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ON WEAKLY RICCI SYMMETRIC MANIFOLDS ADMITTING A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract

The object of the present paper is to investigate the properties of a weakly Ricci symmetric manifold admitting a semi-symmetric metric connection.

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1. Introduction

By a triple (M, g, T), we mean (M, g) is a Riemannian manifold with a tensor T defined of M which is a smooth section of the tensor bundle (T^*M). Also, ∇ and ∇^* denote the Levi-Civita connection and the linear connection on the manifold (M, g, T) for the torsion tensor T, respectively. Here and below, unless otherwise stated, the symbols X, Y and Z stand for arbitrary smooth vector fields on M.

Semi-symmetric metric connections plays an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one defined point, the north pole, then this displacement is semi-symmetric [17]. During the mathematical congress in Moscow in 1934, one evening mathematicians invented the "Moscow displacement". The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the streets always facing the Kremlin, then this displacement is semi-symmetric and metric [17, 19].

In [7], Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold. Then, Hayden introduced the idea of metric connection with torsion on a Riemannian manifold [8]. In 1970, K. Yano considered a semi-symmetric metric connection on a Riemannian manifold which was published in [24]. T. Imai found some properties of a Riemannian manifold and a hypersurface of

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a Riemannian manifold with a semi-symmetric metric-connection [9]. Z. Nakao studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection [13]. Riemannian manifolds with a semi-symmetric metric connection satisfying some special conditions are studied by some authors [12, 14, 3].

The concept of a semi-symmetric manifold has been applied to the Kenmotsu manifold [15], the almost contact manifold [4] and the Sasakian manifold [16].

Let (M_n, g) be an n-dimensional differentiable manifold of class C^∞ with the metric tensor g. A smooth linear connection ∇^* on (M_n, g) is said to be *semi-symmetric* if its torsion tensor T of ∇^* satisfies the relation

$$(1.1) \qquad T(X,Y) = w(Y)X - w(X)Y$$

for any vector fields X and Y on M_n and w is a 1-form associated with the torsion tensor T of the connection ∇^* given by $w(X) = g(X, \rho)$.

If ∇^* further satisfies the condition $\nabla^* g = 0$, then ∇^* is called a *semi-symmetric metric connection* [24].

The relation between the semi-symmetric metric connection ∇^* and the Riemannian connection ∇ of (M_n, g) is given by [24]

$$(1.2) \qquad \nabla^*_X Y = \nabla_X Y + w(Y)X - g(X,Y)\rho$$

for any vector field X, Y on M. In particular, if the 1-form ω vanishes identically then a semi-symmetric metric connection reduces to the Riemannian connection. We denote by $R^*(X,Y)Z$ and $R(X,Y)Z$ the curvature tensor of ∇^* and ∇, respectively. Then, we have [24]

$$(1.3) \qquad R^*(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)LX + g(X,Z)LY,$$

where α is the tensor field of type $(0, 2)$ given by

$$(1.4) \qquad \alpha(X,Y) = g(LX,Y) = (\nabla_X w)(Y) - w(X)w(Y) + \frac{1}{2}w(\rho)g(X,Y)$$

for any vector fields X and Y.

Using (1.3), we get

$$(1.5) \qquad S^*(Y,Z) = S(Y,Z) - (n-2)\alpha(Y,Z) - \theta g(Y,Z),$$

where S^* and S denote respectively the Ricci tensor with respect to ∇^* and ∇, $\theta = g^{ih}\alpha_{ih} = trace\alpha$. The tensor α of type (0,2) given in (1.4) is not symmetric in general and hence from (1.5) it follows that the Ricci tensor S^* is not symmetric. If the 1-form ω is closed, it can be easily seen that S^* is symmetric with the help of (1.4).

