Hacettepe Journal of Mathematics and Statistics
Volume 41 (4) (2012), 527 – 536

ON SUM OF POWERS OF THE SIGNLESS LAPLACIAN EIGENVALUES OF GRAPHS

Muhuo Liu*†‡ and Bolian Liu†§

Received 21 : 07 : 2011 : Accepted 01 : 12 : 2011

Abstract

For a graph G and a real number α ($\alpha \neq 0, 1$), the graph invariant $S_\alpha(G)$ is the sum of the α^{th} power of the signless Laplacian eigenvalues of G. Let $\mathrm{IE}(G)$ denote the incidence energy of G, i.e., $\mathrm{IE}(G) = S_{\frac{1}{2}}(G)$. This note presents some properties and bounds for $S_\alpha(G)$ and $\mathrm{IE}(G)$.

Keywords: Signless Laplacian matrix, Laplacian matrix, Incidence energy.

2000 AMS Classification: 05 C 50, 05 C 75, 05 C 90.

1. Introduction

Let $G = (V, E)$ be a undirected simple graph with n vertices and m edges. Sometimes, G is referred to be an (n, m) graph. Suppose the degree of vertex v_i equals d_i for $i = 1, 2, \ldots, n$, then $(d_1, \ldots, d_n)$ is called the *degree sequence* of G. Throughout this paper, the degrees are enumerated in non-increasing order, i.e., $d_1 \geq d_2 \geq \cdots \geq d_n$. As usual, K_n and $K_{1,n-1}$ denote a complete graph and a star of order n, respectively.

Let $A(G)$ be the adjacency matrix, and $D(G)$ the diagonal matrix of vertex degrees of G, respectively. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. It is well known that both $L(G)$ and $Q(G)$ are symmetric and positive semidefinite, then we can denote the eigenvalues of $L(G)$ and $Q(G)$ by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$, respectively. If there is no confusion, we write $q_i(G)$ as q_i, and $\mu_i(G)$ as μ_i, respectively.

This work is supported by the Foundation for Distinguished Young Talents in Higher Education of Guangdong, China (No. LYM10039), NNSF of China (Nos. 11071088, 11201156), and Project of Graduate Education Innovation of Jiangsu Province (No. CXZZ12-0378).

*School of Mathematical Science, Nanjing Normal University, Nanjing, 210046 P. R. China. E-mail: (M. Liu) liumuhuo@163.com

†School of Mathematics, South China Normal University, Guangzhou, 510631 P. R. China. E-mail: (B. Liu) liubl@scnu.edu.cn

‡Department of Applied Mathematics, South China Agricultural University, Guangzhou, 510642 P. R. China.

§Corresponding Author.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $A(G)$. The *energy* $E(G)$ of G is defined as [7] $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This quantity has a long known application in molecular-orbital theory of organic molecules (see [8, 9]) and has been much investigated. In the sequel, Gutman and Zhou [13] posed the definition of *Laplacian energy* $LE(G)$ of an (n, m) graph G, where $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. There is a great deal of analogy between the properties of $E(G)$ and $LE(G)$, but also some significant differences [13].

Recently, the *Laplacian-energy-like invariant* of G, denoted by $\mathrm{LEL}(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$, has been defined and investigated in [16]. It is proved that $E(G)$ and $\mathrm{LEL}(G)$ have a number of similar properties [10, 14, 16], while also some significant differences [10, 14, 16]. Moreover, Stevanović *et al.* [24] showed that the LEL-invariant is a well designed molecular descriptor, which has great application in chemistry.

Motivated by the definition of $\mathrm{LEL}(G)$, Jooyandeh *et al.* [14] put forward the definition of the *incidence energy* $\mathrm{IE}(G)$ of G, where $\mathrm{IE}(G) = \sum_{i=1}^{n} \sqrt{q_i}$. They called $\mathrm{LEL}(G)$ the *directed incidence energy* $\mathrm{DIE}(G)$ of G to distinguish the notation incidence energy. This new invariant immediately attracted the attention of other scholars [10].

