}+\epsilon\right)} \\
&\geq \text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}\right)}{\bar{\Delta}+\sup_{k,l\geq m,n}\sum_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim_{k,l}\sup_{i,j\geq k,l} y_{i,j}+\epsilon\right)} \\
&- \text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\epsilon}{\bar{\Delta}+\sup_{k,l\geq m,n}\sum_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim_{k,l}\sup_{i,j\geq k,l} y_{i,j}+\epsilon\right)} \\
&\geq \text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}\right)}{\left\{\begin{array}{l}\bar{\Delta}+\epsilon\sup\limits_{k,l\geq m,n}\sum\limits_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j} \\ \qquad +\sup\limits_{k,l\geq m,n}\sum\limits_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim\limits_{k,l}\sup\limits_{i,j\geq k,l} y_{i,j}\right)\end{array}\right\}} \\
&\qquad - \text{P-}\lim_{m,n} \frac{\epsilon\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}}{\left\{\begin{array}{l}\bar{\Delta}+\epsilon\sup\limits_{k,l\geq m,n}\sum\limits_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j} \\ \qquad +\sup\limits_{k,l\geq m,n}\sum\limits_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim\limits_{k,l}\sup\limits_{i,j\geq k,l} y_{i,j}\right)\end{array}\right\}} \\
&\geq \text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}\right)}{\left\{\begin{array}{l}\bar{\Delta}+\epsilon\sup\limits_{k,l\geq m,n}\sum\limits_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j} \\ \qquad +\sup\limits_{k,l\geq m,n}\sum\limits_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim\limits_{k,l}\sup\limits_{i,j\geq k,l} y_{i,j}\right)\end{array}\right\}} - \frac{\epsilon}{\delta} \\
&= \frac{1}{B_1+B_2+B_3} - \frac{\epsilon}{\delta}
\end{aligned}
$$

where $\Delta_1 = \sup_{k,l\geq m,n}\sum_{i,j=0,0}^{M_1,N_1} a_{k,l,i,j}$, $\Delta_2 = \sup_{k,l\geq m,n}\sum_{i,j=M_1+1,0}^{\infty,N_1} a_{k,l,i,j}$, $\Delta_3 = \sup_{k,l\geq m,n}\sum_{i,j=0,N_1+1}^{M_1,\infty} a_{k,l,i,j}$, $\bar{\Delta} = M_1\sum_{l=1}^{3}\Delta_l$,

$$
\begin{aligned}
B_1 &= \frac{M_1\bar{\Delta}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}\right)}, \\
B_2 &= \frac{\epsilon\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}\right)},
\end{aligned}
$$

and

$$
B_3 = \frac{\sup_{k,l\geq m,n}\sum_{i,j>M_1,N_1} a_{k,l,i,j}\left(\text{P-}\lim_{k,l}\sup_{i,j\geq k,l} y_{i,j}\right)}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}\left(\text{P-}\lim_{k,l}\sup_{r,s\geq k,l} x_{r,s}\right)}.
$$

In addition let

$$\bar{B}_1 = \frac{M_1\bar{\Delta}}{\delta \sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}},$$
$$\bar{B}_2 = \frac{\epsilon \sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}}{\delta \sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}},$$

and

$$\bar{B}_3 = \frac{\sup_{k,l\geq m,n}\sum_{i,j>M_1,N_1} a_{k,l,i,j}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}}.$$

It is also clear that

$$\text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}x_{i_r,j_s}}{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}y_{i,j}} \geq \frac{1}{\bar{B}_1+\bar{B}_2+\bar{B}_3} - \frac{\epsilon}{\delta}.$$

Conditions (2) and (3) ensure that

$$\text{P-}\lim_{m,n} \frac{\bar{\Delta}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}} = 0,$$

Thus given $\epsilon > 0$ there exists $N_2 \in \mathbf{N}$, such that if $m, n \geq N_2$ then

$$\frac{\bar{\Delta}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}} < 3\epsilon.$$

Condition (3) grant us the following

$$\frac{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}} < 1+\epsilon$$

and

$$\frac{\sup_{k,l\geq m,n}\sum_{i,j>M_1,N_1} a_{k,l,i,j}}{\sup_{k,l\geq m,n}\sum_{r,s=,0,0}^{\infty,\infty} a_{k,l,i_r,j_s}} < 1+\epsilon.$$

Thus the above inequalities grant us the following

$$\begin{aligned}\text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}x_{i_r,j_s}}{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}y_{i,j}} &\geq \frac{1}{\bar{B}_1+\bar{B}_2+\bar{B}_3} - \frac{\epsilon}{\delta} \\ &\geq \frac{1}{\frac{M_1 3\epsilon}{\delta} + \frac{\epsilon}{\delta}(1+\epsilon)+1+\epsilon} - \frac{\epsilon}{\delta}.\end{aligned}$$

Since ϵ is arbitrary we have that

$$\text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}x_{i_r,j_s}}{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}y_{i,j}} \geq 1.$$

Using a similar argument we can establish the following

$$\text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}x_{i_r,j_s}}{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}y_{i,j}} \leq 1.$$

Hence

$$\text{P-}\lim_{m,n} \frac{\sup_{k,l\geq m,n}\sum_{r,s=0,0}^{\infty,\infty} a_{k,l,i_r,j_s}x_{i_r,j_s}}{\sup_{k,l\geq m,n}\sum_{i,j=0,0}^{\infty,\infty} a_{k,l,i,j}y_{i,j}} = 1.$$

This completes the proof. □

4. Conclusion

This paper extends the concept of rate of P-convergence to asymptotic equivalence. This is accomplished by presenting and proving two Robison-Hamilton type characterizations of asymptotically equivalent sequences. That is, we present necessary and sufficient conditions on the entries of a four-dimensional matrix to preserve asymptotic equivalence under a four-dimensional matrix transformation.

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