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NEAR GROUPS ON NEARNESS APPROXIMATION SPACES

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Abstract

Near set theory provides a formal basis for observation, comparison and classification of perceptual granules. In the near set approach, every perceptual granule is a set of objects that have their origin in the physical world. Objects that have, in some degree, affinities are considered perceptually near each other, i.e., objects with similar descriptions. In this paper, firstly we introduce the concept of near groups, near subgroups, near cosets, near invariant sub-groups, homomorphisms and isomorphisms of near groups in nearness approximation spaces. Then we give some properties of these near structures.

Keywords: Near set, Rough set, Approximation space, Nearness approximation space, Near group.

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1. Introduction

In 1982, the concept of a rough set was originally proposed by Pawlak [13] as a formal tool for modelling incompleteness and imprecision in information systems. The theory of rough sets is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A basic notion in the Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set.

An algebraic approach to rough sets has been given by Iwinski [7]. Afterwards, Biswas and Nanda [1] introduced the notion of rough subgroups. Kuroki in [8], introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [9] gave some properties of

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the lower and upper approximations with respect to normal subgroups, and Davvaz [2], introduced the notion of rough subring (respectively ideal) with respect to an ideal of a ring. In recent years, there has been a fast growing interest in this new emerging theory, ranging from work in pure theory such as topological and algebraic foundations [3, 4, 15, 26, 24, 25, 11, 19, 20] and [21], to diverse areas of applications [5, 6].

In 2002, near set theory was introduced by J. F. Peters as a generalization of rough set theory. In this theory, Peters depends on the features of objects to define the nearness of objects [18] and consequently, the classification of our universal set with respect to the available information of the objects. The concept of near set theory was motivated by image analysis and inspired by a study of the perception of the nearness of familiar physical objects was carried out in cooperation with Pawlak in a purely philosophical manner in a poem entitled “How Near” written in 2002 and published in 2007 [14]. More recent work considers a generalized approach theory in the study of the nearness of nonempty sets that resemble each other [20, 21] and a topological framework for the study of nearness and apartness of sets (see, e.g., [11, 19]).

Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real valued function representing a feature of physical objects such as images or behaviors of individual biological organisms or collections of artificial organisms such as robot societies. But in this paper, in a more general setting that includes data mining, probe functions φ_i will be defined to allow for non-numerical values, i.e., let $\varphi_i : X \longrightarrow V$, where V is the value set for the range of φ_i [22]. This more general definition of $\varphi_i \in \mathcal{F}$ is also better for setting forth the algebra and logic of near sets after the manner of algebra and logic.

This paper begins by introducing the basic concepts of near set theory [16]. Our aim in this article is to improve the concept of near group theory, which extends the notion of a group to include the algebraic structures of near sets. Our definition of near group is similar to the definition of rough groups [10]. Also, we introduce near subgroups, near cosets, near invariant subgroups, homomorphism and isomorphism of near groups in nearness approximation spaces, and we give some properties of these structures.

2. Preliminaries

In this section we give some definitions and properties regarding near sets [16].

2.1. Object Description.

Table 1. Description Symbols

Symbol	Interpretation
$\mathbb{R}$	Set of real numbers,
$\mathcal{O}$	Set of perceptual objects,
X	$X \subseteq \mathcal{O}$, set of sample objects,
x	$x \in \mathcal{O}$, sample objects,
$\mathcal{F}$	A set of functions representing object features,
B	$B \subseteq \mathcal{F}$,
Φ	$\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$, object description,
L	L is a description length,
i	$i \leq L$,
$\Phi(x)$	$\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_i(x), \ldots, \varphi_L(x))$. (1)

Objects are known by their descriptions. An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$. The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe an object of interest. Assume that $B \subseteq \mathcal{F}$ (see Table 1) is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \longrightarrow \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$, a vector containing measurements (returned values) associated with each functional value $\varphi_i(x)$ in (1), where the description length $|\Phi| = L$.

$$\textit{Object Description} : \Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_i(x), \ldots, \varphi_L(x)).$$

The intuition underlying a description $\Phi(x)$ is a recording of measurements from sensors, where each sensor is modelled by a function φ_i.

2.2. Nearness of Objects.

Table 2. Relation and Partition Symbols

Symbol	Interpretation
$\sim_B$	$\sim_B = \{(x, x') \mid f(x) = f(x')\ \forall f \in B\}$, indiscernibility relation,
$[x]_B$	$[x]_B = \{x' \in X \mid x \sim_B x'\}$, elementary set (class),
$\mathcal{O}/\sim_B$	$\mathcal{O}/\sim_B = \{[x]_B \mid x \in \mathcal{O}\}$, quotient set,
ξ_B	Partition $\xi_B = \mathcal{O}/\sim_B$,
Δ_{φ_i}	$\Delta_{\varphi_i} = \lvert\varphi_i(x') - \varphi_i(x)\rvert$, probe function difference.

Sample objects $X \subseteq \mathcal{O}$ are near each if and only if the objects have similar descriptions. Recall that each φ defines a description of an object (see Table 1). Then let Δ_{φ_i} denote

$$\Delta_{\varphi_i} = \left|\varphi_i\left(x'\right) - \varphi_i(x)\right|,$$

where $x, x' \in \mathcal{O}$. The difference Δ_φ leads to a definition of the indiscernibility relation $\sim_B$ introduced by Z. Pawlak [12]. (See Definition 2.1).

2.1. Definition. Let $x, x' \in \mathcal{O}$, $B \subseteq \mathcal{F}$. Then

$$\sim_B = \left\{\left(x, x'\right) \in \mathcal{O} \times \mathcal{O} \mid \forall \varphi_i \in B,\ \Delta_{\varphi_i} = 0\right\}$$

is called the *indiscernibility relation on* $\mathcal{O}$, where the description length $i \leq |\Phi|$.