For the relation between the eigenvalues of $Q(G)$ and $L(G)$, it is well known that

1.1. Proposition. [4]

(i) *If G is connected, then $q_n(G) = 0$ if and only if G is bipartite.*
(ii) *If G is bipartite, then $Q(G)$ and $L(G)$ share the same eigenvalues.*

Since the definitions of $\mathrm{LEL}(G)$ and Kirhhoff index (one can refer to [11] for its definition), Zhou [26] put forward the definition $s_\alpha(G)$, where

$$s_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha(G).$$

In [26], Zhou called $s_\alpha(G)$ the *sum of powers of the Laplacian eigenvalues of* G, and he achieved some properties and bounds for $s_\alpha(G)$. In the sequel, some bounds of s_α for connected bipartite graphs were obtained in [25], which improve some known results of [26]. Moreover, Zhou established some bounds for s_α and for the Estrada index in terms of degree sequences in [27]. Motivated by the definitions of $\mathrm{LEL}(G)$, $\mathrm{IE}(G)$, $s_\alpha(G)$, and Proposition 1.1, the *sum of powers of the signless Laplacian eigenvalues of* G, denoted by $S_\alpha(G)$, was also investigated by other mathematicians [1], where

$$S_\alpha(G) = \sum_{i=1}^{n} q_i^\alpha(G).$$

In this paper, by employing similar techniques to those applied in [26], we establish some properties and bounds for $S_\alpha(G)$ and $\mathrm{IE}(G)$.

2. Bounds for $S_\alpha(G)$

2.1. Lemma. [5] *Let G be an (n, m) graph and e an edge of G. Then,*

$$0 \le q_n(G-e) \le q_n(G) \le q_{n-1}(G-e) \le q_{n-1}(G) \le \cdots \le q_1(G-e) \le q_1(G).$$

Denote by $\overline{G}$ the complement graph of G. By Lemma 2.1, we have

2.2. Theorem. *For any graph G on n vertices and $\alpha > 0$, $S_\alpha(G) \ge 0$, where the equality holds if and only if $G \cong \overline{K_n}$. Moreover, if G has components $G_1, \dots, G_p$, then $S_\alpha(G) = \sum_{i=1}^{p} S_\alpha(G_i)$.* □

Note that $\sum_{i=1}^{n} q_i(G) - \sum_{i=1}^{n} q_i(G-e) = 2$. By Lemma 2.1, it immediately follows that

2.3. Theorem. *Let e be an edge of G. Then, $S_\alpha(G) > S_\alpha(G-e)$ for $\alpha > 0$.* □

Suppose $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ are two non-increasing sequences of real numbers, we say (x) is *majorized* by (y), denoted by $(x) \trianglelefteq (y)$, if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, and $\sum_{i=1}^j x_i \le \sum_{i=1}^j y_i$ for all $j = 1, 2, \ldots, n$. Furthermore, by $(x) \triangleleft (y)$ we mean that $(x) \trianglelefteq (y)$ and (x) is not the rearrangement of (y).

2.4. Lemma. [16, 20] *Suppose $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ are non-increasing sequences of real numbers. If $(x) \trianglelefteq (y)$, then for any convex function ψ, $\sum_{i=1}^n \psi(x_i) \le \sum_{i=1}^n \psi(y_i)$. Furthermore, if $(x) \triangleleft (y)$ and ψ is a strictly convex function, then $\sum_{i=1}^n \psi(x_i) < \sum_{i=1}^n \psi(y_i)$.* □

Denote by $\Phi(G, x) = \det(xI - Q(G))$ the *signless Laplacian characteristic polynomial* of G. Let $SQ(G) = (q_1, q_2, \ldots, q_n)$ be the spectrum of $Q(G)$. Set

$$\begin{aligned}
A_\alpha(n) &= (n-2)(n-2)^\alpha + \Big(\frac{1}{2}\Big)^\alpha \big(3n-6-\sqrt{(n-2)(n+6)}\,\big)^\alpha \\
&\qquad + \Big(\frac{1}{2}\Big)^\alpha \big(3n-6+\sqrt{(n-2)(n+6)}\,\big)^\alpha, \\
B_\alpha(n) &= (n-4)^\alpha + (n-3)(n-2)^\alpha + \Big(\frac{1}{2}\Big)^\alpha \big(3n-6-\sqrt{n^2+4n-28}\,\big)^\alpha \\
&\qquad + \Big(\frac{1}{2}\Big)^\alpha \big(3n-6+\sqrt{n^2+4n-28}\,\big)^\alpha, \\
C_\alpha(n) &= (n-3)^\alpha + (n-3)(n-2)^\alpha + \Big(\frac{1}{2}\Big)^\alpha \big(3n-7-\sqrt{n^2+6n-23}\,\big)^\alpha \\
&\qquad + \Big(\frac{1}{2}\Big)^\alpha \big(3n-7+\sqrt{n^2+6n-23}\,\big)^\alpha.
\end{aligned}$$

2.5. Theorem. *For any connected graph G on n vertices and $\alpha > 0$, we have $S_\alpha(G) \le (n-1)(n-2)^\alpha + (2n-2)^\alpha$, where the equality holds if and only if $G \cong K_n$. Moreover, if $G \ncong K_n$, then $S_\alpha(G) \le A_\alpha(n)$, where equality holds if and only if $G \cong K_n - e$.*

