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SOME DISTANCE-BASED TOPOLOGICAL INDICES OF A NON-COMMUTING GRAPH

M. Mirzargar* and A. R. Ashrafi*†

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Abstract

Let G be a non-abelian group and let $Z(G)$ be the center of G. The noncommuting graph of G, $\Gamma(G)$, is a graph with vertex set $G \setminus Z(G)$ and two distinct vertices x and y are adjacent if and only if $xy \neq yx$. In this paper the Hyper-Wiener, Schultz, Gutman, eccentric connectivity and Zagreb group indices of this graph are computed.

Keywords: Non-commuting graph, AC-group, Wiener index, Hyper-Wiener index, Schultz index, Zagreb index.

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1. Introduction

In this section we recall some definitions that will be used in the paper. Let G be a simple graph without directed and multiple edges and without loops, the vertex and edge-sets of which are represented by $V(G)$ and $E(G)$, respectively. The degree of a vertex v is denoted by $\deg_G(v)$. Suppose Graph is the collection of all graphs. A mapping $Top : \text{Graph} \longrightarrow \mathbb{R}$ is called a topological index, if $G \cong H$ implies that $Top(G) = Top(H)$. If $x, y \in V(G)$ then the distance $d(x, y)$ between x and y is defined as the length of a minimum path connecting x and y. The Wiener index is the first and most studied of the distance-based topological indices, both from a theoretical point of view and applications [26]. It is equal to the sum of distances between all pairs of vertices of the respective graph.

The hyper-Wiener index of acyclic graphs was introduced by Milan Randić in 1993. Then Klein *et al.* [17], generalized Randić's definition for all connected graphs, as a generalization of the Wiener index. It is defined as

$$WW(G) = \frac{1}{2}W(G) + \tfrac{1}{2}\textstyle\sum_{\{u,v\}\subseteq V(G)} d(u,v)^2.$$

*Department of Mathematics, Statistics and Computer Science, Faculty of Science, University of Kashan, Kashan 87317-51167, I. R. Iran.
E-mail: (M. Mirzargar) m.mirzargar@gmail.com (A. R. Ashrafi) ashrafi@kashanu.ac.ir
†Corresponding Author.

The eccentric connectivity index of G, $\xi^c(G)$, was proposed by Sharma, Goswami and Madan [25]. It is defined as $\xi^c(G) = \sum_{v \in V(G)} \deg_G(v)\varepsilon(v)$, where $\varepsilon(v)$ is the largest distance between v and any other vertex of G. The radius and diameter of G are defined as the minimum and maximum eccentricity among vertices of G, respectively. We encourage the interested reader to consult papers [3, 4, 8, 18, 23] for the chemical meaning and [29] for mathematical properties of this new topological index.

Suppose G is a graph. The Zagreb indices of G have been introduced more than thirty years ago by Gutman and Trinajestić [10]. They are defined as:

$$M_1(G) = \sum_{v \in V(G)} (\deg_G(v))^2; \; M_2(G) = \sum_{uv \in E(G)} \deg_G(u) \deg_G(v).$$

We refer to [2, 12, 16, 30, 28, 31] for historical background, computational techniques and mathematical properties of the Zagreb indices. The Schultz index of G, $\mathrm{MTI}(G)$, was introduced by Schultz in 1989, as the molecular topological index [24]. It is defined by

$$\mathrm{MTI}(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)[\deg(u) + \deg(v)].$$

To the best of our knowledge, the first paper on mathematical properties of this graph invariant was published in 1994 by Ivan Gutman [9], who introduced a modification of this index. This modification is named the Gutman index [9, 13] and defined by

$$\mathrm{Gut}(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)[\deg(u) \times \deg(v)].$$

We now assume that $w \in V(G)$ and $e = uv, f = ab \in E(G)$. Define $n_u(e)$ to be the number of vertices lying closer to u than v and $n_v(e)$ is defined analogously. The Szeged index of G is defined as follows:

$$Sz(G) = \sum_{e = uv \in E(G)} n_u(e) n_v(e).$$

Notice that in computing vertex and edge Szeged indices of G, vertices equidistant from both ends of the edge $e = uv$ are not counted [11].

Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate a graph $\Gamma(G)$ with G as follows: Take $G \setminus Z(G)$ as the vertices of $\Gamma(G)$ and join two distinct vertices x and y, whenever $xy \neq yx$. The graph $\Gamma(G)$ is called the non-commuting graph of G [20]. This is a graph with exactly $|G| - |Z(G)|$ vertices and $\frac{|G|}{2}(|G| - k(G))$ edges, where $k(G)$ denotes the number of conjugacy classes of G. The complement of a graph Γ is a graph $\bar{\Gamma}$ on the same vertices such that two vertices of $\bar{\Gamma}$ are adjacent if and only if they are not adjacent in Γ. The complement graph $\bar{\Gamma}(G)$ is called the commuting graph of G. The best paper in this topic is a paper by Abdollahi, Akbari and Maimani [1].