2.2. Definition. Let $B \subseteq \mathcal{F}$ be a set of functions representing features of objects $x, x' \in \mathcal{O}$. Objects x, x' are called *minimally near each other* if there exists $\varphi_i \in B$ such that $x \sim_{\{\varphi_i\}} x'$, $\Delta_{\varphi_i} = 0$. We call it the "Nearness Description Principle - NDP" [16].

2.3. Theorem. *The objects in a class* $[x]_B \in \xi_B$ *are near objects.* □

2.3. Near Sets. The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects X, X' are considered near each other if the sets contain objects with at least partial matching descriptions.

2.4. Definition. Let $X, X' \subseteq \mathcal{O}$, $B \subseteq \mathcal{F}$. Set X is called *near* X', if there exists $x \in X$, $x' \in X'$, $\varphi_i \in B$ such that $x \sim_{\{\varphi_i\}} x'$.

2.5. Remark. If X is near X', then X is a near set relative to X' and X' is a near set relative to X. Notice that if we replace X' by X in Definition 2.4, this leads to what is known as reflexive nearness.

2.6. Definition. Let $X \subseteq \mathcal{O}$ and $x, x' \in X$. If x is near x', then X is called a *near set relative to itself* or the *reflexive nearness* of X.

2.7. Theorem. *A class in a partition ξ_B is a near set.* □

2.8. Theorem. *A partition ξ_B is a near set.* □

2.9. Definition. Let $X \subseteq \mathcal{O}$, $X', X'' \subseteq X$. If X', X'' are near sets, then X is a near set. We call it the "Hierarchy of Near Sets".

2.10. Theorem. *A set containing a near set is itself a near set.* □

2.4. Fundamental Approximation Space. This subsection presents a number of near sets resulting from the approximation of one set by another set. Approximations are carried out within the context of a fundamental approximation space FAS $= (\mathcal{O}, \mathcal{F}, \sim_B)$, where $\mathcal{O}$ is a set of perceived objects, $\mathcal{F}$ is a set of probe functions representing object features, and $\sim_B$ is an indiscernibility relation defined relative to $B \subseteq \mathcal{F}$. The space FAS is considered fundamental because it provided a framework for the original rough set theory [12]. It has also been observed that an approximation space is the formal counterpart of perception.

Approximation starts with the partition ξ_B of $\mathcal{O}$ defined by the relation $\sim_B$. Next, any set $X \subseteq \mathcal{O}$ is approximated by considering the relation between X and the classes $[x]_B \in \xi_B$, $x \in \mathcal{O}$. To see this, consider first the lower approximation of a set.

2.4.1. *Lower Approximation of a Set.*

Table 3. Approximation Notation

Symbol	Interpretation
$(\mathcal{O}, \mathcal{F}, \sim_B)$	Fundamental approximation space (FAS), $B \subseteq \mathcal{F}$,
B_*X	$\bigcup_{x:[x]_B \subseteq X} [x]_B$, B-lower approximation of X,
B^*X	$\bigcup_{x:[x]_B \cap X \neq \varnothing} [x]_B$, B-upper approximation of X,
$\mathrm{Bnd}_B X$	$\mathrm{Bnd}_B X = B^*X \setminus B_*X = \{x \mid x \in B^*X \text{ and } x \notin B_*X\}$.

Affinities between objects of interest in the set $X \subseteq \mathcal{O}$ and classes in the partition ξ_B can be discovered by identifying those classes that have objects in common with X. Approximation of the set X begins by determining which elementary sets $[x]_B \in \mathcal{O}/\sim_B$ are subsets of X. This discovery process leads to the construction of what is known as the B-lower approximation of $X \subseteq \mathcal{O}$, which is denoted by B_*X:

$$B_*X = \bigcup_{x:[x]_B \cap X \neq \varnothing} [x]_B .$$

2.11. Lemma. *The lower approximation B_*X of a set X is a near set.* □

2.12. Theorem. *If a set X has a non-empty lower approximation B_*X, then X is a near set.* □

2.4.2. *Upper Approximation of a Set.* To begin, assume that $X \subset \mathcal{O}$, where X contains perceived objects that are in some sense interesting. Also assume that B contains functions representing features of objects in $\mathcal{O}$. A B-upper approximation of X is defined as follows:

$$B^*X = \bigcup_{x:[x]_B \cap X \neq \varnothing} [x]_B .$$

2.13. Theorem. *The upper approximation B^*X and the set X are near sets.* □

2.4.3. *Boundary Region.* Let $\mathrm{Bnd}_B X$ denote the boundary region of an approximation defined as follows:

$$\mathrm{Bnd}_B X = B^* X \setminus B_* X = \{x \mid x \in B^* X \text{ and } x \notin B_* X\}.$$

2.14. Theorem. *A set X with an approximation boundary $|\mathrm{Bnd}_B X| \geq 0$ is a near set.* □

2.5. Nearness Approximation Spaces.

Table 4. Nearness Approximation Space Symbols

Symbol	Interpretation
B	$B \subseteq \mathcal{F}$,
B_r	$r \leq \lvert B\rvert$ probe functions in B,
$\sim_{B_r}$	Indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, equivalence class,
$\mathcal{O}/\sim_{B_r}$	$\mathcal{O}/\sim_{B_r} = \left\{[x]_{B_r} \mid x \in \mathcal{O}\right\}$, quotient set,
$\xi_{\mathcal{O},B_r}$	Partition $\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r}$,
r	$\binom{\lvert B\rvert}{r}$, i.e. $\lvert B\rvert$ probe functions $\varphi_i \in B$ taken r at a time,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$, set of partitions,
ν_{N_r}	$\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0,1]$, overlap function,
$N_r(B)_* X$	$N_r(B)_* X = \bigcup_{x:[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^* X$	$N_r(B)^* X = \bigcup_{x:[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$, upper approximation,
$\mathrm{Bnd}_{N_r(B)}(X)$	$N_r(B)^* X \setminus N_r(B)_* X = \left\{x \in N_r(B)^* X \mid x \notin N_r(B)_* X\right\}$.