Proof. By an elementary computation, it follows that

$$\begin{aligned}
SQ(K_n - e) = \Big(&\frac{3n-6+\sqrt{(n-2)(n+6)}}{2}, n-2, \ldots \\
&\ldots, n-2, \frac{3n-6-\sqrt{(n-2)(n+6)}}{2}\Big).
\end{aligned}$$

Note that $SQ(K_n) = (2n-2, n-2, \ldots, n-2)$. The result follows from Theorem 2.3. □

Let $G_1 \cup G_2$ be the new graph consisting of two (disconnected) components G_1 and G_2, and kG the new graph consisting of k copies of G. The *join* $G_1 \vee G_2$ of G_1 and G_2 is the graph having vertex set $V(G_1 \vee G_2) = V(G_1 \cup G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. Let $W_1 = K_{n-4} \vee C_4$, $W_2 = K_{n-3} \vee (K_1 \cup K_2)$.

2.6. Theorem. *Suppose G is a connected graph with $n \ge 6$ vertices, and $G \notin \{K_n, K_n - e\}$.*

(i) *If $0 < \alpha < 1$, then $S_\alpha(G) \le B_\alpha(n)$, where equality holds if and only if $G \cong W_1$.*
(ii) *If $\alpha > 1$, then $S_\alpha(G) \le C_\alpha(n)$, where equality holds if and only if $G \cong W_2$.*

Proof. By an elementary computation, we have

$$SQ(W_1) = \Big(\frac{3n-6+\sqrt{n^2+4n-28}}{2}, n-2, \ldots \\ \ldots, n-2, \frac{3n-6-\sqrt{n^2+4n-28}}{2}, n-4\Big),$$

$$SQ(W_2) = \Big(\frac{3n-7+\sqrt{n^2+6n-23}}{2}, n-2, \ldots \\ \ldots, n-2, n-3, \frac{3n-7-\sqrt{n^2+6n-23}}{2}\Big).$$

Observe that for $x > 0$, $-x^\alpha$ is a strictly convex function if $0 < \alpha < 1$, and $SQ(W_1) \lhd SQ(W_2)$. By Lemma 2.4, $B_\alpha(n) = S_\alpha(W_1) > S_\alpha(W_2) = C_\alpha(n)$ if $0 < \alpha < 1$. On the other hand, since W_1 and W_2 are all the graphs on n vertices with $\binom{n}{2} - 2$ edges, by Theorems 2.3 and 2.5, (i) follows.

Observe that for $x > 0$, x^α is a strictly convex function if $\alpha > 1$, and $SQ(W_1) \lhd SQ(W_2)$. By Lemma 2.4, $B_\alpha(n) = S_\alpha(W_1) < S_\alpha(W_2) = C_\alpha(n)$ if $\alpha > 1$. Thus, (ii) follows from Theorems 2.3 and 2.5. $\square$

2.7. Lemma. [18, 21] *Let G be a connected graph with diameter $d(G)$. If $Q(G)$ (resp. $L(G)$) has exactly k distinct eigenvalues, then $d(G) + 1 \le k$.* $\square$

2.8. Lemma. *If G is a connected graph on n vertices, then $q_2 = q_3 = \cdots = q_n$ if and only if $G \cong K_n$.*

Proof. If $q_2 = q_3 = \cdots = q_n$, then $d(G) = 1$ follows from Lemma 2.7. Thus, $G \cong K_n$. Conversely, if $G \cong K_n$, then $q_2 = q_3 = \cdots = q_n = n - 2$. The result follows. $\square$

The *first Zagreb index* $M_1 = M_1(G)$ is defined as [12] $M_1(G) = \sum_{i=1}^{n} d_i^2$.

2.9. Lemma. [17] *Suppose G is a connected (n, m) graph. Then $q_1 \ge \frac{M_1}{m}$, where equality holds if and only if G is a regular graph or a bipartite semiregular graph.* $\square$

2.10. Lemma. *Let G be a connected (n, m) graph, where $n \ge 3$. Then,*

$$\frac{M_1}{m} \ge 2\sqrt{\frac{M_1}{n}} \ge \frac{4m}{n} > \frac{2m}{n-1} > \frac{2m}{n}.$$

Proof. Note that

$$\frac{M_1}{m} = \frac{\sum_{i=1}^{n} d_i^2}{m} \ge \frac{(\sum_{i=1}^{n} d_i)^2}{mn} = \frac{(2m)^2}{mn} = \frac{4m}{n} > \frac{2m}{n-1} > \frac{2m}{n}.$$

Then, $4\frac{M_1}{n} = \frac{M_1}{m} \cdot 4\frac{m}{n} \ge \left(\frac{4m}{n}\right)^2$. The result follows. $\square$

2.11. Theorem. *Let G be a connected non-bipartite (n, m) graph with $n \ge 3$.*

(i) *If $\alpha < 0$ or $\alpha > 1$, then*

$$S_\alpha(G) \ge \left(\frac{M_1}{m}\right)^\alpha + \frac{(2m^2 - M_1)^\alpha}{m^\alpha (n-1)^{\alpha-1}} \tag{2.1}$$

where equality holds if and only if $G \cong K_n$.