If R^* and R denote the scalar curvatures with respect to the linear connection ∇^* and the Levi-Civita connection ∇, respectively; then, they are related by the following form:

$$(1.6) \qquad R^* = R - 2(n-1)\theta.$$

In 1993, Tamassy and Binh [21] introduced the notion of weakly Ricci symmetric manifolds, later studied by many authors, and the existence of such a manifold is given by Shaikh and Jana [10]. Again, Shaikh *et al.* [18] proved the existence of weakly Ricci symmetric manifold admitting a semi-symmmetric metric connection by means of several examples. A non-flat Riemannian manifold (M_n, g) $(n > 2)$ is called *weakly Ricci symmetric* if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(1.7) \qquad (\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(X,Y),$$

where A, B, D are three non-zero 1-forms and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such a manifold is said t be weakly-Ricci symmetric, and a n-dimensional manifold of this kind is denoted by $(WRS)_n$.

In the study of a $(WRS)_n$ an important role is played by the 1-form $\delta(X)$ defined by $\delta(X) = B(X) - D(X)$ for all X such that $\delta(X) \neq 0$.

Now, we can state the following Lemma which will be used in our subsequent work:

1.1. Lemma. [5] *In a $(WRS)_n$ with a defined metric, if $\delta(X) \neq 0$, then the scalar curvature R is non-zero and the Ricci tensor is of the form*

$$S(X,Y) = RT(X)T(Y), T(X) = \frac{\delta(X)}{\| \delta(X) \|}. \tag{1.8}$$

□

A Riemannian manifold is called an *Einstein manifold* if its Ricci tensor is a constant multiple of the metric tensor.

2. Sectional curvatures of a Riemannian manifold having semi-symmetric metric connection

Let $P(x^k)$ be any point of $M_n(\nabla^*, g)$ and let us denote by X^α, Y^α the components of two arbitrary linearly independent vectors $X, Y \in T_p(M_n)$. These vectors determine a two-dimensional subspace (plane) π of $T_p(M_n)$.

The scalar

$$K^*(\pi) = \frac{R^*_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu}{(g_{\beta\lambda} g_{\alpha\mu} - g_{\alpha\lambda} g_{\beta\mu}) X^\alpha Y^\beta X^\lambda Y^\mu} \tag{2.1}$$

is called the *sectional curvature of $M_n(\nabla^*, g)$ at P with respect to the plane π.*

Assume that at any point $P \in M_n(\nabla^*, g)$, the sectional curvature is the same for all planes in $T_p(M)$. The case of a 2-dimensional Riemannian manifold having semi-symmetric metric connection need not to be considered, since it has only one plane at each point.

Since the sectional curvature at the point $P \in M_n(\nabla^*, g)$ is the same for all planes in $T_p(M_n)$, by using (2.1), we have

$$R^*_{\alpha\beta\lambda\mu} = K^*(\pi)(g_{\beta\lambda} g_{\alpha\mu} - g_{\alpha\lambda} g_{\beta\mu}). \tag{2.2}$$

Multiplying the relation (2.2) by $g^{\alpha\mu}$ and summing for α and μ, we get

$$R^*_{\lambda\beta} = K^*(\pi)(n-1) g_{\lambda\beta}. \tag{2.3}$$

Transvecting (2.3) by $g^{\lambda\beta}$, we get

$$R^* = n(n-1)K^*(\pi). \tag{2.4}$$

(2.3) can be rewritten in the following form

$$R^*_{(\lambda\beta)} = (n-1)K^*(\pi) g_{\lambda\beta}, \tag{2.5}$$

where

$$R^*_{(\lambda\beta)} = \frac{R^*_{\lambda\beta} + R^*_{\beta\lambda}}{2}. \tag{2.6}$$

From (1.5), we have

$$R^*_{[\lambda\beta]} = (2-n)\alpha_{[\lambda\beta]}. \tag{2.7}$$

From (2.3) and (2.7), it follows that

$$R^*_{[\lambda\beta]} = 0 \tag{2.8}$$

and

$$\nabla_{[\lambda} w_{\beta]} = 0. \tag{2.9}$$

From (2.9), it is clear that 1-form w is closed.