In the next section, the non-commuting graphs of some well-known finite groups are considered. In Section 3, the commuting and non-commuting graphs of finite groups are investigated in general. We first consider a finite group G and present a condition under which it is possible to decompose the commuting graph $\bar{\Gamma}(G)$ into complete subgraphs. Then we focus on the problem of computing distance-based topological indices of these graphs. Some open questions are also presented.

Throughout this paper our notation is standard and taken mainly from the standard book of graph theory and references [6, 15, 27]. The complete graph on n vertices is denoted by K_n and its complement by $\emptyset_n$. For two graphs with disjoint vertex sets V_1 and V_2 their union is the graph Γ for which $V(\Gamma) = V_1 \cup V_2$ and $E(\Gamma) = E_1 \cup E_2$.

2. Examples

Throughout this paper all groups are presumed to be finite and non-abelian. The aim of this section is to compute the Wiener and Szeged indices of the groups D_{2n}, SD_{2^n}, T_{4n}, U_{6n}, V_{8n}, $A(n,\theta)$ and a non-abelian p-group P of order p^{2r} which will be defined later, $r \geq 1$ [14]. These groups are defined as follows:

$$\begin{aligned}
D_{2n} &= \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\
SD_{2^n} &= \langle a, b \mid a^{2^n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\
T_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\
U_{6n} &= \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle, \\
V_{8n} &= \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle, \\
A(n,\theta) &= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a\theta & 1 \end{bmatrix} \ : \ a, b \in F \right\},
\end{aligned}$$

where $F = GF(2^n)$ is a finite field of order 2^n and θ is an automorphism of F. We start with dihedral groups.

2.1. Example. Consider the dihedral group D_{2n}.

If n is odd then $|Z(D_{2n})| = 1$, $C_{D_{2n}}(a^i) = \langle a \rangle$, $1 \leq i \leq n-1$, and $C_{D_{2n}}(a^i b) = \langle a^i b \rangle$, $0 \leq i \leq n-1$. If n is even then $Z(D_{2n}) = \langle a^{\frac{n}{2}} \rangle$, $C_{D_{2n}}(a^i) = \langle a \rangle$, $1 \leq i \neq \frac{n}{2} \leq n-1$, and $C_{D_{2n}}(a^i b) = \langle a^i b, a^{\frac{n}{2}} \rangle$, $0 \leq i \leq n-1$. By the above calculations the complement of the non-commuting graph $\Gamma(D_{2n})$ is:

i) The union of complete graph K_{n-1} and empty graph $\emptyset_n$, when n is odd;
ii) The union of complete graph K_{n-2} and $\frac{n}{2}$ copies of K_2, when n is even.

In [5], Azad and Eliasi proved that for a finite non-abelian group G,

$$W(\Gamma(G)) = \frac{1}{2}[(|G| - |Z(G)|)(|G| - 2|Z(G)| - 2) + |G|(k(G) - |Z(G)|)],$$

where $k(G)$ denotes the number of conjugacy classes in G. By this formula,

$$W(\Gamma(D_{2n})) = \begin{cases} \frac{5}{2}n^2 - 7n + 6 & n \text{ is even}, \\ \frac{5}{2}n^2 - \frac{9}{2}n + 2 & n \text{ is odd}. \end{cases}$$

The Szeged index of this graph can be calculated directly from the non-commuting graph $W(\Gamma(D_{2n}))$. We have:

$$\text{Sz}(\Gamma(D_{2n})) = \begin{cases} 2n^3 - 6n^2 + 4n & n \text{ is even}, \\ n^3 - \frac{3}{2}n^2 + \frac{1}{2}n & n \text{ is odd}. \end{cases}$$

2.2. Example. In this example the Wiener and Szeged indices of the dicyclic group T_{4n} is computed. Since $Z(T_{4n}) = \langle a^n \rangle$, $C_{T_{4n}}(a^i) = \langle a \rangle$, $1 \leq i \neq n \leq 2n-1$, and $C_{T_{4n}}(a^i b) = \langle a^i b, a^n \rangle$, $0 \leq i \leq 2n-1$. By these calculations, the complement of $\Gamma(T_{4n})$ is the union of the complete graph K_{2n-2} and n copies of K_2. On the other hand, $\Gamma(T_{4n}) \cong \Gamma(D_{4n})$. Therefore,

$$W(\Gamma(T_{4n})) = \begin{cases} 10n^2 - 14n + 6 & n\text{is even}, \\ 10n^2 - 9n + 2 & n\text{is odd}, \end{cases}$$

$$\text{Sz}(\Gamma(T_{4n})) = \begin{cases} 16n^3 - 24n^2 + 8n & n \text{ is even}, \\ 8n^3 - 6n^2 + n & n \text{ is odd}. \end{cases}$$

In the following two examples, the Szeged and Wiener indices of $\Gamma(U_{6n})$ and $\Gamma(V_{8n})$ are computed. These groups were introduced by James and Liebeck in their famous book [15]. In this book, the conjugacy classes and character table of V_{8n} is computed, when n is odd. For the case of even n, the conjugacy classes and character table of V_{8n} was computed by Darafsheh and Poursalavati [7].