(ii) *If $0 < \alpha < 1$, then*

$$S_\alpha(G) \le \left(\frac{M_1}{m}\right)^\alpha + \frac{(2m^2 - M_1)^\alpha}{m^\alpha (n-1)^{\alpha-1}}. \tag{2.2}$$

where equality holds if and only if $G \cong K_n$.

Proof. Here we only prove (i), (ii) can be shown similarly.

Observe that for $x > 0$, x^α is a strictly convex function if $\alpha < 0$ or $\alpha > 1$. Then,

$$\left(\sum_{i=2}^{n} \frac{q_i}{n-1}\right)^\alpha \leq \sum_{i=2}^{n} \frac{1}{n-1} q_i^\alpha.$$

Hence,

$$\sum_{i=2}^{n} q_i^\alpha \geq \frac{1}{(n-1)^{\alpha-1}} \left(\sum_{i=2}^{n} q_i\right)^\alpha = \frac{(2m-q_1)^\alpha}{(n-1)^{\alpha-1}},$$

where equality holds if and only if $q_2 = q_3 = \cdots = q_n$. It follows that

$$S_\alpha(G) \geq q_1^\alpha + \frac{(2m-q_1)^\alpha}{(n-1)^{\alpha-1}}.$$

Let $f(x) = x^\alpha + \frac{(2m-x)^\alpha}{(n-1)^{\alpha-1}}$. If $x \geq \frac{2m}{n}$, then $f'(x) = \alpha\left(x^{\alpha-1} - (\frac{2m-x}{n-1})^{\alpha-1}\right) \geq 0$ whether $\alpha < 0$ or $\alpha > 1$.

Note that $\frac{M_1}{m} > \frac{2m}{n}$ by Lemma 2.10. By Lemma 2.9, we have

$$S_\alpha(G) \geq f(q_1) \geq f(\frac{M_1}{m}) = \left(\frac{M_1}{m}\right)^\alpha + \frac{(2m^2 - M_1)^\alpha}{m^\alpha (n-1)^{\alpha-1}}.$$

If equality (2.1) holds, then $q_2 = q_3 = \cdots = q_n$ and $q_1 = \frac{M_1}{m}$. Thus, Lemmas 2.8 and 2.9 imply that $G \cong K_n$. For the converse, if $G \cong K_n$, it is easy to see that equality (2.1) holds. □

2.12. Lemma. [2] *If $G = (V, E)$ is a connected graph, then $\mu_1 \leq d_1 + d_2$, where equality holds if and only if G is a regular bipartite graph or a semiregular bipartite graph.* □

2.13. Lemma. [22] *If G_1 and G_2 are graphs on k and t vertices, respectively with eigenvalues $0 = \mu_k(G_1) \leq \mu_{k-1}(G_1) \leq \cdots \leq \mu_1(G_1)$ and $0 = \mu_t(G_2) \leq \mu_{t-1}(G_2) \leq \cdots \leq \mu_1(G_2)$ respectively, then the Laplacian eigenvalues of $G_1 \vee G_2$ are given by*

$$0, \mu_{k-1}(G_1) + t, \ldots, \mu_1(G_1) + t, \mu_{t-1}(G_2) + k, \ldots, \mu_1(G_2) + k, t + k. \qquad \square$$

2.14. Lemma. *Let G be a connected graph with $n \geq 3$ vertices. Then $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$*

Proof. If $G \cong K_n$ or $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, it is easy to see that $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$. Conversely, suppose $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$. By Lemma 2.7, the diameter of G satisfies $d(G) \leq 2$. We may suppose that $G \ncong K_n$, and hence $d(G) = 2$ in the following.