A *nearness approximation space* (NAS) is a tuple $\mathrm{NAS} = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ where the approximation space NAS is defined with a set of perceived objects $\mathcal{O}$, a set of probe functions $\mathcal{F}$ representing object features, an indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, a collection of partitions (families of neighbourhoods) $N_r(B)$, and a neighbourhood overlap function ν_{N_r}. The relation $\sim_{B_r}$ is the usual indiscernibility relation from rough set theory restricted to a subset $B_r \subseteq B$. The subscript r denotes the cardinality of the restricted subset B_r, where we consider $\binom{|B|}{r}$, i.e., $|B|$ functions $\phi_i \in \mathcal{F}$ taken r at a time to define the relation $\sim_{B_r}$.

This relation defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$. These classes form a new set called the *quotient set* $\mathcal{O}/\sim_{B_r}$, where $\mathcal{O}/\sim_{B_r} = \left\{[x]_{B_r} \mid x \in \mathcal{O}\right\}$. In effect, each choice of probe functions B_r defines a partition $\xi_{\mathcal{O},B_r}$ on a set of objects $\mathcal{O}$, namely, $\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r}$. Every choice of the set B_r leads to a new partition of $\mathcal{O}$.

Let $\mathcal{F}$ denote a set of features for objects in a set X, where each $\phi_i \in \mathcal{F}$ maps X to some value set V_{ϕ_i} (range of ϕ_i). The value of $\phi_i(x)$ is a measurement associated with a feature of an object $x \in X$. The overlap function ν_{N_r} is defined by $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0,1]$, where $\wp(\mathcal{O})$ is the powerset of $\mathcal{O}$. The overlap function ν_{N_r} maps a pair of sets to a number in $[0,1]$, representing the degree of overlap between sets of objects with features defined by he probe functions $B_r \subseteq B$ [23]. For each subset $B_r \subseteq B$ of probe functions, define the binary relation $\sim_{B_r} = \{(x,x') \in \mathcal{O} \times \mathcal{O} \mid \forall \phi_i \in B_r,\ \phi_i(x) = \phi_i(x')\}$. Since each $\sim_{B_r}$ is, in fact, the usual indiscernibility relation [12], for $B_r \subseteq B$ and $x \in \mathcal{O}$, let $[x]_{B_r}$ denote the equivalence class containing x, i.e., $[x]_{B_r} = \{x' \in \mathcal{O} \mid \forall f \in B_r,\ f(x') = f(x)\}$. If $(x,x') \in \sim_{B_r}$, then x and x' are said to be *B-indiscernible with respect to all feature* *probe functions in* B_r. Then define a collection of partitions $N_r(B)$, where $N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$. Families of neighborhoods are constructed for each combination of probe functions in B using $\binom{|B|}{r}$, i.e., the probe functions $|B|$ taken r at a time.

3. Near groups and near subgroups

3.1. Definition. Let NAS $= (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and let $\cdot$ be a binary operation defined on $\mathcal{O}$. A subset G of perceptual objects $\mathcal{O}$ is called a *near group* if the following properties are satisfied.

(1) $\forall x, y \in G$, $x \cdot y \in N_r(B)^* G$;
(2) $\forall x, y \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^* G$;
(3) $\exists e \in N_r(B)^* G$ such that $\forall x \in G$, $x \cdot e = e \cdot x = x$, e is called the *near identity element* of the near group G;
(4) $\forall x \in G$, $\exists y \in G$ such that $x \cdot y = y \cdot x = e$, y is called the *near inverse element of x in* G.

3.2. Example. Let $\mathcal{O} = \{0, a, b, c, d, e, f, g, h, \imath\}$, $B = \{\phi_1, \phi_2, \phi_3\} \subseteq \mathcal{F}$ denote a set of perceptual objects and a set of functions, respectively. Sample values of the ϕ_1 function $\phi_1 : \mathcal{O} \longrightarrow V_1 = \{\alpha_1, \alpha_2, \alpha_3\}$, ϕ_2 function $\phi_2 : \mathcal{O} \longrightarrow V_2 = \{\alpha_1, \alpha_2\}$ and ϕ_3 function $\phi_3 : \mathcal{O} \longrightarrow V_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ are shown in Table 5.

Table 5

$\mathcal{O}$	ϕ_1	ϕ_2	ϕ_3
0	α_2	α_1	α_3
a	α_3	α_2	α_1
b	α_2	α_1	α_3
c	α_2	α_2	α_3
d	α_1	α_1	α_4
e	α_1	α_1	α_2
f	α_3	α_2	α_2
g	α_1	α_1	α_4
h	α_2	α_1	α_3
$\imath$	α_3	α_2	α_1

And let $\cdot$ be a binary operation of perceptual objects on $\mathcal{O}$ with the following table:

Table 6

$\cdot$	0	a	b	c	d	e	f	g	h	$\imath$
0	0	a	b	c	d	e	f	g	h	$\imath$
a	a	b	c	d	e	f	g	h	$\imath$	0
b	b	c	d	e	f	g	h	$\imath$	0	a
c	c	d	e	f	g	h	$\imath$	0	a	b
d	d	e	f	g	h	$\imath$	0	a	b	c
e	e	f	g	h	$\imath$	0	a	b	c	d
f	f	g	h	$\imath$	0	a	b	c	d	e
g	g	h	$\imath$	0	a	b	c	d	e	f
h	h	$\imath$	0	a	b	c	d	e	f	g
$\imath$	$\imath$	0	a	b	c	d	e	f	g	h