2.3. Example. Consider the group U_{6n} of order $6n$. Clearly, $Z(U_{6n}) = \langle a^2 \rangle$, $C_{U_{6n}}(a^{2r+1}) = \langle a \rangle$, $C_{U_{6n}}(a^{2r+1}b) = \langle a^2 \rangle \cdot \langle \{a^{2s+1}b \mid 0 \leq s \leq n-1\} \rangle$, $C_{U_{6n}}(a^{2r+1}b^2) = \langle a^2 \rangle \cdot \langle \{a^{2s+1}b^2 \mid 0 \leq s \leq n-1\} \rangle$ and $C_{U_{6n}}(a^{2r}b) = \langle a^2 \rangle \cdot \langle \{a^{2s}b, a^{2s}b^2 \mid 0 \leq s \leq n-1\}$, $0 \leq r \leq n-1$. By these calculations, the complement of $\Gamma(U_{6n})$ is the union of the complete graph K_{2n} and 3 copies of K_n. Therefore,

$$W(\Gamma(U_{6n})) = 16n^2 - 5n \text{ and } \mathrm{Sz}(\Gamma(U_{6n})) = 15n^4.$$

2.4. Example. To compute the Szeged and Wiener indices of the non-commuting graph $\Gamma(V_{8n})$, we distinguish two cases that n is odd or even.

We first assume that n is odd. Then $Z(V_{8n}) = \langle b^2 \rangle$, $C_{V_{8n}}(a^r) = C_{V_{8n}}(a^r b^2) = \langle a \rangle \cdot Z(V_{8n})$, $C_{V_{8n}}(a^{2r}b) = C_{V_{8n}}(a^{2r}b^3) = \langle a^{2r}b \rangle$ and $C_{V_{8n}}(a^{2r+1}b) = C_{V_{8n}}(a^{2r+1}b^3) = \langle a^{2r+1}b, b^2 \rangle$. Therefore, the complement of $\Gamma(V_{8n})$ is a union of the complete graph on $4n-2$ vertices and a matching of size $2n$. So, $W(\Gamma(V_{8n})) = 40n^2 - 28n + 6$ and $Sz(\Gamma(V_{8n})) = 64n^2 - 32n$.

Next we assume that n is even. Then $Z(V_{8n}) = \langle b^2, a^n \rangle$, $C_{V_{8n}}(a^r) = C_{V_{8n}}(a^r b^2) = \langle a \rangle \cdot Z(V_{8n})$, $C_{V_{8n}}(a^{2r}b) = C_{V_{8n}}(a^{2r}b^3) = \langle a^{2r}b \rangle \cdot Z(V_{8n})$ and $C_{V_{8n}}(a^{2r+1}b) = C_{V_{8n}}(a^{2r+1}b^3) = \langle a^{2r+1}b, b^2, a^n \rangle$. Therefore, the complement of $\Gamma(V_{8n})$ is the union of a complete graph on $4n-4$ vertices and n copies of the complete graph K_4. Hence in this case, we have: $W(\Gamma(V_{8n})) = 40n^2 - 48n + 20$ and $\mathrm{Sz}(\Gamma(V_{8n})) = 128n(2n^2 + n + 1)$.

Following Issacs [14], we assume that $F = GF(p)$ and $E = GF(p^r)$ are finite fields of orders p and p^r, respectively. Suppose $R = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 \mid \alpha_i \in E\}$, where $x^3 = 0$ and for each $\alpha \in E$, $x\alpha = \alpha^p x$. Then $P = P_2(p, r) = \{1 + \alpha_1 x + \alpha_2 x^2 \mid \alpha_i \in E\}$ is a p-subgroup of the group $R^\times$ of units in R, of order p^{2r}. In the aforementioned paper it is proved that if p is a prime and $r > 1$ is a positive integer then the p-group P is non-abelian of order p^{2r} such that: (1) $|Z(P)| = p^r$; (2) $P/Z(P)$ is an elementary abelian p-group; (3) for every non-central element x of P, $C_P(x) = Z(P)\langle x \rangle$, see also [21, 19] for details.

2.5. Example. In this example the Wiener and Szeged indices of P are computed. We first calculate $k(P)$. To do this, we assume that x is a non-central element of P. Then $|x^P| = |P : C_P(x)| = |\frac{P}{Z(P)\langle x \rangle}| = \frac{p^r |Z(P) \cap \langle x \rangle|}{O(x)}$. Suppose $\langle xZ(P) \rangle$ is an arbitrary subgroup of $\frac{P}{Z(P)}$. Then $\langle x \rangle \cap Z(P) = \langle x^p \rangle$. Thus $|\langle x^p \rangle| = \frac{O(x)}{p}$ and so $|x^P| = p^{r-1}$. To calculate the number of conjugacy classes, we notice that P has exactly p^r one element conjugacy classes and $\frac{p^{2r} - p^r}{p^{r-1}} = p(p^r - 1)$ conjugacy classes of size greater than one. Therefore, $k(P) = p^{r+1} + p^r - p$, $W(P) = \frac{1}{2}(p^r - 1)[p^r(p^{2r} - 2p^r - 2) + p^{2r+1}]$.