It is well known that $\mu_{n-1} \leq d_n$ if $G \ncong K_n$. Note that $\mu_2 \geq d_2$ (see [15]). Then, $\mu_{n-1} \leq d_n \leq d_{n-1} \leq \cdots \leq d_2 \leq \mu_2$. Thus, $d_n = d_{n-1} = \cdots = d_2 = \mu_2 = \mu_{n-1}$. It follows that

$$d_2 = \mu_2 = \frac{2m - \mu_1}{n-2} = \frac{(n-1)d_2 + d_1 - \mu_1}{n-2} = d_2 + \frac{d_2 + d_1 - \mu_1}{n-2}.$$

Thus, $\mu_1 = d_1 + d_2$. By Lemma 2.12, we can conclude that G is a complete regular bipartite graph or a complete semiregular bipartite graph because $d(G) = 2$.

If G is a complete regular bipartite graph, then $G \cong K_{\frac{n}{2},\frac{n}{2}}$. If G is a complete semiregular bipartite graph, then $G \cong K_{1,n-1}$ follows from Lemma 2.13 because $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$. □

2.15. Theorem. *Let G be a connected bipartite (n,m) graph with $n \geq 3$.*

(i) *If $\alpha > 1$, then*

$$s_\alpha(G) = S_\alpha(G) \geq \left(\frac{M_1}{m}\right)^\alpha + \frac{(2m^2 - M_1)^\alpha}{m^\alpha (n-2)^{\alpha-1}} \tag{2.3}$$

where equality holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

(ii) *If $0 < \alpha < 1$, then*

$$s_\alpha(G) = S_\alpha(G) \leq \left(\frac{M_1}{m}\right)^\alpha + \frac{(2m^2 - M_1)^\alpha}{m^\alpha (n-2)^{\alpha-1}}. \tag{2.4}$$

where equality holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. Here we only prove (i), (ii) can be shown similarly.

Note that $q_n = \mu_n = 0$ by Proposition 1.1. Using similar arguments as in the proof of Theorem 2.11 (i), we have

$$\left(\sum_{i=2}^{n-1} \frac{q_i}{n-2}\right)^\alpha \leq \sum_{i=2}^{n} \frac{1}{n-2} q_i^\alpha.$$

Thus, it follows that

$$\sum_{i=2}^{n-1} q_i^\alpha \geq \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} q_i\right)^\alpha = \frac{(2m-q_1)^\alpha}{(n-2)^{\alpha-1}},$$

where equality holds if and only if $q_2 = q_3 = \cdots = q_{n-1}$. Let $g(x) = x^\alpha + (n-2)\left(\frac{2m-x}{n-2}\right)^\alpha$. If $x \geq \frac{2m}{n-1}$, then $g'(x) = \alpha\left(x^{\alpha-1} - \left(\frac{2m-x}{n-2}\right)^{\alpha-1}\right) \geq 0$ for $\alpha > 1$. By Lemmas 2.9 and 2.10,

$$S_\alpha(G) \geq q_1^\alpha + \frac{(2m-q_1)^\alpha}{(n-2)^{\alpha-1}} \geq \left(\frac{M_1}{m}\right)^\alpha + \frac{(2m^2 - M_1)^\alpha}{m^\alpha(n-2)^{\alpha-1}}. \tag{2.5}$$

Note that all the equalities hold in (2.5) if and only if $q_2 = q_3 = \cdots = q_{n-1}$ and $q_1 = \frac{M_1}{m}$. By Lemma 2.9, Lemma 2.14 and Proposition 1.1, the second part of the theorem follows. □

2.16. Remark. With an observation to the proof of Theorem 2.15, it is easy to see that bound (2.3) also holds for $s_\alpha(G)$ when G is a connected bipartite (n,m) graph and $\alpha < 0$.

For a bipartite graph G, Zhou justified [26]

$$s_\alpha(G) = S_\alpha(G) \geq \left(2\sqrt{\frac{M_1}{n}}\right)^\alpha + \frac{\left(2m - 2\sqrt{\frac{M_1}{n}}\right)^\alpha}{(n-2)^{\alpha-1}} \tag{2.6}$$

if $\alpha < 0$ or $\alpha > 1$, and

$$s_\alpha(G) = S_\alpha(G) \leq \left(2\sqrt{\frac{M_1}{n}}\right)^\alpha + \frac{\left(2m - 2\sqrt{\frac{M_1}{n}}\right)^\alpha}{(n-2)^{\alpha-1}}. \tag{2.7}$$

if $0 < \alpha < 1$. When $x > \frac{2m}{n-1}$, $g(x)$ is increasing for $\alpha > 1$ and decreasing for $0 < \alpha < 1$. Thus, by Lemma 2.10 it follows that

2.17. Remark. The bound (2.3) is better than that of (2.6), and the bound (2.4) is better than that of (2.7). Moreover, if we can obtain a new bound $\mu_1 \geq \alpha \geq \frac{M_1}{m}$, then we can improve the bounds in Theorems 2.11 and 2.15.