We can easily shown that $(\mathcal{O},\cdot)$ is a group. Let $G=\{0,a,b,e,h,\imath\}$ be a subset of the perceptual objects. Then let $\cdot$ be a binary operation of perceptual objects on $G\subseteq\mathcal{O}$ with the following table:

Table 7

$\cdot$	0	a	b	e	h	$\imath$
0	0	a	b	e	h	$\imath$
a	a	b	c	f	$\imath$	0
b	b	c	d	g	0	a
e	e	f	g	0	c	d
h	h	$\imath$	0	c	f	g
$\imath$	$\imath$	0	a	d	g	h

Now,

$$\begin{aligned}[0]_{\{\varphi_1\}} &= \{x'\in\mathcal{O}\mid \varphi_1(x')=\varphi_1(0)=\alpha_2\}\\ &=\{0,b,c,h\}=[b]_{\{\varphi_1\}}=[c]_{\{\varphi_1\}}=[h]_{\{\varphi_1\}},\\ [a]_{\{\varphi_1\}} &= \{x'\in\mathcal{O}\mid \varphi_1(x')=\varphi_1(a)=\alpha_3\}\\ &=\{a,f,\imath\}=[f]_{\{\varphi_1\}}=[\imath]_{\{\varphi_1\}},\\ [d]_{\{\varphi_1\}} &= \{x'\in\mathcal{O}\mid \varphi_1(x')=\varphi_1(d)=\alpha_1\}\\ &=\{d,e,g\}=[e]_{\{\varphi_1\}}=[g]_{\{\varphi_1\}}.\end{aligned}$$

Hence $\xi_{(\varphi_1)}=\left\{[0]_{\{\varphi_1\}},[a]_{\{\varphi_1\}},[d]_{\{\varphi_1\}}\right\}$. Also,

$$\begin{aligned}[0]_{\{\varphi_2\}} &= \{x'\in\mathcal{O}\mid \varphi_2(x')=\varphi_2(0)=\alpha_1\}\\ &=\{0,b,d,e,g,h\}=[b]_{\{\varphi 2\}}=[d]_{\{\varphi_2\}}=[e]_{\{\varphi_2\}}=[g]_{\{\varphi_2\}}=[h]_{\{\varphi_2\}},\\ [a]_{\{\varphi_2\}} &= \{x'\in\mathcal{O}\mid \varphi_2(x')=\varphi_2(a)=\alpha_2\}\\ &=\{a,c,f,\imath\}=[c]_{\{\varphi 2\}}=[f]_{\{\varphi_2\}}=[\imath]_{\{\varphi_2\}}.\end{aligned}$$

Thus $\xi_{(\varphi_2)}=\left\{[0]_{\{\varphi_2\}},[a]_{\{\varphi_2\}}\right\}$. Lastly,

$$\begin{aligned}[0]_{\{\varphi_3\}} &= \{x'\in\mathcal{O}\mid \varphi_3(x')=\varphi_1(0)=\alpha_3\}\\ &=\{0,b,c,h\}=[b]_{\{\varphi_3\}}=[c]_{\{\varphi_3\}}=[h]_{\{\varphi_3\}},\\ [a]_{\{\varphi_3\}} &= \{x'\in\mathcal{O}\mid \varphi_3(x')=\varphi_3(a)=\alpha_1\}\\ &=\{a,\imath\}=[\imath]_{\{\varphi_3\}},\\ [d]_{\{\varphi_3\}} &= \{x'\in\mathcal{O}\mid \varphi_3(x')=\varphi_3(d)=\alpha_4\}\\ &=\{d,g\}=[g]_{\{\varphi_1\}},\\ [e]_{\{\varphi_3\}} &= \{x'\in\mathcal{O}\mid \varphi_3(x')=\varphi_3(e)=\alpha_2\}=\{e,f\}=[f]_{\{\varphi_3\}},\end{aligned}$$

and so $\xi_{(\varphi_3)}=\left\{[0]_{\{\varphi_3\}},[a]_{\{\varphi_3\}},[d]_{\{\varphi_3\}},[e]_{\{\varphi_3\}}\right\}$.

Therefore, for $r=1$, a classification of $\mathcal{O}$ is $N_1(B)=\left\{\xi_{(\varphi_1)},\xi_{(\varphi_2)},\xi_{(\varphi_3)}\right\}$.

Then, we can write

$$
\begin{aligned}
N_1(B)^* G &= \bigcup_{x:[x]_{\{\varphi_i\}} \cap G \neq \varnothing} [x]_{\{\varphi_i\}} \\
&= \{0,b,c,h\} \cup \{a,f,\imath\} \cup \{d,e,g\} \cup \{0,b,d,e,g,h\} \\
&\qquad \cup \{a,c,f,\imath\} \cup \{a,\imath\} \cup \{e,f\} \\
&= \{0,a,b,c,d,e,f,g,h,\imath\} = \mathcal{O}.
\end{aligned}
$$

Since

(1) $\forall x,y \in G$, $x \cdot y \in N_r(B)^* G = \{0,a,b,c,d,e,f,g,h,\imath\}$;
(2) the property $\forall x,y \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds in $N_r(B)^* G$;
(3) $\exists\, 0 \in N_r(B)^* G$ such that $\forall x \in G$, $x \cdot 0 = 0 \cdot x = x$ (0 is called the *near identity element of the near group* G);
(4) the properties $\forall x \in G$, $\exists\, y \in G$ such that $x \cdot y = y \cdot x = e$ (y is called a *near inverse element of* x in G) are satisfied,

the subset G of the perceptual objects $\mathcal{O}$ is indeed a near group.