2.6. Example. Consider the semi-dihedral group SD_{2^n}. It is obvious that $Z(SD_{2^n}) = \langle a^{2^{n-2}} \rangle$, $C_{SD_{2^n}}(a^i) = \langle a \rangle$, $1 \leq i \neq 2^{n-2} \leq 2^{n-1} - 1$, $C_{SD_{2^n}}(a^i b) = \{e, a^i b, a^{2^{n-2}}, a^{i + 2^{n-2}} b\}$, $1 \leq i \leq 2^{n-1} - 1$. From these calculations the complement of $\Gamma(SD_{2^n})$ is a union of a complete graph $K_{2^{n-1}-2}$ and 2^{n-2} copies of K_2. The Wiener and Szeged indices of this graph is computed as follows:

$$W(\Gamma(SD_{2^n})) = 2^{2n-1} + 2^{2n-3} - 7 \times 2^{n-1} + 6,$$
$$\mathrm{Sz}(\Gamma(SD_{2^n})) = 2^n(2^{n-1} - 2)(2^{n-1} - 1).$$

2.7. Example. The group $A(n,\theta)$ is a group of order 2^{2n} with the unit element $U(0,0)$. It is easy to see that if θ is non-trivial then $Z(A(n,\theta)) = \{U(0,b) \mid b \in F\}$. To simplify our argument we assume that θ is the Frobenius automorphism of F, $\theta : x \mapsto x^2$.

Suppose that $a \neq 0$ and $U(a,b) \in A(n,\theta)$. Then $C_{A(n,\theta)}(U(a,b)) = \{U(r,s) \mid r = 0 \text{ or } a\}$. Therefore, $C_{A(n,\theta)}(U(a_0,b_0)) = C_{A(n,\theta)}(U(a_1,b_1))$ if and only if $a_0 = a_1$. Therefore, the complement of $\Gamma(A(n,\theta))$ contains $2^n - 1$ copies of the complete graph K_{2^n}. The Wiener and Szeged indices of this graph is $W(\Gamma(A(n,\theta)) = 2^{4n-1} - 5 \times 2^{3n-2} + 2^n - 2^{2n-2}$ and $\mathrm{Sz}(A(n,\theta)) = 2^{5n-2}(2^{n-1} - 1)$.

3. Main results

We say a group G has abelian centralizers, if for each non-central element $x \in G$, $C_G(x)$ is abelian. Such a group is called an AC-group. By [1, Lemma 3.5], for an arbitrary field F, the group $GL(2,F)$ is an AC-group. In [22] Rocke proved that the following are equivalent: (a) G has abelian centralizers; (b) if $xy = yx$, then $C_G(x) = C_G(y)$ whenever $x, y \notin Z(G)$; (c) if $xy = yx$ and $xz = zx$, then $yz = zy$ whenever $x \notin Z(G)$; (d) if U and B are subgroups of G and $Z(G) < C_G(U) \leq C_G(B) < G$ then $C_G(U) = C_G(B)$. In the following lemma we apply the latter result to obtain the general case of Examples 2.1-2.7.

3.1. Proposition. *Let G be a group. Then the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs if and only if G is an AC-group.*

Proof. Suppose the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs. Then by definition of $\bar{\Gamma}(G)$, all proper element centralizers are abelian and so G has abelian centralizers.

Conversely, suppose that G is a group with abelian centralizers. We prove that the intersection of two proper element centralizers is the center of G. To do this, we assume that $y, z \notin Z(G)$, $C_G(y) \neq C_G(z)$ and x is a non-central element of $C_G(y) \cap C_G(z)$. Thus $xy = yx$ and $xz = zx$ and by condition (c) of [22, Lemma 3.2], $yz = zy$. Hence by part (b) of the aforementioned theorem, $C_G(y) = C_G(z)$ which is impossible. So $C_G(y) \cap C_G(z) = Z(G)$. Therefore, if G is a group with abelian centralizers then the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs. □

We notice that there is no finite group G such that $\bar{\Gamma}(G)$ is a complete graph or a union of two complete graphs. But it is possible to find a finite group G such that $\bar{\Gamma}(G)$ is a union of three complete graphs. By Example 2.1, the dihedral group of order 8 is such an example and it can be easily seen that $\bar{\Gamma}(D_8)$ is a union of three complete graphs K_2. It is far from true that each graph has this property. To verify this, it is enough to note that the commuting graph $\bar{\Gamma}(S_4)$ is not a union of complete graphs. In the following lemma a graph theoretical equivalence for a group G to be AC-group or equivalently for $\bar{\Gamma}(G)$ to be a union of complete graphs is obtained.

3.2. Proposition. *Let G be a group. Then the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs K_p, p is prime, if and only if $p = 2$ and $G \cong D_8$ or Q_8.*

Proof. Suppose the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs K_p, p is an odd prime. We also assume that a is a non-central element of G. By definition, $|C_G(a)| = |Z(G)| + p$ and so $|Z(G)| \in \{1, p\}$.