Let $t(G)$ be the number of spanning trees of a connected graph G.

2.18. Lemma. [6] *If G is a connected bipartite graph on n vertices, then $\prod_{i=1}^{n-1} q_i = \prod_{i=1}^{n-1} u_i = nt(G)$. If G is a connected non-bipartite graph on n vertices, then $\prod_{i=1}^{n} q_i = 2\frac{t(G\times K_2)}{t(G)}$.* □

2.19. Theorem. *Let α be a real number with $\alpha \neq 0, 1$, and set $t_1 = 2\frac{t(G\times K_2)}{t(G)}$ and $t_2 = nt(G)$.*

(i) *If G is a connected non-bipartite (n, m) graph with $n \geq 3$, then*

$$S_\alpha(G) \geq \left(\frac{M_1}{m}\right)^\alpha + (n-1)\left(\frac{t_1 m}{M_1}\right)^{\frac{\alpha}{n-1}},$$

where equality holds if and only if $G \cong K_n$.

(ii) *If $\alpha > 0$ and G is a connected bipartite (n, m) graph with $n \geq 3$, then*

$$s_\alpha(G) = S_\alpha(G) \geq \left(\frac{M_1}{m}\right)^\alpha + (n-2)\left(\frac{t_2 m}{M_1}\right)^{\frac{\alpha}{n-2}}, \tag{2.8}$$

where equality holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof. Here we only prove (i), (ii) can be shown similarly.

By Lemma 2.18 and the arithmetic-geometric mean inequality, it follows that

$$S_\alpha(G) = q_1^\alpha + \sum_{i=2}^{n} q_i^\alpha \geq q_1^\alpha + (n-1)\left(\prod_{i=2}^{n} q_i^\alpha\right)^{\frac{1}{n-1}} = q_1^\alpha + (n-1)\left(\frac{t_1}{q_1}\right)^{\frac{\alpha}{n-1}},$$

where equality holds if and only if $q_2 = q_3 = \cdots = q_n$. Let $\varphi(x) = x^\alpha + (n-1)\left(\frac{t_1}{x}\right)^{\frac{\alpha}{n-1}}$. By solving

$$\varphi'(x) = \alpha\left(x^{\alpha-1} - (t_1)^{\frac{\alpha}{n-1}}\, x^{-\frac{\alpha}{n-1}-1}\right) \geq 0,$$

we conclude that $\varphi(x)$ is increasing for $x \geq (t_1)^{\frac{1}{n}}$ whether $\alpha > 0$ or $\alpha < 0$. On the other hand, by Lemmas 2.9 and 2.10 we have

$$q_1 \geq \frac{M_1}{m} > \frac{2m}{n} = \frac{\sum_{i=1}^{n} q_i}{n} \geq \left(\prod_{i=1}^{n} q_i\right)^{\frac{1}{n}} = (t_1)^{\frac{1}{n}}.$$

Thus, $S_\alpha(G) \geq \varphi(\frac{M_1}{m})$, and hence (i) follows. The equality holds in (i) if and only if $q_2 = q_3 = \cdots = q_n$ and $q_1 = \frac{M_1}{m}$, namely, if and only if $G \cong K_n$ by Lemmas 2.8 and 2.9. □

2.20. Remark. With an observation to the proof of Theorem 2.19, it is easy to see that bound (2.8) also holds for $s_\alpha(G)$ when G is a connected bipartite (n, m) graph and $\alpha < 0$.

2.21. Lemma. [18] *Let G be a graph with signless Laplacian spectrum $(q) = (q_1, q_2, \ldots, q_n)$ and degree sequence $(d) = (d_1, d_2, \ldots, d_n)$. Then, $(d) \trianglelefteq (q)$.* □

The *first general Zagreb index* of G, denoted by $Z_\alpha(G)$, is defined as [19] $Z_\alpha(G) = \sum_{i=1}^{n} d_i^\alpha$, where α is an arbitrary real number other than 0 or 1. The first general Zagreb index is also called the general zeroth-order Randić index [23]. Clearly, $Z_2(G) = M_1(G)$. The next result presents a relation between $Z_\alpha(G)$ and $S_\alpha(G)$.

2.22. Theorem. *Let G be a connected graph with $n \geq 2$ vertices.*

(i) *If $0 < \alpha < 1$, then $S_\alpha(G) < Z_\alpha(G)$;*

(ii) *If $\alpha > 1$, then $S_\alpha(G) > Z_\alpha(G)$.*

Proof. Here we only prove (i), (ii) can be shown similarly.