3.3. Example. Let $H = \{0,b,e,h\}$ be a subset of the near group $G = \{0,a,b,e,h,\imath\}$. Let $\cdot$ be a binary operation of perceptual objects on $H \subset G$ with the following table:

Table 8

$\cdot$	0	b	e	h
0	0	b	e	h
b	b	d	g	0
e	e	g	0	c
h	h	0	c	f

We know from Example 3.2, for $r = 1$, a classification of $\mathcal{O}$ is

$$N_1(B) = \left\{\xi_{(\varphi_1)}, \xi_{(\varphi_2)}, \xi_{(\varphi_3)}\right\}.$$

And we can write

$$
\begin{aligned}
N_1(B)^* H &= \bigcup_{x:[x]_{\{\varphi_i\}} \cap H \neq \varnothing} [x]_{\{\varphi_i\}} \\
&= \{0,b,c,h\} \cup \{d,e,g\} \cup \{0,b,d,e,g,h\} \cup \{e,f\} \\
&= \{0,b,c,d,e,f,g,h\} \neq \mathcal{O}.
\end{aligned}
$$

From Theorem 3.8, since the conditions

(1) $\forall\, x,y \in H$, $x \cdot y \in N_r(B)^* H$;
(2) $\forall\, x \in H$, $x^{-1} \in H$

hold, H is a near subgroup of the near group G.

3.4. Example. Let $\mathcal{O} = \{0,a,b,c,d,e,f,g,h,\imath\}$, $B = \{\phi_1,\phi_2,\phi_3\} \subseteq \mathcal{F}$ denote a set of perceptual objects and set of functions, respectively. Sample values of the ϕ_1 function $\phi_1 : \mathcal{O} \longrightarrow V_1 = \{\alpha_1,\alpha_2,\alpha_3\}$, the ϕ_2 function $\phi_2 : \mathcal{O} \longrightarrow V_2 = \{\alpha_1,\alpha_2\}$ and the ϕ_3 function $\phi_3 : \mathcal{O} \longrightarrow V_3 = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ are as shown in Table 5.

Additionally, $\cdot$ is a binary operation of perceptual objects on $\mathcal{O}$ with the following table:

Table 9

$\cdot$	0	a	b	c	d	e	f	g	h	$\imath$
0	0	a	b	c	d	e	f	g	h	$\imath$
a	a	b	c	d	e	f	g	h	$\imath$	0
b	b	c	d	e	f	g	h	$\imath$	0	a
c	c	d	e	f	g	h	$\imath$	0	a	b
d	d	e	f	g	h	$\imath$	0	a	b	c
e	e	f	g	h	$\imath$	0	a	b	c	e
f	f	g	h	$\imath$	0	a	b	c	d	e
g	g	h	$\imath$	0	a	b	c	d	e	f
h	h	$\imath$	0	a	b	c	d	e	f	g
$\imath$	$\imath$	0	a	b	c	e	e	f	g	h

Since $a \cdot (d \cdot \imath) \neq (a \cdot d) \cdot \imath$, $(\mathcal{O}, \cdot)$ is not a group. Let $H = \{0, b, e, h\}$ be a subset of the perceptual objects. Then, let $\cdot$ be a binary operation of the perceptual objects in $H \subseteq \mathcal{O}$ with the following table:

Table 10

$\cdot$	0	b	e	h
0	0	b	e	h
b	b	d	g	0
e	e	g	0	c
h	h	0	c	f

We know from Example 3.2, for $r = 1$, a classification of $\mathcal{O}$ is $N_1(B) = \left\{\xi_{(\varphi_1)}, \xi_{(\varphi_2)}, \xi_{(\varphi_3)}\right\}$.

And, we can write

$$N_1(B)^* H = \{0, b, c, d, e, f, g, h\} \neq \mathcal{O}.$$

From Theorem 3.8, since the conditions

(1) $\forall x, y \in H$, $x \cdot y \in N_r(B)^* H$;
(2) $\forall x \in H$, $x^{-1} \in H$,

hold, H is a near subgroup of the near group G.

3.5. Proposition. *Let G be a near group.*

(1) *There is one and only one identity element in near group G.*
(2) *$\forall x \in G$, there is only one y such that $x \cdot y = y \cdot x = e$; we denote it by x^{-1}.*
(3) $\left(x^{-1}\right)^{-1} = x$.
(4) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$. □

3.6. Proposition. *Let G be a near group. For all $a, x, x', y, y' \in G$,*

(1) *if $a \cdot x = a \cdot x'$ then $x = x'$,*
(2) *if $y \cdot a = y' \cdot a$ then $x = x'$.* □

3.7. Definition. A non-empty subset H of a near group G is called its *near subgroup*, if it is a near group itself with respect to the operation $\cdot$.

There is only one guaranteed trivial near subgroup of near group G, i.e. G itself. A necessary and sufficient condition for $\{e\}$ to be a trivial near subgroup of near group G is $e \in G$.

3.8. Theorem. *A necessary and sufficient condition for a subset H of a near group G to be a near subgroup is that:*

(1) $\forall x, y \in H,\ x \cdot y \in N_r(B)^* H$;
(2) $\forall x \in H,\ x^{-1} \in H$.