We have from (1.5) and (1.6) that

$$\alpha_{ij} = -\lambda_{ij} - \frac{R^*_{ij}}{n-2} + \frac{R^* g_{ij}}{2(n-1)(n-2)}, \tag{2.10}$$

where

$$\lambda_{ij} = -\frac{1}{n-2} R_{ij} + \frac{1}{2(n-1)(n-2)} R g_{ij}. \tag{2.11}$$

From (2.3) and (2.4), we have $R^*_{ih} = \frac{R^* g_{ih}}{n}$. Then, by using (2.10), we find

$$\alpha_{ij} = -\lambda_{ij} - \frac{R^* g_{ij}}{2n(n-1)}. \tag{2.12}$$

By the aid of the equations (1.3), (2.4) and (2.12), we get

$$R^*_{ijkh} = C_{ijkh} + K^*(\pi)(g_{ih} g_{jk} - g_{ik} g_{jh}). \tag{2.13}$$

By using (2.2) and (2.13), we can easily see that this space is conformally flat.

In reference [9], by using a different method, it has been found that "A Riemannian manifold admitting a semi-symmetric metric connection with closed 1-form π constant curvature is conformally flat".

Since this manifold is conformally flat, we then have

$$\begin{aligned} R_{ijkh} = {} & \frac{1}{(n-2)} (g_{jk} R_{ih} - g_{ik} R_{jh} + g_{ih} R_{jk} - g_{jh} R_{ik}) \\ & - \frac{1}{(n-1)(n-2)} R (g_{jk} g_{ih} - g_{jh} g_{ik}). \end{aligned} \tag{2.14}$$

Thus from (1.8) and (2.14), we have

$$R_{ijkl} = b(-g_{jl} T_i T_k + g_{jk} T_i T_l - g_{ik} T_j T_l + g_{il} T_j T_k) + a(g_{il} g_{jk} - g_{jl} g_{ik}), \tag{2.15}$$

where $a = \frac{-R}{(n-1)(n-2)}$ and $b = \frac{R}{(n-2)}$.

D. Smaranda, [20], calls a Riemannian manifold whose curvature tensor satisfies (2.15), a manifold of almost constant curvature.

The notion of "almost constant curvature" is the same notion as "quasi-constant curvature" introduced by Chen and Yano in 1972 [1]. Later, A. L. Mocanu [11] pointed out that both the notions are the same. In addition, G. Vranceanu [22] defined the notion of almost constant curvature by the same expression (2.15).

In this case, we have the following theorem:

2.1. Theorem. *If a $(WRS)_n$ of definite metric admits a semi-symmetric metric connection with constant sectional curvature, then it is a manifold with quasi-constant curvature.* □

The Weyl conformal curvature tensor C_{ijkh} which is conformally invariant in an n-dimensional Riemannian manifold is defined by

$$\begin{aligned} C_{ijkh} = R_{ijkh} & - \frac{1}{(n-2)} (g_{jk} R_{ih} - g_{ik} R_{jh} + g_{ih} R_{jk} - g_{jh} R_{ik}) \\ & + \frac{R}{(n-1)(n-2)} (g_{jk} g_{ih} - g_{jh} g_{ik}) \end{aligned} \tag{2.16}$$

and in Riemannian geometry, the Schouten tensor is a second order tensor which is introduced by J. A. Schouten as follows:

$$H_{ij} = \frac{R_{ij}}{(n-2)} - \frac{Rg_{ij}}{2(n-1)(n-2)}. \tag{2.17}$$

On a conformally flat Riemannian manifold M, if there exist two functions γ and β such that $\gamma > 0$ and

$$H_{ij} = -\frac{\gamma^2}{2} g_{ij} + \beta X_i X_j \tag{2.18}$$

then M is called a *special conformally flat space* [2]. In particular, if β is a function of γ, then a special conformally flat M is called a *subprojective manifold* [2].

From (1.8), (2.17) and (2.18), we have

$$H_{ij} = -\frac{\gamma^2}{2} g_{ij} + \beta T_i T_j \tag{2.19}$$

and

$$\gamma^2 = \frac{\beta}{(n-1)};\ \gamma^2 = \frac{R}{(n-1)(n-2)};\ \beta = \frac{R}{n-2}. \tag{2.20}$$

If $R = 0$ in $(WRS)_n$ of definite metric admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by (2.2), then this manifold is flat. Since $R \neq 0$, γ is not zero. Suppose that $R > 0$ and hence that γ may be taken as positive. From (2.19), we conclude that the $(WRS)_n$ of definite metric under consideration is a special conformally flat manifold. Moreover, from (2.20) we find that β is a function of γ, it follows that the manifold under consideration is a particular kind of a special conformally flat manifold, namely a subprojective manifold. We can therefore state the following theorem:

2.2. Theorem. *If a $(WRS)_n$ of definite metric with $R > 0$ admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by (2.2), then the manifold is a subprojective manifold in the sense of Kagan.* □

It is known from a theorem of Chen's and Yano's paper [2] that every simply connected special conformally flat manifold can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface. This leads to the following result:

2.3. Theorem. *If a simply connected $(WRS)_n$ of definite metric with $R > 0$ admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by (2.2), then the manifold can be isometrically immersed in a Euclidean space as a hypersurface.* □

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [25]. According to them a quasi-conformal curvature tensor W_{ijkh} is defined by

$$\begin{aligned} W_{ijkh} &= aR_{ijkh} + b(g_{jk}R_{ih} - g_{ik}R_{jh} + g_{ih}R_{jk} - g_{jh}R_{ik}) \\ &\quad - \frac{R}{n}\left(\frac{a}{(n-1)} + 2b\right)(g_{jk}g_{ih} - g_{jh}g_{ik}) \end{aligned} \tag{2.21}$$

where a and b are constants. If $a = 1$ and $b = \frac{-1}{(n-2)}$, then (2.21) takes the form $W_{ijkh} = C_{ijkh}$, thus the conformal curvature tensor C is a particular case of the tensor W_{ijkh}. For this reason, W_{ijkh} is called a *quasi-conformal curvature tensor*. A manifold with dimension $n > 3$ is called quasi-conformally flat if the quasi conformal curvature tensor $W_{ijkh} = 0$ [6].

In the reference [10], we have:

2.4. Theorem. *In a quasi-conformally flat* $(WRS)_n$ *(*$n > 3$*) with* $2a-(n-1)(n-4)b \neq 0$ *and* $a+b \neq 0$*, the vector field* ρ *defined by* $g(X,\rho) = T(X)$ *is a unit proper concircular vector field.* □

By using Theorem 2.4, we can state the following theorem:

2.5. Theorem. *If a* $(WRS)_n$ *of definite metric (*$R \neq$ *constant) admitting a semi-symmetric metric connection whose sectional curvature* $K^*(\pi)$ *is given by (2.2), then this manifold has a proper concircular vector field.* □

In [23], K. Yano proved that in order for a Riemannian manifold to admit a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written as

(2.22) $$ds^2 = (dx^1)^2 + c^q g^*_{\alpha\beta} dx^\alpha dx^\beta$$

where

(2.23) $$g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\nu)$$

are the functions of x^ν $(\alpha, \beta, \nu = 2, 3, \ldots, n)$ and $q = q(x^1) \neq$ constant, is a function of x^1 only.

Thus, we have the following theorem:

2.6. Theorem. *If a* $(WRS)_n$ *of definite metric (*$R \neq$ *constant) admitting a semi-symmetric metric connection whose sectional curvature* $K^*(\pi)$ *is given by (2.2), then the first fundamental form of this manifold has the form*

(2.24) $$ds^2 = (dx^1)^2 + c^q g^*_{\alpha\beta} dx^\alpha dx^\beta$$

where $\alpha, \beta, \gamma = 2, \ldots, n$*,* $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$*,* $q = q(x^1) \neq$ *constant.* □

In the reference [10], we have:

2.7. Theorem. *A non-Einstein quasi-conformally flat* $(WRS)_n$ *(*$n > 3$*) with* $a+b \neq 0$ *and* $2a-(n-1)(n-4)b \neq 0$ *can be expressed as a warped product* $IX_{e^q}M^*$ *where* (M^*, g^*) *is an (n-1)-dimensional Riemannian manifold.* □

Theorem 2.7 leads to the following theorem:

2.8. Theorem. *If a* $(WRS)_n$ *of definite metric admits a semi-symmetric metric connection with constant sectional curvature then this manifold can be expressed as a warped product* $IX_{e^q}M^*$ *where* M^* *is an Einstein manifold.* □

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