Let $(q) = (q_1, q_2, \ldots, q_n)$ and $(d) = (d_1, d_2, \ldots, d_n)$. Since G is connected, $q_1 \geq \mu_1 \geq d_1 + 1 > d_1$ (see [21]). Thus, $(d) \lhd (q)$ follows from Lemma 2.21. Observe that for $x > 0$, $-x^\alpha$ is a strictly convex function if $0 < \alpha < 1$. By Lemma 2.4, the result follows. □

3. Bounds for $\mathrm{IE}(G)$

Note that $\mathrm{IE}(G) = S_{\frac{1}{2}}(G)$. By inequalities (2.2) and (2.4), it follows that

3.1. Theorem.

(i) *Let G be a connected non-bipartite (n, m) graph, where $n \geq 3$. Then*

$$\mathrm{IE}(G) \leq \sqrt{\frac{M_1}{m}} + \sqrt{(n-1)\left(2m - \frac{M_1}{m}\right)}.$$

where equality holds if and only if $G \cong K_n$.

(ii) *Let G be a connected bipartite (n, m) graph, where $n \geq 3$. Then*

$$\mathrm{LEL}(G) = \mathrm{IE}(G) \leq \sqrt{\frac{M_1}{m}} + \sqrt{(n-2)\left(2m - \frac{M_1}{m}\right)}.$$

where equality holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$. □

In [10], Gutman *et al.* proved that

$$\mathrm{IE}(G) \leq \sqrt{2}\sqrt[4]{\frac{M_1}{n}} + \sqrt{(n-1)\left(2m - 2\sqrt{\frac{M_1}{n}}\right)}. \tag{3.1}$$

Note that the function $h(x) = \sqrt{x} + \sqrt{(n-1)(2m-x)}$ decreases on $x > \frac{2m}{n}$. By Lemma 2.10 and the fact that $(n-1)\left(2m - \frac{M_1}{m}\right) > (n-2)\left(2m - \frac{M_1}{m}\right)$, we have

3.2. Remark. The bounds of Theorem 3.1 are always better than bound (3.1).

Denote by Δ and δ the maximum and minimum degrees of G, respectively. In the following, we set $\beta = \frac{1}{2}\left(\Delta + \delta + \sqrt{(\Delta-\delta)^2 + 4\Delta}\right)$ for convenience.

3.3. Lemma. [3] *If G is a connected graph of order $n \geq 3$, then $q_1(G) \geq \beta$, where equality holds if and only if $G \cong K_{1,n-1}$.* □

By Lemma 3.3, it can be proved similarly to Theorems 2.11 and 2.15 that

3.4. Theorem.

(i) *Let G be a connected non-bipartite (n, m) graph, where $n \geq 3$. Then,*

$$\mathrm{IE}(G) < \sqrt{\beta} + \sqrt{(n-1)(2m-\beta)}.$$

(ii) *Let G be a connected bipartite (n, m) graph, where $n \geq 3$. Then*

$$\mathrm{LEL}(G) = \mathrm{IE}(G) \leq \sqrt{\beta} + \sqrt{(n-2)(2m-\beta)},$$

where equality holds if and only if $G \cong K_{1,n-1}$.

In [10], the next upper bound for $\mathrm{IE}(G)$ was given as:

$$\mathrm{IE}(G) < \sqrt{1+\Delta} + \sqrt{(n-1)(2m-1-\Delta)}. \tag{3.2}$$

3.5. Remark. Note that $\beta \geq \Delta + 1 > \frac{2m}{n}$ for any connected graph. Thus, the bounds of Theorem 3.4 are always finer than the bound (3.2).

Finally, we shall introduce the lower bounds for $\mathrm{IE}(G)$, which are a consequence of Theorem 2.19:

3.6. Theorem. *Let* $t_1 = 2\frac{t(G\times K_2)}{t(G)}$ *and* $t_2 = nt(G)$.

(i) *If* G *is a connected non-bipartite* (n,m) *graph with* $n \geq 3$*, then*

$$\mathrm{IE}(G) \geq \sqrt{\frac{M_1}{m}} + (n-1)\left(\frac{t_1 m}{M_1}\right)^{\frac{1}{2(n-1)}},$$

where equality holds if and only if $G \cong K_n$.

(ii) *If* G *is a connected bipartite* (n,m) *graph with* $n \geq 3$*, then*

$$\mathrm{LEL}(G) = \mathrm{IE}(G) \geq \sqrt{\frac{M_1}{m}} + (n-2)\left(\frac{t_2 m}{M_1}\right)^{\frac{1}{2(n-2)}},$$

where equality holds if and only if $G \cong K_{1,n-1}$ *or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments and suggestions, which led to an improvement of the original manuscript.