Proof. The necessary condition is obvious. We prove only the sufficient condition. By (1) we have $\forall x, y \in H,\ x \cdot y \in N_r(B)^* H$, by (2) we have $\forall x \in H,\ x^{-1} \in H$, so we have $\forall x \in H,\ x \cdot x^{-1} \in N_r(B)^* H$ by (1) and (2). And, since association holds in $N_r(B)^* G$, so it holds in $N_r(B)^* H$. Hence the theorem is proved. □

An important difference between a near group and group is the following:

3.9. Theorem. *Let H_1 and H_2 be two near subgroups of the near group G. A sufficient condition for the intersection of these two near subgroups of G to be a near subgroup of G is $N_r(B)^* H_1 \cap N_r(B)^* H_2 = N_r(B)^* (H_1 \cap H_2)$.*

Proof. Suppose H_1 and H_2 are two near subgroups of the near group G. It is obvious that $H_1 \cap H_2 \subset G$. Consider $x, y \in H_1 \cap H_2$. Since H_1 and H_2 are near subgroups, we have $x \cdot y \in N_r(B)^* H_1$, $x \cdot y \in N_r(B)^* H_2$, and $x^{-1} \in H_1$, $x^{-1} \in H_2$, i.e. $x \cdot y \in N_r(B)^* H_1 \cap N_r(B)^* H_2$ and $x^{-1} \in H_1 \cap H_2$. Assuming $N_r(B)^* H_1 \cap N_r(B)^* H_2 = N_r(B)^* (H_1 \cap H_2)$, we have $x \cdot y \in N_r(B)^* (H_1 \cap H_2)$ and $x^{-1} \in H_1 \cap H_2$. Thus $H_1 \cap H_2$ is a near subgroup of G. □

3.10. Definition. A near group is called a *commutative near group* if $x \cdot y = y \cdot x$ for all $x, y \in G$.

3.11. Example. Example 3.2 and Example 3.3 are commutative near groups.

4. Near cosets

Let $NAS = (\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space, $G \subset \mathcal{O}$ a near group and H a near subgroup of G. Let us define a relationship on the elements of the near group G as follows:

$$\sim:\ a \sim b \text{ if and only if } a \cdot b^{-1} \in H \cup \{e\}.$$

4.1. Theorem. *"$\sim$" is a compatible relation over the elements of the near group G.*

Proof. $\forall a \in G$, since G is a near group, $a^{-1} \in G$. Since $a \cdot a^{-1} = e$, we have $a \sim a$. Further, $\forall a, b \in G$, if $a \sim b$, then $a \cdot b^{-1} \in H \cup \{e\}$, i.e. $a \cdot b^{-1} \in H$ or $a \cdot b^{-1} \in \{e\}$. If $a \cdot b^{-1} \in H$, then, since H is a near subgroup of G, we have $\left(a \cdot b^{-1}\right)^{-1} = b \cdot a^{-1} \in H$, and thus $b \sim a$. If $a \cdot b^{-1} \in \{e\}$, then $a \cdot b^{-1} = e$. That means $b \cdot a^{-1} = \left(a \cdot b^{-1}\right)^{-1} = e^{-1} = e$, and thus $b \sim a$. Hence, "$\sim$" is compatible. □

4.2. Definition. A compatible category defined by the relation "$\sim$" is called a *near right coset*. The near right coset that contains the element a is denoted by $H \cdot a$, i.e.,

$$H \cdot a = \{h \cdot a \mid h \in H,\ a \in G,\ h \cdot a \in G\} \cup \{a\}.$$

Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space, $G \subset \mathcal{O}$ a near group and H a near subgroup of G. Consider the relation "$\sim'$" on the elements of G defined as follows:

$$\sim':\ a \sim' b \text{ if and only if } a^{-1} \cdot b \in H \cup \{e\}.$$

4.3. Theorem. *"$\sim'$" is a compatible relation over the elements of the near group G.* □

4.4. Definition. A compatible category defined by the relation "$\sim'$" is called a *near left coset*. The near left coset that contains the element a is denoted by $a \cdot H$, i.e.,

$$a \cdot H = \{a \cdot h \mid h \in H,\ a \in G,\ a \cdot h \in G\} \cup \{a\}.$$

4.5. Remark. Generally speaking, the binary operation of a near group does not satisfy the commutative law, so the compatible relations "$\sim$" and "$\sim'$" are different. As a result, the near left and right cosets are different.

4.6. Theorem. *The near left cosets and near right cosets are equal in number.*

Proof. Denote by S_1, S_2 the families of near right and and left cosets, respectively. Define $\varphi : S_1 \longrightarrow S_2$ such that $\varphi(H \cdot a) = a^{-1} \cdot H$. We prove that φ is a bijection.

(1) If $H \cdot a = H \cdot b$ $(a \neq b)$, then $a \cdot b^{-1} \in H$. Because H is a near subgroup, we have $b \cdot a^{-1} \in H$, that means $a^{-1} \in b^{-1} \cdot H$, i.e. $a^{-1} \cdot H = b^{-1} \cdot H$. Hence, φ is a mapping.

(2) Any element $a \cdot H$ of S_2 is the image of $H \cdot a^{-1}$, an element of S_1, hence φ is an onto mapping.

(3) If $H \cdot a \neq H \cdot b$, then $a \cdot b^{-1} \notin H$, i.e. $a^{-1} \cdot H \neq b^{-1} \cdot H$. Hence, φ is a one-to-one mapping.

Thus the near left cosets and near right cosets are equal in number. $\square$

4.7. Definition. The number of both near left cosets and near right cosets is called the *index* of the subgroup H in G.

5. Near normal subgroups

5.1. Definition. A near subgroup N of a near group G is called a *near normal subgroup* if $a \cdot N = N \cdot a$ for all $a \in G$.

5.2. Theorem. *A necessary and sufficient condition for a near subgroup N of near group G to be a near normal subgroup is that $a \cdot N \cdot a^{-1} = N$ for all $a \in G$.*

Proof. Suppose N is a near normal subgroup of G. By definition, $\forall\, a \in G$ we have $a \cdot N = N \cdot a$. Because G is a near group, we have

$$(a \cdot N) \cdot a^{-1} = (N \cdot a) \cdot a^{-1},$$
$$a \cdot N \cdot a^{-1} = N \cdot \left(a \cdot a^{-1}\right),$$

i.e. $a \cdot N \cdot a^{-1} = N$.