We first assume that $|Z(G)| = 1$. Then $|C_G(a)| = p+1$ and $|a^G| = \frac{|G|}{p+1}$. If there are t conjugacy classes of non-central elements then $\frac{t|G|}{p+1} + |Z(G)| = |G|$ and so $(p+1-t)|G| = p+1 = |C_G(a)|$, which is impossible. If $|Z(G)| = p$ then for each non-central element $a \in G$, $|C_G(a)| = 2p$. This implies that G is a group of order $2p$ or $2p^2$. In the first case, G

is a dihedral group of order $2p$ and by Example 2.1, $\bar{\Gamma}(G)$ is a union of a complete graphs K_{p-1} and p isolated vertices, contradicting our assumption. Thus $|G| = 2p^2$. From elementary group theory we know that up to isomorphism there are three non-abelian groups of order $2p^2$ for an odd prime p. These are as follows:

$$G_1 = \langle a, b \mid a^{p^2} = b^2 = 1; b^{-1}ab = a^{-1}\rangle,$$
$$G_2 = \langle a, b, c \mid a^p = b^p = c^2 = 1; c^{-1}ac = a^{-1}; c^{-1}bc = b^{-1}; a^{-1}b^{-1}ab = 1\rangle,$$
$$G_3 = \langle a, b, c \mid a^p = b^p = c^2 = 1; a^{-1}b^{-1}ab = a^{-1}c^{-1}ac = 1; c^{-1}bc = b^{-1}\rangle.$$

Since G cannot be a dihedral group, $G \cong G_2$ or G_3. By a simple calculation, we can see that G_2 is centerless and the element centralizers of the non-central elements are $\langle a, b\rangle$ and $\langle a^ib^jc\rangle$, $1 \le i, j \le p-1$, which is impossible. Finally, $Z(G_3) = \langle a\rangle$, $C_{G_3}(a^ib^j) = \langle a, b\rangle$ and $C_G(a^ib^jc) = Z(G) \cdot \langle b^jc\rangle$, $1 \le i, j \le p-1$. Again G has at least two element centralizers of different orders, which leads to our final contradiction. Therefore, $p = 2$ and $G \cong D_8$ or Q_8. □

Suppose that G is a finite group G such that the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs K_n, n is not prime. Then $|C_G(a)| = n+|Z(G)|$, where a is a non-central element of G. In this case, one can easily seen that $|C_G(a)| = \frac{|G|(k(G)-|Z(G)|}{|G|-|Z(G)|}$.

3.3. Question. Is there any classification of finite groups G such that the commuting graph $\bar{\Gamma}(G)$ is a union of complete graphs K_n, where n is not necessary prime.

From now on, we compute exact formulas for the Schultz, Gutman, hyper-Wiener, Eccentric connectivity and Zagreb indices of $\Gamma(G)$. We begin with the hyper-Wiener index of a non-commuting graph of a finite group G.

Before going into the calculation of hyper-Wiener index of an arbitrary non-abelian group G, the hyper-Wiener index of the dihedral group D_{2n} and the group U_{6n} introduced in Section 2, are computed. If n is odd then $WW(\Gamma(D_{2n})) = 3(n-1)^2$ and if n is even $WW(\Gamma(D_{2n})) = 3(n^2 - 3n + 3)$. The hyper-Wiener index of the group U_{6n} is $WW(\Gamma(U_{6n})) = \frac{57}{2}n^2 - \frac{15}{2}n$.

3.4. Proposition. *Suppose G is a non-abelian finite group. Then*

$$WW(\Gamma(G)) = 3\binom{|G| - |Z(G)|}{2} - |G|(|G| - k(G)).$$

Proof. By a result of [1] the diameter of non-commuting graph is two and so,

$$\begin{aligned} WW(\Gamma(G)) &= \frac{1}{2}\sum_{\{u,v\}\subseteq V(\Gamma(G))}(d(u,v))^2 + \frac{1}{2}\sum_{\{u,v\}\subseteq V(\Gamma(G))}(d(u,v)) \\ &= \frac{1}{2}\Bigg[\sum_{uv\in E(\Gamma(G))}(d(u,v))^2 + \sum_{uv\in E(\bar{\Gamma}(G))}(d(u,v))^2 + \sum_{uv\in E(\Gamma(G))}(d(u,v)) \\ &\quad + \sum_{uv\in E(\bar{\Gamma}(G))}(d(u,v))\Bigg] \\ &= 3|E(\bar{\Gamma}(G))| + |E(\Gamma(G))| \\ &= 3\binom{|G| - |Z(G)|}{2} - |G|(|G| - k(G)), \end{aligned}$$

proving our proposition. □

Suppose n is an odd positive integer, then $\mathrm{MTI}(\Gamma(D_{2n})) = n(n-1)(7n-8)$. If n is even, then $\mathrm{MTI}(\Gamma(D_{2n})) = n(n-2)(7n-10)$ and $\mathrm{MTI}(\Gamma(U_{6n})) = 114n^3 - 36n^2$, in general. In what follows a general formula for the MTI index of $\Gamma(G)$, G non-abelian, is computed.