References

[1] Akbari, S., Ghorbani, E., Koolen, J.H. and Oboudi, M.R. *On sum of powers of the Laplacian and signless Laplacian eigenvalues of graphs*, Electron. J. Combin. **17**, R115, 2010.
[2] Anderson, W.N. and Morley, T.D. *Eigenvalues of the Laplacian of a graph*, Linear Multilinear Algebra **18**, 141–145, 1985.
[3] Chen, Y.Q. and Wang, L.G. *Sharp bounds for the largest eigenvalue of the signless Laplacian of a graph*, Linear Algebra Appl. **433**, 908–913, 2010.
[4] Cvetković, D., Rowlinson, P. and Simić, S.K. *Signless Laplacians of finite graphs*, Linear Algebra Appl. **423**, 155–171, 2007.
[5] Cvetković, D., Rowlinson, P. and Simić, S.K. *Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math. (Beograd), **81** (95), 11–27, 2007.
[6] Cvetković, D. and Simić, S.K. *Towards a spectral theory of graphs based on the signless Laplacian II*, Linear Algebra Appl. **432**, 2257–2272, 2010.
[7] Gutman, I. *The energy of a graph*, Ber. Math Statist. Sekt. Forschungsz. Graz. **103**, 1–22, 1978.
[8] Gutman, I. *Total π-electron energy of benzenoid hydrocarbons*, Topics Curr. Chem. **162**, 29–63, 1992.
[9] Gutman, I. *The energy of a graph: odd and new results*, in : A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.) (Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001), 196–211.
[10] Gutman, I., Kiani, D., Mirzakhah, M. and Zhou, B. *On incidence energy of a graph*, Linear Algebra Appl. **431**, 1223–1233, 2009.
[11] Gutman, I. and Mohar, B. *The quasi-Wiener and the Kirchhoff indices coincide*, J. Chem. Inf. Comput. Sci. **36**, 982–985, 1996.
[12] Gutman, I. and Trinajstić, N. *Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17**, 535–538, 1972.
[13] Gutman, I. and Zhou, B. *Laplacian energy of a graph*, Linear Algebra Appl. **414**, 29–37, 2006.
[14] Jooyandeh, M.R., Kiani, D. and Mirzakhah, M. *Incidence energy of a graph*, MATCH Commun. Math. Comput. Chem. **62**, 561–572, 2009.
[15] Li, J. and Pan, Y. *A note on the second largest eigenvalue of the Laplacian matrix of a graph*, Linear Multilinear Algebra **48**, 117–121, 2000.
[16] Liu, J.P. and Liu, B.L. *A Laplacian-energy-like invariant of a graph*, MATCH Commun. Math. Comput. Chem. **59**, 355–372, 2008.
[17] Liu, M.H. and Liu, B.L. *New sharp upper bounds for the first Zagreb index*, MATCH Commun. Math. Comput. Chem. **62**, 689–698, 2009.

[18] Liu, M. H. and Liu, B. L. *The signless Laplacian spread*, Linear Algebra Appl. **432** (2-3), 505–514, 2010.

[19] Li, X. L. and Zheng, J. *A unified approach to the extremal trees for different indices*, MATCH Commun. Math. Comput. Chem. **54**, 195–208, 2005.

[20] Marshall, A. W. and Olkin, I. *Inequalities, Theory of Majorization and its Applications* (Academic Press, New York, 1979).

[21] Merris, R. *Laplacian matrices of graphs: a survey*, Linear Algebra Appl. **197-198**, 143–176, 1994.

[22] Merris, R. *Laplacian graph eigenvectors*, Linear Algebra Appl. **278**, 221–236, 1998.

[23] Pavlović, L. *Maximal value of the zeroth-order Randić index*, Discrete Appl. Math. **127**, 615–626, 2003.

[24] Stevanović, D., Ilić, A., Onisor, C. and Diudea, M. *LEL–a newly designed molecular descriptor*, Acta Chim. Slov. **56**, 410–417, 2009.

[25] Tian, G. X., Huang, T. Z. and Zhou, B. *A note on sum of powers of the Laplacian eigenvalues of bipartite graphs*, Linear Algebra Appl. **430**, 2503–2510, 2009.

[26] Zhou, B. *On sum of powers of the Laplacian eigenvalues of graphs*, Linear Algebra Appl. **429**, 2239–2246, 2008.

[27] Zhou, B. *On sum of powers of Laplacian eigenvalues and Laplacian Estrada index of graphs*, MATCH Commun. Math. Comput. Chem. **62**, 611–619, 2009.