Suppose N is a near subgroup of G and $\forall\, a \in G$, $a \cdot N \cdot a^{-1} = N$. Then $\left(a \cdot N \cdot a^{-1}\right) \cdot a = N \cdot a$, i.e. $a \cdot N = N \cdot a$. Thus N is a near invariant subgroup of G. $\square$

5.3. Theorem. *A necessary and sufficient condition for a near subgroup N of the near group G to be a near normal subgroup is that $a \cdot n \cdot a^{-1} \in N$ for all $a \in G$ and $n \in N$.*

Proof. Suppose N is a near normal subgroup of the near group G. We have $a \cdot N \cdot a^{-1} = N$ for all $a \in G$. For any $n \in N$, therefore, we have $a \cdot n \cdot a^{-1} \in N$.

Suppose N is a near subgroup of the near group G. Suppose $a \cdot n \cdot a^{-1} \in N$ for all $a \in G$ and $n \in N$. We have $a \cdot n \cdot a^{-1} \subset N$. Because $a^{-1} \in G$, we further have $a \cdot n \cdot a^{-1} \subset N$. It follows that $a \cdot \left(a^{-1} \cdot n \cdot a\right) \cdot a^{-1} \subset a \cdot n \cdot a^{-1}$, i.e. $N \subset a \cdot n \cdot a^{-1}$. Since $a \cdot n \cdot a^{-1} \subset N$ and $N \subset a \cdot n \cdot a^{-1}$, we have $a \cdot n \cdot a^{-1} = N$. Thus N is a near normal subgroup. $\square$

6. Homomorphisms of near groups

Let $(\mathcal{O}_1, \mathcal{F}_1, \sim_{B_{r_1}}, N_{r_1}, \nu_{N_{r_1}})$ and $(\mathcal{O}_2, \mathcal{F}_2, \sim_{B_{r_2}}, N_{r_2}, \nu_{N_{r_2}})$ be two nearness approximation spaces, and let $\cdot, \circ$ be binary operations over $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively.

6.1. Definition. Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups. If there exists a surjection $\varphi : N_{r_1}(B)^* G_1 \to N_{r_2}(B)^* G_2$ such that $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in N_{r_1}(B)^* G_1$ then φ is called a *near homomorphism* and G_1, G_2 are called *near homomorphic groups.*

Throughout this section φ is a near homomorphism such that $\varphi : N_{r_1}(B)^* G_1 \to N_{r_2}(B)^* G_2$, $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in N_{r_1}(B)^* G_1$.

6.2. Theorem. *Let G_1 and G_2 be near homomorphic groups. If $\cdot$ satisfies the commutative law, then $\circ$ also satisfies it.*

Proof. Consider G_1, G_2, and φ such that $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in N_{r_1}(B)^* G_1$. For every $\varphi(x)$, $\varphi(y) \in N_{r_2}(B)^* G_2$ since φ is surjective, there exist $x, y \in N_{r_1}(B)^* G_1$ such that $x \mapsto \varphi(x)$, $y \mapsto \varphi(y)$. Thus $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$, and $\varphi(y \cdot x) = \varphi(y) \circ \varphi(x)$. Now, assuming $x \cdot y = y \cdot x$, we obtain $\varphi(x) \circ \varphi(y) = \varphi(y) \circ \varphi(x)$. That means that $\circ$ satisfies the commutative law. □

6.3. Theorem. *Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic and let $N_r(B)^* \varphi(G_1) = N_r(B)^* G_2$. Then $\varphi(G_1)$ is a near group.*

Proof. (1) $\forall x', y' \in \varphi(G_1)$, consider $x, y \in G_1$ such that $x \mapsto x'$, $y \mapsto y'$. We have $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y) \in N_r(B)^* G_2 = N_r(B)^* \varphi(G_1)$, that is $x' \circ y' \in N_r(B)^* \varphi(G_1)$.

(2) Since $e \in N_r(B)^* G_1$, $\varphi(e) \in N_r(B)^* G_2$ and $\forall \varphi(x) \in \varphi(G_1)$, $\varphi(e) \circ \varphi(x) = \varphi(x \cdot e) = \varphi(x)$.

(3) G_1 is a near group, so $\forall x, y, z \in G_1$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Hence,

$$\varphi(x \cdot (y \cdot z)) = \varphi(x) \circ \varphi(y \cdot z) = \varphi(x) \circ (\varphi(y) \circ \varphi(z)),$$
$$\varphi((x \cdot y) \cdot z) = \varphi(x \cdot y) \circ \varphi(z) = (\varphi(x) \circ \varphi(y)) \circ \varphi(z),$$

i.e., $(\varphi(x) \circ \varphi(y)) \circ \varphi(z) = \varphi(x) \circ (\varphi(y) \circ \varphi(z))$.

(4) $\forall x' \in \varphi(G_1)$, consider $x \in G_1$ such that $x \mapsto x'$. Since G_1 is a near group, $x^{-1} \in G_1$. Hence, $\varphi(x^{-1}) \in \varphi(G_1)$ and $\varphi(x) \circ \varphi(x^{-1}) = \varphi(x^{-1}) \circ \varphi(x) = \varphi(e)$. Therefore, we can put $(x')^{-1} = \varphi(x^{-1})$. Consequently, we can conclude that $\varphi(G_1)$ is a near group. □

6.4. Theorem. *Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic. Let e and e' be the near identity elements of G_1 and G_2, respectively. Then $\varphi(e) = e'$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$, for all $a \in N_r(B)^* G_1$.* □

6.5. Definition. Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic. Let e and e' be the near identity elements of G_1 and G_2 respectively. The set $\{x \mid \varphi(x) = e',\ x \in G_1\}$ is called the near homomorphism kernel, denote by N.