3.5. Proposition. *Suppose G is a non-abelian finite group. Then*

$$\mathrm{MTI}(\Gamma(G)) = 4|E(\Gamma(G))|(|G| - |Z(G)| - 1) - M_1(\Gamma(G)).$$

Proof. Suppose $\Gamma(G)$ has exactly n vertices. Then by [1, Proposition 2.1] and the definition of the Schultz index, we have:

$$\begin{aligned}
\mathrm{MTI}(\Gamma(G)) &= \sum_{\{u,v\}\subseteq V(\Gamma(G))} d(u,v)[\deg(u)+\deg(v)] \\
&= \sum_{uv\in E(\Gamma(G))} [\deg(u)+\deg(v)] + 2\sum_{uv\in E(\bar{\Gamma}(G))} [\deg(u)+\deg(v)] \\
&= \sum_{u\in V(\Gamma(G))} (\deg(u))^2 \\
&\quad + 2\sum_{uv\in E(\bar{\Gamma}(G))} [2(|G|-|Z(G)|-1) - (\deg_{\bar{\Gamma}(G)}(u) + \deg_{\bar{\Gamma}(G)}(v))] \\
&= M_1(\Gamma(G)) + 4(|G|-|Z(G)|-1)|E(\bar{\Gamma}(G))| \\
&\quad - 2\sum_{u\in V(\Gamma(G))} (\deg_{\bar{\Gamma}(G)}(u))^2 \\
&= M_1(\Gamma(G)) + 4(|G|-|Z(G)|-1)|E(\bar{\Gamma}(G))| \\
&- 2\sum_{u\in V(\Gamma(G))} [|G|-|Z(G)|-1-\deg_{\Gamma(G)}(u)]^2 \\
&= M_1(\Gamma(G)) + 4(|G|-|Z(G)|-1)|E(\bar{\Gamma}(G))| \\
&\quad - 2(|G|-|Z(G)|)(|G|-|Z(G)|-1)^2 \\
&\qquad + 8(|G|-|Z(G)|-1)|E(\Gamma(G))| - 2M_1(\Gamma(G)) \\
&= 4|E(\Gamma(G))|(|G|-|Z(G)|-1) - M_1(\Gamma(G)).
\end{aligned}$$

This completes our argument. □

Consider the dihedral group D_{2n} and the group U_{6n} introduced in Section 2. The Zagreb group indices of the non-commuting graph of these groups are computed as follows:

Table 1. The Zagreb Group Indices of $\Gamma(D_{2n})$ and $\Gamma(U_{6n})$

n is odd	$M_1(\Gamma(D_{2n})) = n(n-1)(5n-4)$	$M_2(\Gamma(D_{2n})) = 2n(n-1)^2(2n-1)$
n is even	$M_1(\Gamma(D_{2n})) = n(n-2)(5n-8)$	$M_2(\Gamma(D_{2n})) = 4n(n-1)(n-2)^2$
for each n	$M_1(\Gamma(U_{6n})) = 66n^3$	$M_2(\Gamma(U_{6n})) = 120n^4$.

3.6. Proposition. *Suppose G is a non-abelian finite group. Then*

$$M_1(\Gamma(G)) = 2|G||E(\Gamma(G))| - |G|^2(k(G) - |Z(G)|) + |G|^2\beta(G),$$

where $\beta(G) = \frac{1}{|G|^2}\sum_{x\in V(\Gamma(G))} |C_G(x)|^2$.

Proof. By definition,

$$\begin{aligned}
M_1(\Gamma(G)) &= \sum_{x\in V(\Gamma(G))} (\deg(x))^2 \\
&= \sum_{xy\in E(\Gamma(G))} (\deg(x)+\deg(y)) \\
&= \sum_{xy\in E(\Gamma(G))} (|G|-|C_G(x)|+|G|-|C_G(y)|) \\
&= 2|G||E(\Gamma(G))| - \sum_{xy\in E(\Gamma(G))} (|C_G(x)|+|C_G(y)|) \\
&= 2|G||E(\Gamma(G))| - \sum_{x\in V(\Gamma(G))} \deg(x)|C_G(x)| \\
&= 2|G||E(\Gamma(G))| - |G|^2(k(G)-|Z(G)|) + |G|^2\beta(G),
\end{aligned}$$

as desired. □

To compute an exact formula for $M_1(\Gamma(G))$, we must calculate $\sum_{x\in V(\Gamma(G))}|C_G(x)|^2$. Since $\beta(G) = \sum_{i=1}^{k(G)-|Z(G)|}\frac{1}{|x_i^G|}$, where the x_i's are representatives of the non-central conjugacy classes of G, $\sum_{x\in V(\Gamma(G))}|C_G(x)|^2 = |G|^2\sum_{i=1}^{k(G)-|Z(G)|}\frac{1}{|x_i^G|} = |G|^2\beta(G)$.

3.7. Question. Is there any simple closed formula for $\sum_{x\in V(\Gamma(G))}|C_G(x)|^2$?

The Gutman index of the non-commuting graph of D_{2n} and U_{6n} is computed as follows:

$$\begin{aligned}
&\mathrm{Gut}(\Gamma(D_{2n})) = n(n-1)(5n^2-8n+2);\ n \text{ is odd},\\
&\mathrm{Gut}(\Gamma(D_{2n})) = n^2(n-2)(5n-11);\ n \text{ is even},\\
&\mathrm{Gut}(\Gamma(U_{6n})) = 204n^4 - 66n^3.
\end{aligned}$$

3.8. Proposition. *Suppose G is a non-abelian finite group. Then*

$$\mathrm{Gut}(\Gamma(G)) = 4|E(\Gamma(G))|^2 - M_2(\Gamma(G)) - M_1(\Gamma(G)).$$

Proof. It follows from [2] that $\bar{M}_2(\Gamma(G)) = \sum_{uv\in\bar{E}(\Gamma(G))}\deg_{\Gamma(G)}(u)\deg_{\Gamma(G)}(v)$, and $\bar{M}_2(\Gamma(G)) = 2|E(\Gamma(G)|^2 - M_2(\Gamma(G)) - \frac{1}{2}M_1(\Gamma(G))$. Therefore,

$$\begin{aligned}
\mathrm{Gut}(\Gamma(G)) &= \sum_{\{u,v\}\subseteq V(\Gamma(G))} d(u,v)[\deg(u)\times\deg(v)] \\
&= \sum_{uv\in E(\Gamma(G))}[\deg(u)\times\deg(v)] + 2\sum_{uv\in E(\bar{\Gamma}(G))}[\deg(u)\times\deg(v)] \\
&= M_2(\Gamma(G)) + 2\bar{M}_2(\Gamma(G)) \\
&= -M_2(\Gamma(G)) + 4|E(\Gamma(G))|^2 - M_1(\Gamma(G)),
\end{aligned}$$

proving the result. □

Suppose $u\in G\setminus Z(G)$ is a vertex of $\Gamma(G)$. Then for the eccentricity of u we have:

$$\varepsilon(u) = \begin{cases} 1 & \forall\, v\in G\setminus Z(G)\cup\{u\},\ uv\neq vu,\\ 2 & \text{otherwise.}\end{cases}$$

It is clear that for odd n, $\xi^c(\Gamma(D_{2n})) = 4n(n-1)$, and $\xi^c(\Gamma(D_{2n})) = 6n(n-2)$, when n is even. Also, $\xi^c(\Gamma(U_{6n})) = 36n^2$. In the following proposition a closed formula for the eccentric connectivity index of $\Gamma(G)$, G non-abelian, is computed.

3.9. Proposition. *Suppose G is a non-abelian finite group, then*

$$\xi^c(\Gamma(G)) = \begin{cases} \frac{3}{2}|G|^2 + |G| - 2|G|k(G) & \exists u \in G \setminus Z(G) \ s.t. \ \varepsilon(u) = 1, \\ 2|G|^2 - 2|G|k(G) & otherwise. \end{cases}$$

Proof. We first prove that if $u \in G \setminus Z(G)$ and $\varepsilon(u) = 1$ then u has order two, $|Z(G)| = 1$ and $C_G(u) = \langle u \rangle$. To do this, we assume that $O(u) \neq 2$. Then u^2 and u^3 commute with u and so $u^2, u^3 \in Z(G)$. Suppose r and s are integers such that $2r + 3s = 1$. Then $u = (u^2)^r(u^3)^s \in Z(G)$, contradicting our assumption. If $C_G(u) = Z(G) \bigcup uZ(G)$ then $|C_G(u)| = 2|Z(G)|$, as desired. If not, there exists another coset $yZ(G)$ of $C_G(u)$ such that $y \notin Z(G)$. By assumption $yu \neq uy$ and so $y \notin C_G(u)$. Therefore $|C_G(u)| = 2|Z(G)|$.

On the other hand, $\varepsilon(u) = 1$ implies that $\deg(u) = |G| - |Z(G)| - 1 = |G| - |C_G(u)|$. Thus $|C_G(u)| = |Z(G)| + 1$ and it follows that $|Z(G)| = 1$.

Next we assume that there exists an element $u \in G \setminus Z(G)$ such that $\varepsilon(u) = 1$. It is clear that $|u^G| = \frac{|G|}{|C_G(u)|} = \frac{1}{2}|G|$. If there is an element $x \notin u^G$ with $\varepsilon(x) = 1$ then $G = u^G \bigcup x^G$, which is impossible. Thus, all of such elements are in the same conjugacy class of G.