6.6. Theorem. *Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic. The near homomorphism kernel N is a near invariant subgroup of G_1.*

Proof. Let φ be an onto mapping from $N_{r_1}(B)^* G_1$ to $N_{r_2}(B)^* G_2$. Then $\forall x, y \in N$ we have $\varphi(x) = e'$, $\varphi(y) = e'$. Thus $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y) = e' \circ e' = e'$, i.e. $x \cdot y \in N$. Moreover, $\forall x \in N$, we have $\varphi(x) = e'$. Because $\varphi(x^{-1}) = \varphi(y)^{-1} = e'^{-1} = e'$, we get $x^{-1} \in N$. We can conclude that N is a near invariant subgroup of G_1. □

6.7. Theorem. *Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic. Let H_1, N_1 be a near subgroup and a near normal subgroup of G_1, respectively. Then;*

(1) *$\varphi(H_1)$ is near subgroup of G_2 if $\varphi(N_{r_1}(B)^* H_1) = N_{r_2}(B)^* \varphi(H_1)$;*
(2) *$\varphi(N_1)$ is near normal subgroup of G_2 if $\varphi(G_1) = G_2$ and $\varphi(N_{r_1}(B)^* N_1) = N_{r_2}(B)^* \varphi(N_1)$.*

Proof. (1) Consider an onto mapping φ from $N_{r_1}(B)^* G_1$ to $N_{r_2}(B)^* G_2$ such that $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in N_{r_1}(B)^* G_1$. For all $\varphi(x)$, $\varphi(y) \in \varphi(H_1)$, by the definition of φ, there exists $x, y \in H_1$ such that $x \mapsto \varphi(x)$ and $\varphi(x) \circ \varphi(y) = \varphi(x \cdot y) \in \varphi(N_{r_1}(B)^* H_1)$. Because $\varphi(N_{r_1}(B)^* H_1) = N_{r_1}(B)^* \varphi(H_1)$, we have $\varphi(x) \circ \varphi(y) \in \varphi(N_{r_1}(B)^* H_1)$.

Further, for all $\varphi(x) \in \varphi(H_1)$, by the definition of φ there exists $x \in H_1$ such that $x \mapsto \varphi(x)$, $y \mapsto \varphi(y)$. Because H_1 is a near subgroup of G_1, we have $x^{-1} \in H_1$. Thus $\varphi(a)^{-1} = \varphi(a^{-1}) \in \varphi(H_1)$. We can conclude that $\varphi(H_1)$ is a near subgroup of G_2.

(2) By (1), it is easy to see that $\varphi(N_1)$ is a near subgroup of G_2 if $\varphi(N_{r_1}(B)^* N_1) = N_{r_2}(B)^* \varphi(N_1)$. For all $\varphi(x) \in G_2$, because $\varphi(G_1) = G_2$, we have $\varphi(x) \in \varphi(G_1)$.

Thus $x \in G_1$, $x^{-1} \in G_1$ and $\varphi(x^{-1}) \in \varphi(G_1) = G_2$. Because for all $\varphi(x) \in G_2$, $\varphi(n) \in \varphi(N_1)$ we have $\varphi(x) \circ \varphi(n) \circ \varphi(x^{-1}) = \varphi(x \cdot n \cdot x^{-1})$ and N_1 is near normal subgroup of G_1, we have $x \cdot n \cdot x^{-1} \in N_1$. Hence $\varphi(x) \circ \varphi(n) \circ \varphi(x^{-1}) \in \varphi(N_1)$. We can conclude that $\varphi(N_1)$ is a near normal subgroup of G_2. □

6.8. Theorem. *Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be near groups that are near homomorphic. Let H_2, N_2 be a near subgroup and a near normal subgroup of G_2, respectively. Then;*

(1) *H_1, which is the inverse image of H_2, is a near subgroup of G_1 if $\varphi(N_{r_1}(B)^* H_1) = N_{r_2}(B)^* H_2$.*
(2) *N_1, which is the inverse image of N_2, is a near normal subgroup of G_1 if $\varphi(G_1) = G_2$ and $\varphi(N_{r_1}(B)^* N_1) = N_{r_2}(B)^* N_2$.*

Proof. (1) H_1 is certainly the inverse image of H_2, and moreover we have $\varphi(H_1) = H_2$. That is, we have $\varphi(x)$, $\varphi(y) \in H_2$ for all $x, y \in H_1$. Because H_2 is a near subgroup of G_2, we have $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y) \in N_{r_2}(B)^* H_2 = \varphi(N_{r_1}(B)^* H_1)$. Thus $x * y \in N_{r_1}(B)^* H_1$. We have $\varphi(x) \in H_2$ for all $x \in H_1$. Because H_2 is a near subgroup of G_2, we have $\varphi(x)^{-1} = \varphi(x^{-1}) \in H_2$. Thus $x^{-1} \in H_1$.

(2) From (1), we can easily shown that N_1 is a near subgroup of G_2 if $\varphi(N_{r_1}(B)^* N_1) = N_{r_2}(B)^* \varphi(N_1)$. We have $\varphi(x) \in \varphi(G_1) = G_2$, $\varphi(x)^{-1} = \varphi(x^{-1}) \in \varphi(G_1) = G_2$, $\varphi(n) \in N_2$ for all $x \in G_1$, $n \in N_1$. Because N_2 is a near normal subgroup of G_2, we have $\varphi(x) \circ \varphi(n) \circ \varphi(x^{-1}) = \varphi(x \cdot n \cdot x^{-1}) \in N_2$. Thus $x \cdot n \cdot x^{-1} \in N_1$. Hence, N_1, which is the inverse image of N_2, is a near normal subgroup of G_1 if $\varphi(G_1) = G_2$ and $\varphi(N_{r_1}(B)^* N_1) = N_{r_2}(B)^* N_2$. □

7. Conclusion

In this paper, we have studied near groups and the algebraic properties of near groups. This work is focused on near groups, near subgroups, near cosets, near invariant subgroups and homomorphism of near groups. To extend this work, one could study the properties of other algebraic structures arising from near set theory.

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