If there exists $u \in G \setminus Z(G)$ with $\varepsilon(u) = 1$ then we have:

$$\begin{aligned}
\xi^c(\Gamma(G)) &= \sum_{u \in V(\Gamma(G))} \deg_{\Gamma(G)}(u)\varepsilon(u) \\
&= \sum_{u \in V(\Gamma(G)), \varepsilon(u)=1} \deg_{\Gamma(G)}(u) + 2 \sum_{u \in V(\Gamma(G)), \varepsilon(u)=2} \deg_{\Gamma(G)}(u) \\
&= \sum_{u \in V(\Gamma(G)), \varepsilon(u)=1} (|G| - |C_G(u)|) + 2 \sum_{u \in V(\Gamma(G)), \varepsilon(u)=2} (|G| - |C_G(u)|) \\
&= \sum_{u \in V(\Gamma(G)), \varepsilon(u)=1} (|G| - 2) + 2|G|\Big(\frac{1}{2}|G| - |Z(G)|\Big) \\
&\qquad - 2 \sum_{u \in V(\Gamma(G)), \varepsilon(u)=2} |C_G(u)| \\
&= \frac{1}{2}|G|^2 - |G| + |G|^2 - |G||Z(G)| \\
&\qquad - 2\Big(\sum_{u \in V(\Gamma(G))} |C_G(u)| - \sum_{u \in V(G), \varepsilon(u)=1} |C_G(u)| \Big) \\
&= \frac{3}{2}|G|^2 + |G| - 2|G|k(G),
\end{aligned}$$

as desired. Otherwise, for any $u \in G \setminus Z(G)$, $\varepsilon(u) = 2$ and $\xi^c(\Gamma(G)) = 2|G|^2 - 2|G|k(G)$. This completes our proof. □

3.10. Proposition. *If G is a non-abelian finite group then* $M_2(\Gamma(G)) = -|G|^2|E(\Gamma(G))| + |G|M_1(\Gamma(G)) + \sum_{xy \in E(\Gamma(G))} |C_G(x)||C_G(y)|$.

Proof. By Proposition 3.5, we have:

$$\begin{aligned}
M_2(\Gamma(G)) &= \sum_{xy\in E(\Gamma(G))} \deg(x)\deg(y)\\
&= \sum_{xy\in E(\Gamma(G))} (|G|-|C_G(x)|)(|G|-|C_G(y)|)\\
&= |G|^2|E(\Gamma(G))| - |G| \sum_{xy\in E(\Gamma(G))} (|C_G(x)|+|C_G(y)|)\\
& 4em + \sum_{xy\in E(\Gamma(G))} |C_G(x)||C_G(y)|\\
&= |G|^2||E(\Gamma(G))| - |G| \sum_{x\in V(\Gamma(G))} \deg(x)|C_G(x)|\\
&\quad + \sum_{xy\in E(\Gamma(G))} |C_G(x)||C_G(y)|\\
&= |G|^2|E(\Gamma(G))| - |G|^2 \sum_{x\in V(\Gamma(G))} |C_G(x)| + |G| \sum_{x\in V(\Gamma(G))} |C_G(x)|^2\\
&\quad + \sum_{xy\in E(\Gamma(G))} |C_G(x)||C_G(y)|\\
&= |G|^2|E(\Gamma(G))| - |G|^3(k(G)-|Z(G)|) + |G|(M_1(\Gamma(G))\\
&\quad - 2|G||E(\Gamma(G))| + |G|^2(k(G)-|Z(G)|))\\
&\quad + \sum_{xy\in E(\Gamma(G))} |C_G(x)||C_G(y)|\\
&= -|G|^2|E(\Gamma(G))| + |G|M_1(\Gamma(G)) + \sum_{xy\in E(\Gamma(G))} |C_G(x)||C_G(y)|
\end{aligned}$$

This completes our proof. □

We have several groups that they have the same size and the same Wiener index. For the first example, we consider non-abelian groups of order p^3, p is prime. Then the Wiener index is $\frac{1}{2}(p^6+p^5-p^4-3p^3+2p^2+2p)$. For the second example, we have groups of order $4p$, which $p \equiv 3(mod 4)$, the Wiener index is $10p^2-14p+6$. But it is possible to find two groups of the same order and different Wiener index. To do this, suppose $p<q<r$ are prime numbers such that $p|q-1$ and $p|r-1$. Define $G = Z_r \times T_{p,q}$ and $H = Z_q \times T_{p,r}$, where $T_{p,q}$ and $T_{p,r}$ are non-abelian groups of order pq and pr, respectively. Then G and H have exactly $r(1+\frac{q-1}{p})$ and $q(1+\frac{r-1}{p})$ conjugacy classes, respectively. We now apply the main result of [5] to prove:

$$\begin{aligned}
W(\Gamma(G)) &= \frac{1}{2}\left[p^2q^2r^2 - 3pqr^2 - 2pqr + q^2r^2 - qr^2 + 2r^2 + 2r\right],\\
W(\Gamma(H)) &= \frac{1}{2}\left[p^2q^2r^2 - 3pq^2r - 2pqr + q^2r^2 - q^2r + 2q^2 + 2q\right].
\end{aligned}$$

Now an easy calculation shows that $|G| = |H|$ but $W(\Gamma(G)) < W(\Gamma(H))$.

We end this paper with the following conjecture:

3.11. Conjecture. *Suppose n is a given positive integer. If for any two non-abelain groups G and H of order n, $W(\Gamma(G)) = W(\Gamma(H))$ then $n = p^3$ or $4p$, where $p \cong 3$ (* mod 4).

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